ON THE GLOBAL SOLUTIONS OF THE PARABOLIC OBSTACLE PROBLEM

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Abstract. In this note we consider a parabolic obstacle problem with zero constraint. The exact representation of the global solutions (i.e., solutions in the entire half-space $\{(x,t) \in \mathbb{R}^{n+1} : x_1 > 0\}$) is established. Based on this representation we characterize the behaviour of the free boundary near contact points.

§0. Introduction

Our objective in this paper is to find the exact representation for all possible solutions to the parabolic obstacle problem in the half-space $\{x_1 > 0\}$ with zero Dirichlet condition on the boundary of the half-space. It is also assumed that near the infinity solutions could have quadratic growth with respect to the space variables and linear growth in time. Such a global analysis is essential in studying the local properties of a free boundary.

The idea to use information about global solutions in conjunction with blow-up technique for studying the local properties of surfaces has its origin in investigations of minimal surfaces in the seventies. For free boundary problems such an approach has been used in the papers [Ca1-Ca2], [CKS] and [SU]. It should be noted that the development of techniques based on global analysis has made it possible to give a complete description of the regularity properties of the free boundary for the elliptic problem without "sign-restriction" on the solution.

For parabolic equations the simplest obstacle problem can be formulated as the following variational inequality:

let $\mathcal{D}$ be a domain in $\mathbb{R}^n$, $\quad \mathcal{Q} = \mathcal{D} \times]0, T[$,

$$\mathcal{K} = \{w \in H^1(\mathcal{Q}) : \quad w \geqslant 0 \text{ a.e. in } \mathcal{Q}, \quad w = \phi \text{ on } \partial'\mathcal{Q}\},$$

where ϕ be a nonnegative function defined on the parabolic boundary $\partial'\mathcal{Q}$ of the cylinder $\mathcal{Q}$. It is required to find a function $u \in \mathcal{K}$ such that

$$\int_{\mathcal{D}} \partial_t u(w-u)dx + \int_{\mathcal{D}} DuD(w-u)dx + \int_{\mathcal{D}} (w-u)dx \geqslant 0$$

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a.e. in $t \in]0,T[$, and for all $w \in \mathcal{K}$.

It is known that a solution u of the problem, formulated above, satisfies (in the sense of distributions) the equation

$$\Delta u - \partial_t u = \chi_\Omega \quad \text{in } \mathcal{Q},$$

where $\Omega = \{(x,t) \in \mathcal{Q} : \ u(x,t) > 0\}$.

The regularity of the free boundary for this problem has been investigated earlier only in the special case of the Stefan problem ([Ca1]), where boundary conditions guarantee the additional information $\partial_t u \geqslant 0$. The results of the present paper enable us to avoid any assumptions on the time-derivative of solutions.

In studying global solutions we replace $\mathcal{D}$ by the half-space $\{x_1 > 0\}$ and assume that a solution u is defined for all $t \in]-\infty, +\infty[$ and satisfies the boundary condition $u|_{x_1=0} = 0$.

Once classifying global solutions we can apply it, in Section 3, to analyse the local behaviour of the free boundary near the fix one. The interior counterpart of this problem is under investigation in [CPS].

Notations and definitions.

Throughout the paper we will use the following notations:
$z = (x,t)$ are points in $\mathbb{R}^{n+1}$, where $x = (x_1, x') = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $t \in \mathbb{R}^1$;
$\mathbb{R}^{n+1}_a = \{(x,t) \in \mathbb{R}^{n+1} : x_1 > a\}$, where $a \in \mathbb{R}$;
$\mathbb{R}^{n+1}_+ = \mathbb{R}^{n+1}_0$,
$\mathbb{R}^{n+1}_- = \{(x,t) \in \mathbb{R}^{n+1} : x_1 < 0\}$;
$\Pi_a = \{(x,t) \in \mathbb{R}^{n+1} : x_1 = a\}$;
$\Pi = \Pi_0$;
$e_1, \ldots, e_n$ is the standard basis in $\mathbb{R}^n$;
χ_Ω denotes the characteristic function of the set Ω ($\Omega \subset \mathbb{R}^{n+1}$);
$v_+ = \max\{v, 0\}$; $\quad v_- = \max\{-v, 0\}$;
$B_r(x^0)$ is the open ball in $\mathbb{R}^n$ with center x^0 and radius r;
$B_r = B_r(0)$;
$B^+_r = B_r \cap \mathbb{R}^{n+1}_+$;
$Q_r(x^0,t^0) = B_r(x^0) \times]t^0 - r^2, t^0[$ denotes the open cylinder in $\mathbb{R}^{n+1}$;
$Q_r = Q_r(0,0)$;
$Q^+_r(x^0,t^0) = Q_r(x^0,t^0) \cap \mathbb{R}^{n+1}_+$.

If $Q = \mathbb{R}^{n+1}_a \cap Q_r(x^0,t^0)$ then $\hat{Q} = \{(x,t_0) : x_1 > a, \ |x - x^0| < r\}$ is the top of Q and $\partial' Q$ is it's parabolic boundary, i.e. the topological boundary minus the top of the cylinder.

D_i denotes the differential operator with respect to x_i; $\partial_t = \frac{\partial}{\partial t}$;
$D = (D_1, D') = (D_1, D_2, \ldots, D_n)$ denotes the spatial gradient;
D_ν denotes the operator of differentiation along the direction $\nu \in \mathbb{R}^n$, i.e., $|\nu| = 1$ and

$$D_\nu u = \sum_{i=1}^{n} D_i u \cdot \nu_i$$

$H = \Delta - \partial_t$ is the heat operator;
$\| \cdot \|_{p,\Omega}$ denotes the norm in $L_p(\Omega)$, $1 \leqslant p \leqslant \infty$.

$W^{2,1}_p(\Omega)$ is the anisotropic Sobolev space with the norm

$$\|u\|_{W^{2,1}_p(\Omega)} = \|\partial_t u\|_{p,\Omega} + \|D(Du)\|_{p,\Omega} + \|u\|_{p,\Omega}.$$

For a nonnegative $C^1_x \cap C^0_t$- function u defined in $\mathbb{R}^{n+1}_a \cup \Pi_a$ we introduce the sets
$\Lambda(u) = \{(x,t) \in \mathbb{R}^{n+1}_a \cup \Pi_a : u(x,t) = |Du(x,t)| = 0\}$;
$\Omega(u) = \{(x,t) \in \mathbb{R}^{n+1}_a : u(x,t) > 0\} = \mathbb{R}^{n+1}_a \setminus \Lambda(u)$;
$\Gamma(u) = \partial\Omega(u) \cap \Lambda(u)$ is the free boundary;
$\Gamma(u) \cap \Pi_a$ is the set of contact points.

