REGULARITY OF A FREE BOUNDARY IN PARABOLIC POTENTIAL THEORY

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ABSTRACT. We study the regularity of the free boundary in a Stefan-type problem

$$\Delta u - \partial_t u = \chi_\Omega \quad \text{in } D \subset \mathbb{R}^n \times \mathbb{R}, \qquad u = |\nabla u| = 0 \quad \text{on } D \setminus \Omega$$

with no sign assumptions on u and the time derivative $\partial_t u$.

CONTENTS

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1. INTRODUCTION

1.1. **Background.** In the last few years the free boundary regularity of both variational and nonvariational type has gained a renewed attention. Due to developments of the so-called monotonicity formulas for elliptic and parabolic PDEs on one side and developments of new techniques in free boundary regularity on the other side several longstanding questions have been answered.

One of these questions, treated in this paper and with roots in parabolic potential theory, concerns the nature of those boundaries that allow caloric continuation of the heat potential from the free space into the space occupied by the density function. To clarify this let U^f be the heat potential of a density function f:

$$U^f(x,t) = \int_{\mathbb{R}^n \times \mathbb{R}} f(y,s)\, G(x-y, s-t)\, dyds,$$

where $G(x,t)$ is the heat kernel. Then it is known that

$$HU^f = c_n f,$$

where $H = \Delta - \partial_t$ is the heat operator and $c_n < 0$ is some constant. Now suppose

$$f(x,t) = \frac{1}{c_n} \chi_\Omega$$

for some domain Ω and denote the corresponding potential by U^Ω. Then

$$HU^\Omega = \chi_\Omega.$$

Suppose now that there exist v such that

$$\begin{cases} Hv = 0 & \text{in } Q_r(x_0,t_0) \\ v = U^\Omega & \text{in } Q_r(x_0,t_0) \setminus \Omega \end{cases}$$

for some $(x_0,t_0) \in \partial\Omega$ and $r > 0$, where $Q_r(x_0,t_0) = B_r(x_0) \times (t_0 - r^2, t_0 + r^2)$. Then we call v caloric continuation of U^Ω. Moreover, the function

$$u = U^\Omega - v$$

satisfies

$$\begin{cases} Hu = \chi_\Omega & \text{in } Q_r \\ u = |\nabla u| = 0 & \text{in } Q_r \setminus \Omega. \end{cases} \tag{1.1}$$

So our question is when does the boundary of a domain allow a caloric continuation of the potential.

It is well known, through the Cauchy-Kowalevskaya theorem, that analytic boundaries do allow such a continuation locally. Hence we ask the reverse of the Cauchy-Kowalevskaya theorem in the sense that the existence of the caloric continuation implies the regularity of the boundary.

In a particular case when $u \geq 0$ and $\partial_t u \geq 0$ problem (1.1) is the well-known *Stefan problem* (see e.g. [Fri88]), describing the melting of ice, and is treated extensively in the literature. However, even the variational inequality case $u \geq 0$ (and not necessarily $\partial_t u \geq 0$) has not been considered earlier.

In this paper we treat (1.1) in its full generality without any sign assumptions on either u or $\partial_t u$. The stationary case, i.e. when u is independent of t was studied in [CKS00]. The results of this paper generalize those of [CKS00] to the time dependent case.

1.2. **Problem.** For a function $u(x,t)$, continuous with its spatial derivatives in a domain D of $\mathbb{R}^n \times \mathbb{R}$, define the *coincidence set* as

$$\Lambda := \{u = |\nabla u| = 0\}$$

and suppose that

$$Hu = \chi_\Omega \quad \text{in } D, \qquad \Omega := D \setminus \Lambda. \tag{1.2}$$

Here $H = \Delta - \partial_t$ is the heat operator and we assume that the equation is satisfied in the weak (distributional) sense, i.e.

$$\int_D u(\Delta\eta + \partial_t \eta)\, dxdt = \int_{D\cap\Omega} \eta\, dxdt$$

for all C^∞ test functions η with compact support in D. Then we are interested in the regularity of the so-called *free boundary* Γ, which consists of all $(x,t) \in \partial\Omega \cap D$, that are not parabolically interior for Λ, i.e. such that

$$Q^-_\varepsilon(x,t) \cap \Omega \neq \emptyset$$

for any small $\varepsilon > 0$, where $Q^-_\varepsilon(x,t) = B_\varepsilon(x) \times (t - \varepsilon^2, t]$ is the lower parabolic cylinder.