For a point $z = (x,t) \in \Omega(u)$ we define the **parabolic distance** to the free boundary as follows:

$$d(z) = \operatorname{dist}\big\{z, \Gamma(u)\big\} := \sup\big\{r > 0 : Q_r(z) \cap \Gamma(u) = \emptyset\big\}.$$

Let M be a constant, $M \geqslant 1$. We denote by $P^+_\infty(M,a)$, the **class of "global nonnegative solutions"** to the variational problem in the entire "half space" $\mathbb{R}^{n+1}_a$ with quadratic growth in x and linear growth in t, i.e., solutions in $\mathbb{R}^{n+1}_a$ satisfying

$$|u(x,t)| \leqslant M(1 + |x|^2 + |t|). \tag{1}$$

More precisely, we say a continuous function u (not identically zero) belongs to the class $P^+_\infty(M,a)$ if u satisfies:

(a) $H[u] = \chi_\Omega$ in $\mathbb{R}^{n+1}_a$, for some open set $\Omega = \Omega(u)$,
(b) $u \geqslant 0$ in $\mathbb{R}^{n+1}_a$, $\quad u|_{\Pi_a} = 0$,
(c) u satisfies inequality (1) in $\mathbb{R}^{n+1}_a$

and equation in (a) is understood in the sense of distributions. For simplicity of notation, we will write $P^+_\infty(M)$ instead of $P^+_\infty(M,0)$.

We also define the class $P^+_\infty(M,-\infty)$ corresponding formally to $a = -\infty$. In this case the whole space $\mathbb{R}^{n+1}$ is considered instead of $\mathbb{R}^{n+1}_a$, $\Pi_a = \emptyset$ and we omit the condition $u|_{\Pi_a} = 0$.

Remark. Let $u \in P^+_\infty(M,a)$, $z^0 = (x^0,t^0) \in \Omega(u)$ and $d(z^0) < \infty$. Although points of $\Gamma(u)$ may occur on the top $\hat{Q}$ of the cylinder $Q = \mathbb{R}^{n+1}_a \cap Q_{d(z_0)}(z_0)$, any derivatives of u are continuous in Q up to $\hat{Q}$, that is $u \in C^\infty(\overline{Q} \setminus \partial' Q)$. Besides, if $d(z^0) > x^0_1 - a$, then u is C^∞–function on the set $\{(x,t) \in Q_{d(z_0)-\varepsilon}(z_0) : x_1 \geqslant a\}$ for any positive ε.

We denote by $P^+_r(M)$, the **class of "local nonnegative solutions"** to the variational problem, i.e., we say a continuous function u (not identically zero) belongs to the class $P^+_r(M)$ if u satisfies:

(a') $H[u] = \chi_\Omega$ in Q^+_r, for some open set $\Omega = \Omega(u) \subset Q^+_r$, and $u = |Du| = 0$ in $Q^+_r \setminus \Omega(u)$,
(b') $u \geqslant 0$ in Q^+_r, $\quad u = 0$ on $\Pi \cap Q_r$,
(c') $\|u(x,t)\|_{\infty,Q^+_r} \leqslant M$

and equation in (a') is understood in the sense of distributions.

Let $\mathcal{E}$ be a domain in $\mathbb{R}^{n+1}$ and $f \in L_{1,loc}(\mathcal{E})$. We say that f is **sub-caloric** (**super-caloric**) if

$$\int_{\mathcal{E}} f(\Delta\varphi + \partial_t\varphi)dxdt \geqslant 0 \qquad (\leqslant 0)$$

for each nonnegative C^∞-function φ with compact support in $\mathcal{E}$.

Finally, we say that f is **caloric** if it is sub-caloric and super-caloric in $\mathcal{E}$.

Useful facts.

For the readers convenience and for the future references we will recall and explain some general facts.

Fact 1: Let $u \in P^+_\infty(M)$. Then

$$|D_iD_ju(x,t)| + |\partial_t u(x,t)| \leqslant C(n)M \qquad \text{for all} \quad (x,t) \in \mathbb{R}^{n+1}_+.$$

The proof of this is given in [ASU].

Similarly, if $u \in P^+_r(M)$ then

$$\sup_{Q_{r/8}\cap\Omega(u)} (|D_iD_ju(x,t)| + |\partial_t u(x,t)|) \leqslant C(n)M.$$

Observe that hence Du is Hölder continuous w.r.t. t with the exponent $1/2$.

Fact 2 (Nondegeneracy): Let $u \in P^+_\infty(M)$. Then for all $z^0 = (x^0,t^0) \in \overline{\Omega}(u)$ we have the estimate

$$\sup_{Q_\rho(z^0)} u \geqslant u(x^0,t^0) + \frac{\rho^2}{2n+1} \qquad \forall \rho > 0.$$

Proof. The proof of this statement is similar to the proof of [Ca2, Lemma 1]. We sketch some details.

Define

$$w(x,t) = u(x,t) - u(x^0,t^0) - \frac{1}{2n+1}(|x-x^0|^2 - (t-t^0)).$$

Then w is caloric in $\Omega(u)\cap Q_\rho(z^0)$ and $w(x^0,t^0) = 0$. By the maximum principle

$$\sup_{\Omega(u)\cap Q_\rho(z^0)} w = \sup_{\partial'(\Omega(u)\cap Q_\rho(z^0))} w \geqslant 0.$$

But w is strictly negative on $\partial\Omega(u)$, hence the nonnegative supremum is attained at some point (x^*,t^*) from the parabolic boundary of the cylinder $Q_\rho(z^0)$. In particular

$$u(x^*,t^*) \geqslant u(x^0,t^0) + \frac{\rho^2}{2n+1}.$$

This completes the proof. □

Fact 3: Let $u \in P^+_\infty(M)$. Then for any $R \geqslant 1$ the free boundary $\Gamma(u) \cap Q_R$ has $(n+1)$—dimensional Lebesgue measure zero.

Proof. This is shown in a very standard way.

Take a point $z^0 = (x^0, t^0) \in \Gamma(u)$ and an arbitrary $\rho > 0$. Using **Fact 2** we obtain another point $z^* = (x^*, t^*) \in \partial' Q_{\rho/2}(z^0)$ such that

$$u(x^*, t^*) \geqslant \frac{\rho^2}{8n+4}.$$

On the other hand, by **Fact 1** we have

$$|u| \leqslant C(n)M(\varepsilon\rho)^2 \qquad \text{in} \quad Q_{\varepsilon\rho}(z^*),$$

where ε is a small parameter which will be chosen later.

Observe that

$$\inf_{Q_{\varepsilon\rho}(z^*)} u \geqslant \frac{\rho^2}{8n+4} - 2C(n)M(\varepsilon\rho)^2$$

and the right-hand side of the above inequality is strictly positive if

$$\varepsilon = \frac{1}{4\sqrt{C(n)M(2n+1)}}.$$

Thus, for all $z^0 = (x^0, t^0) \in \Gamma(u)$ and all $\rho > 0$ the set $\Omega(u) \cap Q_{3\rho/4}(z^0)$ contains the subset $Q_{\varepsilon\rho}(z^*)$ of proportional volume. So $\Gamma(u)$ does not contain density points, i.e. the free boundary has zero Lebesgue measure. □

Fact 4: Assume that we extend $u \in P^+_\infty(M)$ across the plane Π to the whole space $\mathbb{R}^{n+1}$ by the odd reflection, i.e., by setting it as $-u(-x_1, x_2, \dots, x_n, t)$ for $(x,t) \in \mathbb{R}^{n+1}_-$ and preserve the notation u for the extended function. Suppose also that $z^0 = (x^0, t^0) \in \mathbb{R}^{n+1}_+ \cap \Gamma(u)$ and, for a sequence $r_m \nearrow \infty$, define

$$u_m(x,t) = \frac{u(r_m x + x^0, r_m^2 t + t^0)}{r_m^2}.$$

Then u_m converges (for a subsequence) in $W^{2,1}_{q,loc}(\mathbb{R}^{n+1})$ with any $q < \infty$ to a limit function u_∞.

Proof. According to **Fact 1**, it is sufficient to show

$$D_i D_j u_m \to D_i D_j u_\infty \qquad \text{a.e. in} \quad \mathbb{R}^{n+1}_+.$$

Let a point $z = (x,t) \in \Omega(u_\infty)$. Then u_∞ and u_m (for all sufficiently large m) are positive in $\mathcal{B}_\rho(z)$, where $\mathcal{B}_\rho(z)$ stands for the open ball in $\mathbb{R}^{n+1}$ with center (x,t) and radius ρ. Therefore $H[u_m - u_\infty] = 0$ in $\mathcal{B}_\rho(z)$ and, by general parabolic theory, $D_i D_j u_m$ converges to $D_i D_j u_\infty$ uniformly in $\mathcal{B}_{1/\rho}(z)$.

If z is an interior point of $\Lambda(u_\infty)$ then, by **Fact 2**, z also belongs to interior of $\Lambda(u_m)$ for all sufficiently large m, and in this case we also have the same convergence. Together with **Fact 3** it completes the proof. □

Main results.

The prime goal of this paper is to show the following results, each of them separately formulated later as theorems.

- If $u \in P_\infty^+(M)$ and $\Gamma(u) \neq \emptyset$ then u is t-independent and one-space dimensional, i.e.,

$$u(x,t) = \frac{((x_1 - a)_+)^2}{2}, \qquad \text{for some} \qquad a \geqslant 0.$$

- For $u \in P_1^+(M)$ a free boundary $\Gamma(u)$ touches the fix boundary tangentially with respect to the space directions in the neighbourhoods of contact points.

Plan of the paper.

This paper is organized as follows. Section 1 is the heart of the paper. It deals with the one-space dimensional global solutions and characterizes them geometrically in Theorem I. Analysis of this case is somewhat simpler and it is carried out in detail, in favour of a more clear exposition.

Geometric classification of global solutions, in the general case $n \geqslant 2$, is considered in Section 2. This result is formulated in Theorem II which follows by dimensional reduction based on the monotonicity formula (Lemma 2.1) due to L. A. Caffarelli and C. Kenig.

Finally, in Section 3 we analyze the behaviour of the free boundary near contact points with the fix boundary (Theorem III), using the characterization of global solutions obtained in Theorem II.

§1. Geometric classification of global solutions in the one-space dimensional case ($n = 1$)

Lemma 1.1. *Let $n = 1$, u be a continuous function in $\{(x,t) : x \geqslant 0, t \leqslant t_0\}$ satisfying condition (1) and the equations*

$$H[u] = 1 \quad in \quad \mathbb{R}^2_+ \cap \{t < t^0\},$$

$$u = 0 \quad on \quad \Pi \cap \{t \leqslant t^0\}.$$

Then

$$u(x,t) \equiv \frac{x^2}{2} + ax \qquad in \quad \mathbb{R}^2_+ \cap \{t \leqslant t^0\} \tag{2}$$

with $a \geqslant 0$.

Proof. We define v as

$$v(x,t) = \begin{cases} u(x,t) - \dfrac{x^2}{2}, & \text{if} \quad x \geqslant 0, \ t \leqslant t^0 \\[2ex] -u(-x,t) + \dfrac{x^2}{2}, & \text{if} \quad x < 0, \ t \leqslant t^0. \end{cases}$$

Then

$$v = 0 \qquad \text{on} \quad \Pi \cap \{t \leqslant t^0\}. \tag{3}$$

In addition, v is caloric in $\mathbb{R}^2 \cap \{t < t^0\}$, and it has quadratic growth with respect to x and linear growth with respect to t. Moreover, the estimates on derivatives of solutions to the heat equation (see Theorem 9, Section 2.3 [E]) guarantee that Liouville's theorem (see Lemma 2.1 [ASU]) holds true for caloric functions defined in the half-space $\{t \leqslant t^0\}$ only.

So, by Liouville's theorem v is a polynomial of degree two, i.e., there exist constants a_1, a, a_0 such that

$$v(x,t) = a_1 x^2 + 2a_1 t + ax + a_0.$$

Obviously, condition (3) implies $a_1 = a_0 = 0$ and the function v takes the form

$$v(x,t) = ax. \tag{4}$$

Using (4) and the definition of v we get (2). $\square$

Lemma 1.2. *Let $n = 1$, $u \in P_\infty^+(M)$. Then the following hold:*

(i) $-1 \leqslant \partial_t u \leqslant 0$;

(ii) $D_1 u \geqslant 0$;

(iii) *if $x^0 > 0$ and $(x^0, t^0) \in \Lambda(u)$, then the whole infinite rectangle $\{(x,t) : 0 \leqslant x \leqslant x^0, t^0 \leqslant t\}$ lies in $\Lambda(u)$.*

Proof. The proof of the first part follows using a contradictory argument. Suppose

$$\sup_{\Omega(u)} \partial_t u = m > 0.$$

Then there exists a sequence $z^j = (x^j, t^j) \in \Omega(u)$ such that

$$\lim_{j\to\infty} \partial_t u(x^j, t^j) = m. \tag{5}$$

It should be noted that $d_j = d(z^j) < \infty$, for all sufficiently large j, otherwise, Lemma 1.1 gives

$$\partial_t u(z^j) = 0$$

which contradicts (5). Now we extend u to $\mathbb{R}^2_-$ as an odd function of x, and we preserve the notation u for the extended function. We also define

$$u_j(x,t) = \frac{u(d_j x + x^j, d_j^2 t + t^j)}{d_j^2}.$$

By **Fact 1**, the derivatives $D_{11}u_j$ and $\partial_t u_j$ are uniformly bounded on $\mathbb{R}^2$ while the functions u_j are bounded on each compact subset of $\mathbb{R}^2$. Hence u_j converges (for a subsequence) to a function u_0 and $D_1 u_j \to D_1 u_0$ uniformly on compact subsets of $\mathbb{R}^2$. By the parabolic theory the convergence of u_j is actually stronger, in particular, $\partial_t u_j$ also converges to $\partial_t u_0$ uniformly on $Q_{1-\varepsilon}$ for an arbitrary small $\varepsilon > 0$.