1.3. **Notations.** Points in $\mathbb{R}^n \times \mathbb{R}$ are denoted by (x,t), where $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$.
Generic constants are denoted by $C, C_0, C_n, \dots$;
$\mathbb{R}^-_a = (-\infty, a]$; $\mathbb{R}^- = \mathbb{R}^-_0$;
$a_\pm = \max(\pm a, 0)$ for any $a \in \mathbb{R}$;
$B_r(x)$ is the open ball in $\mathbb{R}^n$ with center x and radius r; $B_r = B_r(0)$;
$Q_r(x,t) = B(x,r) \times (t - r^2, t + r^2)$ (parabolic cylinder); $Q_r = Q_r(0,0)$;
$Q^+_r(x,t) = B_r(x) \times [t, t + r^2)$ (the upper half-cylinder); $Q^+_r = Q^+_r(0,0)$;
$Q^-_r(x,t) = B_r(x) \times (t - r^2, t]$ (the lower half-cylinder); $Q^-_r = Q^-_r(0,0)$;
$\partial_p Q_r(x,t)$ is the parabolic boundary, i.e., the topological boundary minus the top of the cylinder.
∇ denotes the spatial gradient, $\nabla = (\partial_1, \dots, \partial_n)$;
$\Delta = \sum_{i=1}^n \partial_{ii}$ (the spatial Laplacian);
$H = \Delta - \partial_t$ (the heat operator);
χ_Ω is the characteristic function of the set Ω;
$E(t) = \{x : (x,t) \in E\}$ is the t-section of the set E in $\mathbb{R}^n \times \mathbb{R}$.

Below we define classes of local and global solutions of (1.2) that we study in this paper.

1.4. **Local solutions.**

Definition 1.1. For given $r, M > 0$ and $(x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}$ let $\mathcal{P}^-_r(x_0, t_0; M)$ be the class of functions u in $Q^- = Q^-_r(x_0, t_0)$ such that

(i) u satisfies (1.2) in $D = Q^-$;
(ii) $|u| \le M$ in Q^-;
(iii) $(x_0, t_0) \in \Lambda$.

In the case $(x_0, t_0) = (0,0)$ we will denote the corresponding class $\mathcal{P}^-_r(0,0;M)$ also by $\mathcal{P}^-_r(M)$.

Similarly, define the class $\mathcal{P}_r(x,t;M)$ by replacing $Q^- = Q^-_r(x_0,t_0)$ with $Q = Q_r(x_0,t_0)$ in (i)–(iii) above.

1.5. Global solutions.

Definition 1.2. For a given $M > 0$ let $\mathcal{P}_\infty^-(M)$ be the class of functions u in $\mathbb{R}^n \times \mathbb{R}^-$ such that

(i) u satisfies (1.2) in $D = \mathbb{R}^n \times \mathbb{R}^-$;
(ii) $|u(x,t)| \le M(1 + |x|^2 + |t|)$;
(iii) $(0,0) \in \Lambda$.

Similarly, define the class $\mathcal{P}_\infty(M)$ by replacing $\mathbb{R}^n \times \mathbb{R}^-$ with $\mathbb{R}^n \times \mathbb{R}$.

The elements of $\mathcal{P}_\infty^-(M)$ and $\mathcal{P}_\infty(M)$ will be called *global solutions*.

It is also noteworthy that elements in $\mathcal{P}_\infty^-(M)$ can be extended, in a natural way, to $\mathbb{R}^n \times \mathbb{R}^+$ by solving the Cauchy problem for the equation $Hu = 1$. In particular, we may consider each element in $\mathcal{P}_\infty^-(M)$ as an element of $\mathcal{P}_\infty(M)$, and vice versa.

The following operations will be extensively used throughout the paper.

1.6. Scaling.

For a function $u(x,t)$ set

$$u_r(x,t) = \frac{1}{r^2} u(rx, r^2 t),$$

the *parabolic scaling* of u around $(0,0)$. This scaling preserves equation (1.2) with

$$\Omega(u_r) = \Omega_r := \{(x,t) : (rx, r^2 t) \in \Omega\}.$$

Also, $u \in \mathcal{P}_r(M)$ implies $u_r \in \mathcal{P}_1(M/r^2)$.

Similarly, one can scale u around any point (x_0, t_0) by

$$\frac{1}{r^2} u(rx + x_0, r^2 t + t_0).$$

1.7. Blow-up.

As we show in Theorem 4.1, solutions $u \in \mathcal{P}_1^-(M)$ are locally $C_x^{1,1} \cap C_t^{0,1}$ regular in Q_1^-. Then the scaled functions u_r are defined and uniformly bounded in Q_R^- for any $R < 1/r$. Since $Hu_r = \chi_{\Omega_r}$, by standard compactness methods in parabolic theory (see e.g. [Fri64]), we may let $r \to 0$ and obtain (for a subsequence) a global solution (see the stability discussion below). This process is referred to as *blowing-up*, and the global solution thus obtained is called a *blow-up* of u.

Similarly, we can define the blow-up of a local solution u at any free boundary point (x_0, t_0) by considering the parabolic scalings of u around (x_0, t_0).

Also, if u is a global solution, we can define the blow-up at infinity, by considering the scaled functions u_r and letting $r \to \infty$. The blow-up at infinity will be called *shrink-down*.

2. Main results

Before stating our main results, we would like to illustrate the problem with the following examples.

2.1. Examples.

1. *Stationary* (i.e. t-independent) *solutions.* Those include *halfspace solutions*

$$u(x,t) = \frac{1}{2}(x \cdot e)_+^2,$$

where e is a spatial unit vector, as well as other global stationary solutions of the obstacle problem that have ellipsoids and paraboloids as coincidence sets.

2. *Space-independent* (i.e. x-independent) *solutions*

$$u(x,t) = -t, \quad u(x,t) = -t_+, \quad u(x,t) = t_-.$$

In fact, it is easy to see that the solutions depending only on t have the form

$$u(x,t) = \begin{cases} -(t - T_2), & t > T_2 \\ 0, & T_1 \le t \le T_2 \\ -(t - T_1), & t < T_1 \end{cases}$$

for some constants $-\infty \le T_1 \le T_2 \le \infty$. This is a particular case of our Theorem I below.

3. *Polynomial solutions* of the type

$$u(x,t) = P(x) + m\,t,$$

where $P(x)$ is a quadratic polynomial satisfying $\Delta P = m + 1$. In particular, for a given constant c, the function

$$u(x,t) = c|x|^2 + (2nc - 1)\,t$$

is a solution of (1.2) in $\mathbb{R}^n \times \mathbb{R}$. The only free boundary point of this solution is the origin $(0,0)$, unless $c = 0$ or $c = 1/2n$. In the former case the free boundary is $\mathbb{R}^n \times \{0\}$ and in the latter case it is $\{0\} \times \mathbb{R}$.

4. For the next example we modify the solution above for $t \ge 0$ by solving the one phase free boundary problem: find a function $f(\xi)$ on $[0,\infty)$ such that

$$u(x,t) = t\, f\left(\frac{|x|}{\sqrt{t}}\right) \tag{2.1}$$

satisfies (1.2) for $t > 0$. This will be so if f vanishes on $[0,a]$ for some $a > 0$ and satisfies an ordinary differential equation

$$f''(\xi) + \left(\frac{n-1}{\xi} + \frac{\xi}{2}\right) f'(\xi) - f(\xi) - 1 = 0$$

on (a,∞) with boundary conditions

$$f(a) = f'(a) = 0.$$

The solution can be given explicitly as

$$f(\xi) = (2n + \xi^2)\left(\frac{1}{2n + a^2} - 2a^n e^{a^2/4} \int_a^\xi \frac{e^{-s^2/4}}{s^{n-1}(2n + s^2)^2}\, ds\right).$$

It is easy to see that the limit

$$c = \lim_{\xi \to \infty} \frac{f(\xi)}{\xi^2} \tag{2.2}$$

exists and satisfies

$$0 < c < \frac{1}{2n}.$$

Moreover, changing a between 0 and ∞ we can get all values from $(0, 1/2n)$. Now (2.2) implies that

$$u(x,0) = c|x|^2.$$

Hence, if we define

$$u(x,t) = \begin{cases} t\, f\left(\frac{|x|}{\sqrt{t}}\right) & \text{for } t > 0 \\ c|x|^2 + (2nc - 1)\,t & \text{for } t \le 0, \end{cases}$$

we again obtain a solution of (1.2) in $\mathbb{R}^n \times \mathbb{R}$. The free boundary in this case is the paraboloid $\{|x|^2 = a^2 t\}$. The solution u is identically 0 inside and positive outside.