Consider now the restriction u_0 on $\mathbb{R}^2_b$ with

$$b = \lim_{j\to\infty} \frac{-x^j}{d_j} \leqslant 0.$$

Obviously u_0 is a global solution in $\mathbb{R}^2_b$ and

$$\Omega(u_0) \supset \{\mathbb{R}^2_b \cap Q_1\}. \tag{6}$$

Observe that the function $v = \partial_t u_0$ is caloric in $\Omega(u_0)$,

$$v = 0 \quad \text{on} \quad \Pi_b \cap \{-1 < t \leq 0\} \quad \text{if} \quad b > -1, \tag{7}$$

while

$$v(0,0) = \lim_{j\to\infty} \partial_t u_j(0,0) = \lim_{j\to\infty} \partial_t u(x^j, t^j) = m. \tag{8}$$

It follows from (7) and (8) that $(0,0) \notin \Pi_b$, i.e. $b < 0$. Besides, for all $(x,t) \in \Omega(u_0)$ we have

$$v(x,t) \leqslant m.$$

Thus v takes a local maximum at the origin which does not lie on the parabolic boundary of $\mathbb{R}^2_b \cap Q_1$. But then v being caloric it is identically equal to m in $\mathbb{R}^2_b \cap Q_1$, i.e. $b \leqslant -1$. Moreover,

$$v(x,t) \equiv m > 0 \qquad \text{in} \quad \Omega_0, \tag{9}$$

where $\Omega_0 \supset Q_1$ is a connected component of the set $\Omega(u_0)$.

According to our definitions of d_j there exists a point (x^*, t^*) on

$$\partial' Q_1 = \{|x| = 1, -1 < t \leqslant 0\} \cup \{|x| \leq 1, t = -1\},$$

such that u_0 and $D_1 u_0$ vanish at (x^*, t^*).

Integration now gives $u_0(x,t) = mt + F(x)$ in $\overline{\Omega_0}$, for some C^1 function F satisfying

$$F'(x^*) = 0, \qquad F(x^*) + mt^* = 0,$$

$$F''(x) = m + 1 \qquad \text{in} \quad Q_1,$$

where the latter follows from (6) and (9).

Further, if $|x^*| = 1$, $t^* > -1$, then elementary integrations for F imply

$$F(x) = \frac{m+1}{2}(x - x^*)^2 - mt^*, \tag{10}$$

and we obtain the exact representation for the limit function u_0

$$u_0(x,t) = \frac{m+1}{2}(x - x^*)^2 + m(t - t^*) \qquad \text{in} \quad \overline{\Omega}_0 \supset Q_1. \tag{11}$$

Otherwise, if $\{(x,t) : |x| = 1, -1 < t \leqslant 0\} \subset \Omega(u_0)$, we deduce from (9) that

$$u_0(x^*, t) = m(t+1) \qquad \text{for} \quad -1 \leqslant t \leqslant 0,$$

and arrive at (10)-(11) again.

Observe that $u_0 = |D_1 u_0| = 0$ on $\mathbb{R}^2_b \cap \partial\Omega_0$. But it follows from (11) that (x^*, t^*) is the only point of $\overline{\Omega}_0$ where both u_0 and $D_1 u_0$ vanish. Besides, representation (11) gives $b = -\infty$. Hence we can conclude that Ω_0 is the whole space $\mathbb{R}^2$.

So, representation (11) takes place in the whole $\mathbb{R}^2$ which guarantees that u_0 takes negative values on the set $\left\{t < t^* - \frac{m+1}{2m}(x-x^*)^2\right\}$. But the latter contradicts the non-negativity of u_0 and, consequently,

$$\partial_t u \leqslant 0 \qquad \text{in} \quad \Omega.$$

Next we suppose

$$\inf_{\Omega(u)} \partial_t u = -\widetilde{m}, \qquad 0 < \widetilde{m} \neq 1$$

and repeat the above argument, with m replaced by $-\widetilde{m}$, to obtain the estimate from below for $\partial_t u$. Unfortunately for $\widetilde{m} = 1$ the limit function u_0 can take the form $u_0 = (-t)_+ \geqslant 0$ and we have no contradiction. So our argument works only for $\widetilde{m} \neq 1$.

The proof of statement (i) is completed.

The second statement of the lemma is now an easy consequence of the inequalities

$$D_1 u \geqslant 0 \quad \text{on} \quad \Pi,$$

$$0 \leqslant D_{11} u \leqslant 1 \quad \text{in} \quad \Omega(u), \tag{12}$$

where the latter follows from statement (i) of the lemma and the equation for u.

For the final part of the lemma we consider some point (x^0, t^0) in $\Lambda(u)$ assuming that $x^0 > 0$. According to statement (ii) of the lemma it follows that

$$u(x, t^0) = 0 \qquad \text{on the segment} \quad [0, x^0].$$

Now, using this equality and the information $u \geqslant 0$, $\partial_t u \leqslant 0$ we will have for any $t > t^0$ that $u(x,t) = 0$ for $x \in [0, x^0]$. The latter means that $\Lambda(u)$ contains an infinite rectangle $\{0 \leqslant x \leqslant x^0\} \times \{t^0 \leqslant t\}$ which is the desired statement (iii). The proof is completed. □

Lemma 1.3. *Let* $n = 1$, $u \in P_\infty^+(M)$, $\Gamma(u) \cap \mathbb{R}^2_+ = \emptyset$ *and* $\Gamma(u) \cap \Pi \neq \emptyset$.
Then $\Gamma(u) = \Pi$ *and* $u = \frac{x^2}{2}$.

Proof. Consider a point $z_0 = (0, t^0) \in \Gamma(u)$. By Lemma 1.1 we have $u(x,t) = \frac{x^2}{2}$ as $t \leq t^0$ that is $\Pi \cap \{t \leq t^0\}$ is contained in $\Gamma(u)$. On the other hand it follows from Lemma 1.2 that

$$u(x,t) \leqslant u(x, t^0) = \frac{x^2}{2} \quad \text{as} \quad t \geqslant t^0,$$

that is all points of $\Pi \cap \{t \geq t^0\}$ belong to $\Gamma(u)$ as well. □

Let us turn now to the situation where $\Gamma(u) \cap \mathbb{R}^2_+ \neq \emptyset$.

Lemma 1.4. *Let $n = 1$, $u \in P_\infty^+(M)$ and $\Gamma(u) \cap \mathbb{R}^2_+ \neq \emptyset$. Then the set of contact points $\Gamma(u) \cap \Pi$ is empty.*

Proof. Let
$$t^0 = \inf\{t : \text{there exists a point}(x,t) \in \Gamma(u) \cap \mathbb{R}^2_+\}.$$

According to statement (iii) of Lemma 1.2, it suffices to prove $t^0 = -\infty$.