5. Finally, we point out that at any time $t = T$ we have the freedom to choose not to have a free boundary. Namely, fix $T > 0$ and let u be, for instance, as in the previous example. Now solve the Cauchy problem

$$Hv = 1 \quad \text{in } \mathbb{R}^n \times (T, \infty); \qquad v(\cdot, T) = u(\cdot, T)$$

and let

$$w(x,t) = \begin{cases} v(x,t), & t > T \\ u(x,t), & t \le T. \end{cases}$$

Then w is a solution of (1.2) in $\mathbb{R}^n \times \mathbb{R}$. Its free boundary is the truncation of the paraboloid $\{|x|^2 = a^2 t\}$ for $t \le T$. We remark that the disk $B_{a^2T} \times \{T\}$ is not a part of the free boundary, even though it is the part of $\partial\Omega$.

As we will see later (Section 7), the points on $B_{a^2T} \times \{T\}$ have zero (balanced) energy, the tip (0, 0) has high energy, and rest of the points on the truncated paraboloid have low energy. We show in this paper that, in a sense, the regular free boundary points are the ones with low energy.

2.2. **Main theorems.** The solution that we constructed in the example above has the property that it is polynomial for $t < 0$, nonnegative and convex in space for $0 \le t \le T$ and solves $Hw = 1$ for $t > T$. Our first main theorem states that something similar is true for every global solution.

Theorem I (Classification of global solutions)**.** *Let u be a solution of* (1.2) *in $D = \mathbb{R}^n \times (-\infty, a]$ with at most quadratic growth at infinity:*

$$|u(x,t)| \le M\left(|x|^2 + |t| + 1\right)$$

for some constant $M > 0$. Then there exist $-\infty \le T_1 \le T_2 \le a$ with the following properties:

(i) *if $-\infty < T_1$, then*

$$u(x,t) = P(x) + m\,t \quad \textit{for } t < T_1$$

for a quadratic polynomial $P(x)$ and a constant m;

(ii) *if $T_1 < T_2$, then*

$$u \ge 0, \quad \partial_{ee} u \ge 0, \quad \partial_t u \le 0 \quad \textit{for } t < T_2,$$

where e is any spatial unit vector;

(iii) *if $T_2 < a$, then u satisfies*

$$Hu = 1 \quad \textit{for } T_2 < t < a.$$

Similar to the obstacle problem (see [Caf98]) the classification of global solutions implies the regularity of the free boundary for local solutions at points satisfying a certain density condition. Such a condition can be given in the terms of the minimal diameter.

Definition 2.1 (Minimal Diameter)**.** The *minimal diameter* of a set E in $\mathbb{R}^n$, denoted $\mathrm{md}(E)$, is the infimum of distances between two parallel planes such that E is contained in the strip between these planes. The *lower density function* for the solution of u of (1.2) at (0, 0) is defined by

$$\delta_r^-(u) = \frac{\mathrm{md}\left(\Lambda(-r^2) \cap B_r\right)}{r}.$$

Theorem II (Regularity of local solutions). *Let $u \in \mathcal{P}_1^-(M)$ be a local solution, such that $(0,0) \in \Gamma$. Then there is a universal modulus of continuity $\sigma(r)$ and a constant $c > 0$ such that if for one value of r, say r_0, we have*

$$\delta_{r_0}^-(u) > \sigma(r_0)$$

then $\Gamma \cap Q_{c r_0}^-$ is a C^∞ surface (in space and time.)

Remark 2.2. If we replace $\delta_r^-(u)$ by a weaker density function

$$\delta_r^*(u) = \sup_{-r^2 \le t \le -r^2/2} \frac{\mathrm{md}(\Lambda(t) \cap B_{2r})}{r}$$

then the conclusion of the theorem still remains true (perhaps with different constants.)

3. Monotonicity formulas

So-called *monotonicity formulas* will play an important role in this paper and will appear in almost every section.