Suppose it fails, i.e. $t^0 > -\infty$. Then the set $\mathbb{R}^2_+ \cap \{t < t^0\}$ lies in $\Omega(u)$ and for each $t^* > t^0$ there exists $\delta(t^*) > 0$ such that

$$\{(x,t) : 0 \leqslant x \leqslant \delta(t^*), t \geqslant t^*\} \subset \Lambda(u).$$

Therefore $(0, t^0) \in \Gamma(u)$.

This together with Lemma 1.1 gives

$$u = \frac{x^2}{2} \quad \text{in} \quad \mathbb{R}^2_+ \cap \{t \leqslant t^0\}. \tag{13}$$

Next, we extend the function u across the plane Π by even reflection, i.e., by setting it as $u(-x,t)$ for points (x,t) with $x < 0$. We will use the notation $\widetilde{u}$ for the extended function.

Define now U in $\mathbb{R}^2$ as
$$U = -G \ * \ \chi_{\Lambda(\widetilde{u})}, \tag{14}$$

where

$$G(x,t) = \frac{\exp(-|x|^2/4t)}{(4\pi t)^{1/2}} \quad \text{for} \quad t > 0 \qquad \text{and} \qquad G(x,t) = 0 \quad \text{for} \quad t \leqslant 0. \tag{15}$$

$$\Lambda(\widetilde{u}) = \{(x,t) \in \mathbb{R}^2 : t \geqslant t^0, \ (|x|, t) \in \Lambda(u)\}.$$

Then $H[U + \widetilde{u}] = 1$ almost everywhere in $\mathbb{R}^2 \cap \{t > t^0\}$ and

$$(U + \widetilde{u})|_{t=t^0} = \frac{x^2}{2}. \tag{16}$$

The reader should notice that in (16) we have used the representation in (13) and the definition of U. The uniqueness for Cauchy's problem now gives

$$U + \widetilde{u} = \frac{x^2}{2} \qquad \text{in} \quad \mathbb{R}^2 \cap \{t \geqslant t^0\}.$$

Hence we have
$$U = \frac{x^2}{2} \qquad \text{in} \quad \Lambda(\widetilde{u}). \tag{17}$$

On the other hand, the function U by definition is negative in $\mathbb{R}^2 \cap \{t > t^0\}$. This gives the contradiction with (17) and completes the proof. □

We proceed to consider the situation $\Gamma(u) \cap \mathbb{R}^2_+ \neq \emptyset$.

Lemma 1.5. *Let $n = 1$, $u \in P^+_\infty(M)$ and for some $a \geqslant 0$ the free boundary $\Gamma(u)$ be defined by*

$$\Gamma(u): \quad x = g(t) > a \tag{18}$$

with a nondecreasing function g satisfying the conditions

$$g(t) \to a \quad as \quad t \to -\infty. \tag{19}$$

and

$$g(t^*) = a + \varepsilon$$

for some t^* and sufficiently small positive ε.

Then

$$0 < \frac{(x-a)^2}{2} - u(x,t) \leqslant \frac{\varepsilon^2}{2} \qquad in \qquad \mathbb{R}^2_a. \tag{20}$$

Remark. Under assumptions of Lemma 1.5 the function g is not assumed to be continuous.

Proof. We prove this lemma in four steps. Without loss of generality we can assume $a = 0$ and $t^* = 0$.

Step 1. We claim that for every $\delta > 0$, and $R > 1$ we have $\partial_t u(x,t) \to 0$, and $D_{11}u(x,t) \to 1$ uniformly with respect to $x \in [\delta, R]$ as $t \to -\infty$.

To prove this, observe that it follows from (12) and the assumptions of this lemma that

$$u(x,t) < \frac{x^2}{2} \qquad \text{in} \quad \mathbb{R}^2_0. \tag{21}$$

Hence we get the inequality

$$-\int_{-\infty}^{\infty} \partial_t u(x,t)dt \leqslant \frac{x^2}{2}$$

which, together with statement (i) of Lemma 1.2, provides an existence of a sequence $t_k = t_k(x) \to -\infty$ such that

$$\partial_t u(x, t_k) \to 0 \quad \text{as} \quad k \to \infty. \tag{22}$$

Consider the functions

$$u_m(x,t) := u(x, t-m), \qquad\qquad m = 1, 2, \ldots$$

For each $x \geqslant \delta$ and $t \leqslant 0$ it is an increasing bounded sequence, so it has a limit $u_\infty(x,t)$ as $m \to \infty$.

Moreover, if m is large enough then

$$H[u_m] = 1 \quad \text{in} \quad \{(x,t) : x > \delta, t \leqslant 0\}.$$

Thus by Harnack's inequality the convergence of u_m is uniform on compact subsets of $\{(x,t) : x > \delta, t \leqslant 0\}$. The same convergence takes place for the derivatives of u_m. In particular, taking (22) into account, we easily obtain

$$\partial_t u(x, -\infty) = 0 \quad \text{as} \quad \delta < x < \infty.$$

Combining the latter with the equation $H[u_\infty] = 1$ we get the equality $D_{11}u_\infty = 1$.

Step 2. There exists a continuous function w defined in $\{(x,t) : x \geqslant \varepsilon; t \leqslant 0\}$ which satisfies the following conditions

$$H[w] = 0, \quad 0 \leqslant w \leqslant \frac{\varepsilon^2}{2} \qquad \text{in} \qquad \mathbb{R}^2_\varepsilon \cap \{t \leqslant 0\},$$

$$w = \frac{\varepsilon^2}{2} - u \quad \text{on} \quad \Pi_\varepsilon. \tag{23}$$

One can find such a function as a limit as $R \to \infty$ of solutions w_R to the following Dirichlet problem in $Q_{\varepsilon,R} = \{(x,t) : \varepsilon < x < R, -R < t \leqslant 0\}$

$$H[w_R] = 0 \quad \text{in} \quad Q_{\varepsilon,R}, \quad w_R(\varepsilon, t) = [\frac{\varepsilon^2}{2} - u(\varepsilon,t)]\zeta_R(t) \quad \text{as} \quad t \in]-R, 0[,$$

$$w_R = 0 \quad \text{on the last part of} \quad \partial' Q_{\varepsilon,R}.$$

Here $\zeta_R(t)$ is a cut-off function which equals 1 as $t \in [0, -R+2]$ and vanishes for $t \leqslant -R+1$.

It is clear that $0 \leqslant w_R \leqslant \frac{\varepsilon^2}{2}$ in $Q_{\varepsilon,R}$ and that $\{w_R\}$ is a monotonously increasing sequence.