We will use two different kinds of monotonicity formulas, the first due to Caffarelli [Caf93] and the second due to Weiss [Wei99], both in global and local forms.

Let

$$G(x,t) := \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t} \quad \text{for } (x,t) \in \mathbb{R}^n \times (0,\infty)$$

be the heat kernel. Then for a function v and any $t > 0$ define

$$I(t; v) = \int_{-t}^{0} \int_{\mathbb{R}^n} |\nabla v(x,s)|^2 G(x,-s)\, dx ds.$$

Theorem 3.1 (Caffarelli [Caf93]). *Let h_1 and h_2 be nonnegative subcaloric functions in the strip $\mathbb{R}^n \times [-1,0]$ with a polynomial growth at infinity such that*

$$h_1(0,0) = h_2(0,0) = 0 \quad \text{and} \quad h_1 \cdot h_2 = 0.$$

Then the functional

$$\Phi(t) = \Phi(t; h_1, h_2) := \frac{1}{t^2} I(t; h_1) I(t; h_2)$$

is monotone nondecreasing in t for $0 < t < 1$. □

For the proof see Theorem 1 in [Caf93]. This theorem is a generalization of the Alt-Caffarelli-Friedman monotonicity formula from [ACF84].

Remark 3.2. As it follows from the proof, if $\Phi(t) > 0$ and the supports of $h_1(\cdot,t)$ and $h_2(\cdot,t)$ are not complementary halfspaces, then $\Phi'(t) > 0$.

We will also use the following local counterpart of the monotonicity theorem above. It takes the form of an estimate.

Theorem 3.3 (Caffarelli [Caf93]). *Let h_1 and h_2 be nonnegative subcaloric functions in Q_1^- such that*

$$h_1(0,0) = h_2(0,0) = 0 \quad \text{and} \quad h_1 \cdot h_2 = 0.$$

Let also $\psi(x) \ge 0$ be a C^∞ cut-off function with $\operatorname{supp} \psi \subset B_{3/4}$ and $\psi|_{B_{1/2}} = 1$ and set $w_i = h_i \psi$. Then there exist a constant $C = C(n,\psi) > 0$ such that

$$\Phi(t; w_1, w_2) \le C \|h_1\|^2_{L^2(Q_1^-)} \|h_2\|^2_{L^2(Q_1^-)}$$

for any $0 < t < 1/2$. □

For the proof see Theorem 2 in [Caf93] and the remark after it. See also Theorem 2.1.3 in [CK98] for the generalization of this estimate for parabolic equations with variable coefficients.

To formulate the second monotonicity formula, we define Weiss' functional for a function u by

$$W(r;u)=\frac{1}{r^4}\int_{-4r^2}^{-r^2}\int_{\mathbb{R}^n}\left(|\nabla u(x,t)|^2+2u(x,t)+\frac{u(x,t)^2}{t}\right)G(x,-t)\,dxdt.$$

Theorem 3.4 (Weiss [Wei99])**.** *Let u be a solution of* (1.2) *in* $\mathbb{R}^n\times(-4,0]$ *with a polynomial growth at infinity. Then $W(r;u)$ is monotone nondecreasing in r for* $0<r<1$. □

The proof can be found in [Wei99]. An easy proof can be given using the following scaling property of W:

$$W(r;u_r)=W(1,u)$$

where $u_r(x,t)=(1/r^2)u(rx,r^2t)$ is the parabolic scaling of u. It can be shown that

$$W'(r;u)=\frac{1}{r^5}\int_{-4r^2}^{-r^2}\int_{\mathbb{R}^n}(\mathcal{L}u)^2\,\frac{G(x,-t)}{-t}\,dxdt\geq 0$$

for every $0<r<1$, where

$$\mathcal{L}u(x,t):=x\cdot\nabla u(x,t)+2t\,\partial_t u(x,t)-2u(x,t)=\frac{d}{dr}u_r(x,t)\Big|_{r=1}.$$

Remark 3.5. In Weiss' monotonicity theorem $W'(r;u)=0$ iff $\mathcal{L}u=0$ a.e. in $\mathbb{R}^n\times[-4r^2\times,-r^2]$. In particular $W(r;u)\equiv const =: W(u)$ iff u is homogeneous, i.e. $u(x,t)=u_r(x,t)=(1/r^2)u(rx,r^2t)$ for $0<r\leq 1$.

Before we state a local form of Weiss' monotonicity theorem, we remark that it will not be used in most of the paper and will appear only in the last sections.

Theorem 3.6. *Let $u\in\mathcal{P}_1^-(M)$ and $\psi(x)\geq 0$ be a C^∞ cut-off function in $\mathbb{R}^n$ with* $\operatorname{supp}\psi\subset B_{3/4}$ *and $\psi|_{B_{1/2}}=1$. Then there exists $C=C(n,\psi,M)>0$ such that for $w=u\,\psi$ the function*

$$W(r;w)+CF_n(r)$$

is monotone nondecreasing in r for $0<r<1/2$, where $F_n(r)=\int_0^r s^{-n-3}e^{-1/(16s^2)}ds$.

The proof is based on the following lemma.

Lemma 3.7. *Let w be of the Sobolev class $W^{2,p}_x\cap W^{1,p}_t(Q_R^-)$ for some $p\geq 2$ and* $\operatorname{supp} w(\cdot,t)\subset\subset B_R$ *for every $-R^2\leq t\leq 0$. Then*

$$W'(r;w)=\frac{1}{r^5}\int_{-4r^2}^{-r^2}\int_{\mathbb{R}^n}\mathcal{L}w(x,t)\left(\frac{\mathcal{L}w(x,t)}{-t}-2(Hw(x,t)-1)\right)G(x,-t)\,dxdt.$$

for $0<r<R/2$.

Proof. The computations below are formal but well justified, since w is a $W^{2,p}_x\cap W^{1,p}_t$ function. Using the scaling property $W(r;w)=W(1;w_r)$, we obtain for $r=1$

$$\begin{aligned}W'(1;w)=\frac{d}{dr}W(1;w_r)&=\int_{-4}^{-1}\int_{\mathbb{R}^n}\left(\mathcal{L}(|\nabla w|^2)+2\,\mathcal{L}w+2\,\frac{w}{t}\,\mathcal{L}w\right)G(x,-t)\,dxdt\\&=\int_{-4}^{-1}\int_{\mathbb{R}^n}\left(2\,\nabla w\cdot\nabla(\mathcal{L}w)+2\,\mathcal{L}w+2\frac{w}{t}\,\mathcal{L}w\right)\,G(x,-t)\,dxdt,\end{aligned}$$

where we have used the (easily verified) identity

$$\mathcal{L}(|\nabla w|^2) = 2\,\nabla w \cdot \nabla(\mathcal{L}w).$$

Now integrating by parts the term

$$2\,\nabla w \cdot \nabla(\mathcal{L}w)\,G(x,-t)$$

and using that

$$\nabla G(x,-t) = -\frac{1}{2t}\,x \cdot G(x,-t)$$

we obtain

$$\begin{aligned} W'(1;w) &= 2\int_{-4}^{-1}\int_{\mathbb{R}^n} \mathcal{L}w\left(-\Delta w - \frac{1}{2t}\,x\cdot\nabla w + 1 + \frac{w}{t}\right) G(x,-t)\,dxdt \\ &= \int_{-4}^{-1}\int_{\mathbb{R}^n} \mathcal{L}w\left(\frac{\mathcal{L}w(x,t)}{-t} - 2(Hw(x,t)-1)\right) G(x,-t)\,dxdt, \end{aligned}$$

which proves the lemma for $r = 1$ and by rescaling argument, for all r. □

Proof of Theorem 3.6. By standard parabolic estimates (see e.g. [Lie96], Chapter VII) we have that u is of class $W_x^{2,p} \cap W_t^{1,p}$ locally in Q_1^- for any $1 < p < \infty$, since $\chi_\Omega \in L^\infty$. As an immediate corollary from Lemma 3.7 we obtain that

$$W'(r;w) \geq -\frac{2}{r^5}\int_{-4r^2}^{-r^2}\int_{\mathbb{R}^n} \mathcal{L}w\,(Hw(x,t)-1)\,G(x,-t)\,dxdt. \tag{3.1}$$

Next, from the representation $w(x,t) = u(x,t)\,\psi(x)$ in Q_1^-, we have the following identities

$$\begin{aligned} \mathcal{L}w &= u\,\mathcal{L}\psi + \psi\,\mathcal{L}u \\ Hw &= u\,\Delta\psi + \psi\,Hu + 2\nabla\psi\cdot\nabla u. \end{aligned}$$