Step 3. We claim

$$\frac{x^2}{2} - u(x,t) \leqslant \frac{\varepsilon^2}{2} \qquad \text{in} \qquad \mathbb{R}^2_0 \cap \{t \leqslant 0\}. \tag{24}$$

To prove this we consider the function

$$v(x,t) = u(x,t) + w(x,t) - \frac{\varepsilon^2}{2}$$

with the same w as in **Step 2**. According to Lemma 1.1

$$v(x,t) = \frac{(x-\varepsilon)^2}{2} + c(x-\varepsilon) \quad \text{in} \quad \mathbb{R}^2_\varepsilon \cap \{t \leqslant 0\},$$

i.e.,

$$u(x,t) = \frac{x^2}{2} + bx - b\varepsilon - w(x,t)$$

for some nonnegative constant c and $b = c - \varepsilon$.

On the other hand, it follows from **Step 1** that $u(x, -\infty) = \frac{x^2}{2}$. Thus one can conclude that $b = 0$. So we obtain the desired estimate for u in $\mathbb{R}^2_\varepsilon \cap \{t \leqslant 0\}$. The same estimate in $\{0 < x < \varepsilon\}$ is evident.

Step 4. We can prove (20) now assuming as above that $a = 0$. The proof is just the same as in Lemma 1.4 with $t^0 = 0$. The only difference is that instead of the exact representation (13) for $t \leqslant 0$ we now have estimate (24). More precisely, we consider the function

$$\widetilde{v} := \frac{x^2}{2} - U - \widetilde{u},$$

where $\widetilde{u}$ is the same function as in the proof of Lemma 1.4 and U is defined by (14) with

$$\Lambda(\widetilde{u}) = \{(x,t) \in \mathbb{R}^2 : t \geqslant 0,\ (|x|,t) \in \Lambda(u)\}.$$

Note that $\widetilde{v}$ satisfies the equation $H[\widetilde{v}] = 0$ and condition (1) for $t \geqslant 0$. Moreover, inequality (24) gives us the estimate

$$\widetilde{v}(x,0) \leqslant \frac{\varepsilon^2}{2}.$$

It follows then from the maximum principle for the Cauchy problem that

$$\widetilde{v} \leqslant \frac{\varepsilon^2}{2}$$

in the whole space $\mathbb{R}^2 \cap \{t \geqslant 0\}$. This, in conjunction with (21) and **Step 3**, completes the proof of Lemma 1.5. $\square$

From Lemmas 1.3–1.5 one can easily deduce the following result.

Theorem I. *If $n = 1$, $u \in P_\infty^+(M)$ and $\Gamma(u) \neq \emptyset$ then*

$$u = \frac{((x-a)_+)^2}{2} \tag{25}$$

for some $a \geqslant 0$.

Proof. According to Lemma 1.3 it is sufficient to consider the case $\Gamma(u) \cap \mathbb{R}^2_+ \neq \emptyset$. Suppose representation (25) fails. Then, according to Lemmas 1.2 and 1.4 there exists $a \geqslant 0$ such that $\Omega(u)$ lies in the half-space $\mathbb{R}^2_a$ and its boundary $\Gamma(u)$ tends asymptotically to the vertical line $\{x = a\}$ as $t \to -\infty$. In other words, there exists an increasing function g satisfying (19) and such that $\Gamma(u)$ is given by (18).

Then there exist $\{\varepsilon_k\} \searrow 0$ and $\{t_k\} \searrow -\infty$ such that $g(t_k) = a + \varepsilon_k$. Now applying Lemma 1.5 we reach a contradiction. $\square$

§2. Geometric classification of global solutions in the general case ($n \geqslant 2$)

Before discussing the main result of the paper we formulate some tools which would be used throughout the section. Our first tool is the following version of the monotonicity formula due to L. A. Caffarelli and C. Kenig (see [Ca3], [CK]).

Lemma 2.1. *Let h_1, h_2 be non-negative, sub-caloric, and continuous functions in $\mathbb{R}^{n+1} \cap \{t \leqslant t^0\}$, satisfying*

$$h_1(z^0) = h_2(z^0) = 0, \qquad z^0 = (x^0, t^0), \quad x^0 \in \mathbb{R}^n,$$

$$|Dh_i| \in L_2(\mathbb{R}^{n+1} \cap \{t \leqslant t^0\}), \quad i = 1, 2,$$

$$h_1(x,t) \cdot h_2(x,t) = 0 \quad \textit{in} \quad \mathbb{R}^{n+1} \cap \{t \leqslant t^0\}.$$

Suppose also that Dh_1 and Dh_2 have at most polynomial growth in x as $|x| \to \infty$.

Then the following function is monotone in r

$$\Phi(r) = \Phi(r, h_1, h_2, z^0) =$$

$$= \frac{1}{r^4} \int_{t^0 - r^2}^{t^0} \int_{\mathbb{R}^n} |Dh_1(x,t)|^2 G(x - x^0, t^0 - t) dx dt \times$$

$$\times \int_{t^0 - r^2}^{t^0} \int_{\mathbb{R}^n} |Dh_2(x,t)|^2 G(x - x^0, t^0 - t) dx dt,$$

where $G(x,t)$ is the function defined in (15).

More exactly, either both of the sets

$$\mathcal{S}_i(r) = \{x \in \mathbb{R}^n : h_i(x, t^0 - r^2) \neq 0\}, \qquad i = 1, 2,$$

coincide, up to sets of measure zero, with half-spaces containing x^0 on their boundary or $\Phi'(r) > 0$, or $\Phi(\tau) \equiv 0$ for $\tau \in (0, r]$.

Proof. The proof of this statement is essentially given in [CK]. For the reader's convenience and for the completeness, we provide some detail's here.

It is sufficient to assume $z^0 = (0,0)$ and carry out the proof for the case that all the functions involved in $\Phi(r)$ are smooth enough, including the support of h_i, $i = 1, 2$. Then an approximation procedure will give the result for the general case.

Let

$$I_i(r^2) = \int_{-r^2}^{0} \int_{\mathbb{R}^n} |Dh_i(x,t)|^2 G(x, -t) dx dt, \qquad \text{for} \quad i = 1, 2.$$

Differentiating Φ we obtain

$$\Phi'(r) = 2r\Phi(r) \left(\frac{I_1'(r^2)}{I_1(r^2)} + \frac{I_2'(r^2)}{I_2(r^2)} - \frac{2}{r^2} \right). \tag{26}$$

Observe that since $H[h_i^2/2] \geqslant |Dh_i|^2$ we get for $\rho = r^2$

$$I_i(\rho) \leqslant \int_{-\rho}^{0} \int_{\mathbb{R}^n} H[\frac{h_i^2}{2}(x,t)] G(x, -t) dx dt = \int_{\mathbb{R}^n} \frac{h_i^2}{2}(x, -\rho) G(x, \rho) dx,$$

where the last equality follows from integration by parts and the assumptions $h_i(0,0) = 0$.