Since u satisfies (1.2) and $\operatorname{supp}\psi \subset B_{3/4}$, it is easy to see that the integrand in (3.1) vanishes a.e. in $B_{1/2} \times [-1,0]$ and $B_{3/4}^c \times [-1,0]$. Hence we obtain

$$W'(r;w) \geq -\frac{1}{r^5}\int_{-4r^2}^{-r^2}\int_{B_{3/4}\setminus B_{1/2}} f(x,t)G(x,-t)\,dxdt$$

with $\|f\|_{L^1(Q_{3/4}^-)} \leq C = C(n,\psi,M) < \infty$ and consequently

$$W'(r;w) \geq -\frac{C}{r^{n+3}}e^{-1/(16r^2)}.$$

Therefore the function

$$W(r;w) + CF_n(r)$$

is nondecreasing, where

$$F_n(r) = \int_0^r s^{-n-3}e^{-1/(16s^2)}ds.$$

The proof is complete. □

4. Uniform $C_x^{1,1} \cap C_t^{0,1}$ regularity of solutions

In this section we establish uniform local $C_x^{1,1} \cap C_t^{0,1}$ regularity of bounded solutions of (1.2).

Theorem 4.1. *Let $u \in \mathcal{P}_1^-(x_0, t_0; M)$. Then $u \in C_x^{1,1} \cap C_t^{0,1}(Q_{1/4}^-(x_0, t_0))$, uniformly. More precisely, there exists a universal constant $C_0 = C_0(n)$ such that if $u \in \mathcal{P}_1^-(x_0, t_0; M)$, then*

$$\sup_{\Omega \cap Q_{1/4}^-(x_0,t_0)} \big(|\partial_{ij} u(x,t)| + |\partial_t u(x,t)|\big) \le C_0 M.$$

In the general theory of the Stefan problem (where the additional assumptions $u \ge 0$ and $\partial_t u \ge 0$ are imposed by the problem) it can be show that $\partial_t u$ is continuous with logarithmic modulus of continuity. In fact, if we knew more regularity of $\partial_t u$ (C^α is enough) we could threat the problem as an elliptic one writing

$$\Delta u = \chi_\Omega f(x,t),$$

where $f(x,t) = (1 + \partial_t u)$.

Here we choose to approach the problem in its parabolic setting. The core of the proof of Theorem 4.1 is the following lemma, establishing the quadratic growth of solutions near the free boundary.

Lemma 4.2. *Let $u \in \mathcal{P}_1^-(M)$. Then there exist a constant $C = C(n)$ such that*

$$\sup_{Q_r^-} |u| \le C M r^2 \tag{4.1}$$

for any $0 \le r \le 1$.

Proof. We use the method adopted from [CKS00]. Set

$$S_j(u) = \sup_{Q_{2^{-j}}^-} |u| \tag{4.2}$$

and define $N(u)$ to be the set of all nonnegative integers satisfying the following doubling condition

$$2^2 S_{j+1}(u) \ge S_j(u). \tag{4.3}$$

Suppose now for some universal constant $C_0 \ge 1$

$$S_{j+1}(u) \le C_0 M\, 2^{-2j} \qquad \text{for all } j \in N(u). \tag{4.4}$$

Then we claim

$$S_j(u) \le C_0 M\, 2^{-2j+2} \qquad \text{for all } j \in \mathbb{N}. \tag{4.5}$$

Obviously (4.5) holds for $j = 1$. Next, let (4.5) hold for some j. Then it holds also for $j + 1$. Indeed, if $j \in N(u)$ it follows from (4.4). If $j \notin N(u)$, (4.3) fails and we obtain

$$S_{j+1}(u) \le 2^{-2} S_j(u) \le C_0 M\, 2^{-2j}.$$

Therefore (4.5) holds for all $j \in \mathbb{N}$. This implies

$$\sup_{Q_r^-} |u| \le 8 C_0 M r^2$$

for any $r \le 1$, and the lemma follows with $C = 8C_0$.