Also

$$I_i'(\rho) = \int_{\mathbb{R}^n} |Dh_i(x, -\rho)|^2 G(x, \rho) dx.$$

Therefore

$$\frac{I_i'(\rho)}{I_i(\rho)} \geqslant 2 \frac{\int_{\mathbb{R}^n} |Dh_i(x, -\rho)|^2 G(x, \rho) dx}{\int_{\mathbb{R}^n} h_i^2(x, -\rho) G(x, \rho) dx} =$$

$$= \frac{1}{\rho} \frac{\int_{\mathbb{R}^n} |D\widetilde{h}_i(y, -1/2)|^2 G(y, 1/2) dy}{\int_{\mathbb{R}^n} \widetilde{h}_i^2(y, -1/2) G(y, 1/2) dy}, \tag{27}$$

where $\widetilde{h}_i(y, \tau) := h_i(\sqrt{2\rho} y, 2\rho\tau)$. Observe that $\widetilde{h}_i(y, -1/2) = h_i(x, -\rho)$.

Next, we define $\lambda(\widetilde{\mathcal{S}}_i)$ as

$$\lambda(\widetilde{\mathcal{S}}_i) = \inf \frac{\int_{\widetilde{\mathcal{S}}_i} |Dw(y)|^2 d\mu(y)}{\int_{\widetilde{\mathcal{S}}_i} w^2(y) d\mu(y)},$$

where

$$\widetilde{\mathcal{S}}_i = \{y \in \mathbb{R}^n : \widetilde{h}_i(y, -1/2) \neq 0\},$$

$d\mu(y) = G(y, 1/2)dy$ is the Gaussian measure and the infimum has been taken over nonzero functions in $\mathbb{R}^n$ with compact support in $\widetilde{\mathcal{S}}_i$. Then it follows from (27) that

$$\frac{I_i'(\rho)}{I_i(\rho)} \geqslant \frac{1}{\rho} \lambda(\widetilde{\mathcal{S}}_i). \tag{28}$$

Now, combining (26), (28) and taking into account that $\rho = r^2$, we get

$$\varPhi'(r) \geqslant \frac{2}{r} \varPhi(r) \big(\lambda(\widetilde{\mathcal{S}}_1) + \lambda(\widetilde{\mathcal{S}}_2) - 2 \big). \tag{29}$$

By results of Beckner-Kenig-Pipher [BKP] (cf. [CK, Corollary 2.4.6]) we have

$$\lambda(\widetilde{\mathcal{S}}_1) + \lambda(\widetilde{\mathcal{S}}_2) - 2 \geqslant 0. \tag{30}$$

It is for this inequality that the hypothesis of disjoint support of the h_i's is needed.

Also, in [CK, Remark 2.4.8] it is shown that equality holds in (30) if and only if $\widetilde{\mathcal{S}}_1 = \{y \in \mathbb{R}^n : y_1 > 0\}$ and $\widetilde{\mathcal{S}}_2 = \{y \in \mathbb{R}^n : y_1 < 0\}$, or a rotation of it. This, together with (29) and the observation that $\widetilde{\mathcal{S}}_i = \mathcal{S}_i(r)$, proves the lemma. □

Lemma 2.2. *Let h be a caloric function in a (not necessarily bounded) domain $\mathcal{E}$ in $\mathbb{R}^{n+1} \cap \{t > t^0\}$. Suppose moreover that h is continuous in $\overline{\mathcal{E}}$, and h belongs to the Tikhonov class.*

If, in addition, $h \geqslant 0$ on $\partial\mathcal{E}$ then $h \geqslant 0$ in $\mathcal{E}$.

The proof of this statement is elementary, but probably not obvious. We sketch some details.

Proof. Suppose $\mathcal{E}^- = \{h < 0\}$ is nonempty, otherwise there is nothing to prove.

Define $\widetilde{h}$ as $\widetilde{h}(x,t) = h(x,t)$ in $\mathcal{E}^-$ and $\widetilde{h}(x,t) = 0$ outside $\mathcal{E}^-$.

Then $\widetilde{h}$ is super-caloric in $\{t > t^0\}$, it belongs to the Tikhonov's class and $\widetilde{h} = 0$ on $\{t = t^0\}$. Hence by the standard maximum principle (see Theorem 2.3 [KL]) $\widetilde{h} \geqslant 0$. This is a contradiction. □

Theorem II. *Let $u \in P^+_\infty(M)$ and $\Gamma(u) \neq \emptyset$. Then*

$$u(x,t) = \frac{((x_1 - a)_+)^2}{2} \qquad \textit{for some} \quad a \geqslant 0.$$

Proof. We prove this theorem in three steps.

Step 1. Let ν be any spatial direction such that $\nu \cdot e_1 > 0$. Observe that $u \geqslant 0$ implies

$$D_\nu u \geqslant 0 \qquad \text{on} \quad \Pi. \tag{31}$$

We claim that

$$D_\nu u \geqslant 0 \qquad \text{in} \quad \mathbb{R}^{n+1}_+. \tag{32}$$

To prove this, we first extend u across the plane Π to the whole space $\mathbb{R}^{n+1}$ by the odd reflection and preserve the notation u for the extended function. Also we define w as

$$w(x,t) = \begin{cases} D_\nu u(x,t), & \text{if} \quad (x,t) \in \mathbb{R}^{n+1}_+ \\ (D_\nu u(x,t))_+, & \text{if} \quad (x,t) \in \mathbb{R}^{n+1}_-. \end{cases} \tag{33}$$

By this definition we will have the following properties of w:

(1) w and the derivatives $D_e w$, for $e \perp e_1$, are continuous across $\Pi \setminus \Lambda(u)$,
(2) the jump of $D_1 w$ on $\Pi \setminus \Lambda(u)$ equals $2\cos(\widehat{\nu, e_1}) > 0$,
(3) $H[w] = 0$ on the set $\{(x,t) \in \mathbb{R}^{n+1} \setminus \Pi : w(x,t) \neq 0\}$.

It is easy to deduce from these facts that w_+ will be sub-caloric in $\{(x,t) \in \mathbb{R}^{n+1} : w(x,t) > 0\}$ and continuously zero outside this set. Therefore it must be sub-caloric in the whole space $\mathbb{R}^{n+1}$. Taking into account definition (33), we see that $w_-(x,t) = 0$ for $x_1 \leqslant 0$ and it is sub-caloric in $\mathbb{R}^{n+1}$ too.

Now we take a point $z^0 = (x^0, t^0) \in \Lambda(u)$ with $x_1 \geqslant 0$. According to Lemma 2.1, for $0 < r < r_j$ with $r_j \to \infty$, we have

$$0 \leqslant \Phi(r, w_+, w_-, z^0) \leqslant \Phi(r_j, w_+, w_-, z^0) \leqslant \lim_{r_j \to \infty} \Phi(r_j, w_+, w_-, z^0) =: C_\nu. \tag{34}$$

Observe that C_ν exists by Lemma 2.1 and boundness of Dw.

Notice that according to **Fact 4** the scaling

$$u_j(x,t) = \frac{u(r_j x + x^0, r_j^2 t + t^0)}{r_j^2}$$

converges (for a subsequence) in $W^{2,1}_{q,loc}(\mathbb{R}^{n+1})$, $q < \infty$, to a limit function u_∞ which is odd with respect to the hyperplane Π.

Next we define w_∞ by (33) with u_∞ instead of u and set

$$w_j(x,t) = \begin{cases} D_\nu u_j(x,t), & \text{if} \quad x_1 \geqslant -\dfrac{x_1^0}{r_j} \\[2ex] (D_\nu u_j(x,t))_+, & \text{if} \quad x_1 < -\dfrac{x_1^0}{r_j}. \end{cases}$$

Then, letting $j \to \infty$ (see **Fact 4**) and using (34) we obtain for every $s > 0$ that

$$C_\nu = \lim_{r_j \to \infty} \Phi(sr_j, w_+, w_-, z^0) = \lim_{r_j \to \infty} \Phi(s, (w_j)_+, (w_j)_-, 0) = \\ = \Phi(s, (w_\infty)_+, (w_\infty)_-, 0), \tag{35}$$

i.e., $\Phi(s, (w_\infty)_+, (w_\infty)_-, 0)$ is constant for all $s > 0$.

Now we proceed to show that C_ν could not be positive. Suppose $C_\nu > 0$. Then, according to (35) and Lemma 2.1 both of the sets

$$\mathcal{S}^+(s) = \{x \in \mathbb{R}^n : D_\nu u_\infty(x, -s^2) > 0\},$$

$$\mathcal{S}^-(s) = \{x \in \mathbb{R}^n : D_\nu u_\infty(x, -s^2) < 0\},$$

must be half-spaces $\mathbb{R}^n_\pm$. However, by (33) $w_\infty \geqslant 0$ in $\mathbb{R}^{n+1}_-$. Then the only possibility for $\mathcal{S}^-(s)$ to coincide with a half-space for any $s > 0$ is the condition $w_\infty < 0$ in $\mathbb{R}^{n+1}_+ \cap \{t \leqslant 0\}$ which contradicts the condition $u_\infty \geqslant 0$ in $\mathbb{R}^{n+1}_+$.

It thus follows that $C_\nu = 0$ and consequently by (34) either $w_+ \equiv 0$ or $w_- \equiv 0$ in $\mathbb{R}^{n+1}_+ \cap \{t \leqslant t^0\}$. Since $u \geqslant 0$ in $\mathbb{R}^{n+1}_+$ the second is true.

Thus we have

$$D_\nu u \geqslant 0 \qquad \text{in} \quad \mathbb{R}^{n+1}_+ \cap \{t \leqslant t^0\}.$$

Hence by Lemma 2.2, where we take $\mathcal{E} = \Omega(u) \cap \{t > t^0\}$, we get (32).

Step 2. We proceed to show that u, in fact, is one-space dimensional, i.e.

$$u(x,t) = u(x_1, t). \tag{36}$$

Since $D_e u = 0$ on Π for e orthogonal to e_1, we may continue $D_e u$ as zero across Π to $\mathbb{R}^{n+1}_-$. Next we denote the extended function by w and repeat the arguments from **Step 1**. This gives that $D_e u$ does not change sign in $\mathbb{R}^{n+1}_+$. Hence by the strong maximum principle either $D_e u > 0$ (or $D_e u < 0$) in connected components of $\Omega(u)$ or $D_e u \equiv 0$. If for the all directions e orthogonal to e_1 we have $D_e u \equiv 0$, then u depends only on x_1 and t, and we immediately arrive at (36).

So suppose (for the definitness) there exists a point (x^*, t^*) such that $D_e u(x^*, t^*) > 0$ for some e orthogonal to e_1. Then $D_{-e} u(x^*, t^*) < 0$ and there exists $a > 0$ such that $D_\nu u(x^*, t^*) < 0$ for $\nu = ae_1 - \sqrt{1-a^2}e$ which contradicts **Step 1**.

Step 3. Now the one-space dimensional case (Theorem I) applies and finishes the proof. $\square$

§3. Application to the behaviour of the free boundary

The partial regularity of a free boundary near contact points with the fix boundary will now follow easily by using the characterization of global solutions. Following [SU] we consider first the auxiliary result.

Lemma 3.1. *Given $\varepsilon > 0$, there exists $\rho = \rho_\varepsilon$ such that if $u \in P_1^+(M)$ and $(0,0) \in \Gamma(u)$, then for $(x^*, t^*) \in \partial\Omega(u) \cap Q^+_{\rho_\varepsilon}(0, \rho_\varepsilon^2/2)$ we have*

$$(x^*, t^*) \in Q^+_{\rho_\varepsilon}(0, \rho_\varepsilon^2/2) \setminus K_\varepsilon, \tag{37}$$

where

$$K_\varepsilon = \{(x,t) : x_1 > \varepsilon\sqrt{x_2^2 + \cdots + x_n^2 + |t|}\}.$$

Proof. Suppose, towards a contradiction, that the conclusion of the lemma fails. Then for every $j \in \mathbb{N}$ there exist $u_j \in P_1^+(M)$ and $(x^j, t^j) \in \partial\Omega(u_j)$ such that $r_j := \sqrt{|x^j|^2 + |t^j|} \to 0$ and (37) fails for (x^j, t^j).

Then define $\widetilde{u}_j$ as

$$\widetilde{u}_j(x,t) = \frac{u(r_j x, r_j^2 t)}{r_j^2}.$$

Observe that for each function $\widetilde{u}_j$ we have a point $(\widetilde{x}^j, \widetilde{t}^j) \in \Gamma(\widetilde{u}_j)$ with $|\widetilde{x}^j|^2 + |\widetilde{t}^j| = 1$ and

$$\widetilde{x}_1^j \geqslant \varepsilon\sqrt{(\widetilde{x}_2^j)^2 + \cdots + (\widetilde{x}_n^j)^2 + |\widetilde{t}^j|} = \varepsilon\sqrt{1-(x_1^j)^2}.$$

Now for a subsequence $\widetilde{u}_j$ and $(\widetilde{x}^j, \widetilde{t}^j)$ converge to u_0 and (x^0, t^0), respectively, where

$$x_1^0 \geqslant \frac{\varepsilon}{\sqrt{1+\varepsilon^2}} > 0.$$

Thus, both (x^0, t^0) and the origin are on the free boundary $\Gamma(u_0)$. Since u_0 is a global solution this contradicts Theorem II. □

Theorem III. *There exist a universal constant $r_0 = r_0(n, M)$, and a modulus of continuity σ ($\sigma(0^+) = 0$) such that if $u \in P_1^+(M)$ and $(0,0) \in \Gamma(u)$ then*

$$\partial\Omega(u) \cap Q_{r_0}(0, r_0/2) \subset \{(x,t) : x_1 \leqslant \sigma(|x| + \sqrt{|t|}) \cdot (|x| + \sqrt{|t|})\}.$$

Proof. Consider the modulus of continuity $\sigma(\rho)$ given by the inverse of the relation $\varepsilon \to \rho_\varepsilon$ in Lemma 3.1. Let now $r_0 = \rho_{\{\varepsilon=1\}}$. □

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