Now to complete the proof we need to show (4.4). Suppose it fails. Then there exist sequences $u_j \in \mathcal{P}_1^-(M)$, and $k_j (\in N(u_j))$, $j = 1, 2, \ldots$, such that

$$S_{k_j+1}(u_j) \geq jM\, 2^{-2k_j}. \tag{4.6}$$

Define $\widetilde{u}_j$ as

$$\widetilde{u}_j(x,t) = \frac{u_j(2^{-k_j}x, 2^{-2k_j}t)}{S_{k_j+1}(u_j)} \qquad \text{in } Q_1^-.$$

Then

$$\sup_{Q_1^-} |H(\widetilde{u}_j)| \leq \frac{2^{-2k_j}}{S_{k_j+1}(u_j)} \leq \frac{1}{jM} \to 0, \tag{4.7}$$

$$\sup_{Q_{1/2}^-} |\widetilde{u}_j| = 1, \quad \text{(by (4.2))} \tag{4.8}$$

$$\sup_{Q_1^-} |\widetilde{u}_j| \leq \frac{S_{k_j}(u_j)}{S_{k_j+1}(u_j)} \leq 4 \qquad \text{(by (4.3))} \tag{4.9}$$

$$\widetilde{u}_j(0,0) = |\nabla \widetilde{u}_j(0,0)| = 0 \tag{4.10}$$

Now by (4.7)–(4.10) we will have a subsequence of $\widetilde{u}_j$ converging in $C_x^{1,\alpha} \cap C_t^{0,\alpha}(Q_1^-)$ to a non-zero caloric function u_0 in Q_1^-, satisfying $u_0(0,0) = |\nabla u_0(0,0)| = 0$. Moreover, from (4.8), we will have

$$\sup_{Q_{1/2}^-} |u_0| = 1. \tag{4.11}$$

For any spatial unit vector e define

$$v = \partial_e u_0, \qquad v_j = \partial_e u_j, \qquad \widetilde{v}_j = \partial_e \widetilde{u}_j.$$

Then, over a subsequence, $\widetilde{v}_j$ converges in $C_x^{0,\alpha} \cap C_t^{0,\alpha}(Q_1^-)$ to v. Moreover $H(v) = 0$. Now, for a fixed cut-off function $\psi(x)$ with $\psi|_{B_{1/2}} = 1$ and $\operatorname{supp}\psi \subset B_{3/4}$ and $u \in \mathcal{P}_1(M)$ consider

$$\Phi(t; (\partial_e u)\psi) = \frac{1}{t^2}\, I(t; (\partial_e u)^+\psi)\, I(t; (\partial_e u)^-\psi).$$

Then to apply [Caf93] monotonicity formula (see Theorem 3.3 above), we need to verify that the functions $(\partial_e u)^\pm$ are sub-caloric; we leave this to the reader. Then, for all $0 < t < t_0$, we obtain

$$\Phi(t; (\partial_e u)\psi) \leq C \|\nabla u\|^4_{L^2(Q_1^-)} \leq C_0, \tag{4.12}$$

for a universal constant C_0, which, by classical estimates, depends on the class only.

Now choose ψ as above and set $\psi_j(x) = \psi(2^{-k_j}x)$. Then estimate (4.12) applied to $\widetilde{v}_j \psi_j$ gives

$$\Phi(1; \widetilde{v}_j\psi_j) \leq \left(\frac{2^{-2k_j}}{S_{k_j+1}}\right)^4 \Phi(2^{-2k_j}; v_j\psi) \leq C_0 \left(\frac{2^{-2k_j}}{S_{k_j+1}}\right)^4 \tag{4.13}$$

for k_j large enough. Since $\psi_j = 1$ in $B_{2^{k_j-1}}$ we will have

$$|\nabla(\widetilde{v}_j\psi_j)|^2 \geq |\nabla \widetilde{v}_j|^2 \chi_{B_1}.$$

Hence for $\varepsilon > 0$ (small and fixed) we have

$$C_{n,\varepsilon} \int_{-1}^{-\varepsilon}\int_{B_1} |\nabla \widetilde{v}_j^{\pm}|^2\,dxdt \le \int_{-1}^{0}\int_{B_1} |\nabla \widetilde{v}_j^{\pm}\psi_j|^2 G(x,-t)\,dxdt = I(1,\widetilde{v}_j^{\pm}\psi_j).$$

This estimate, in combination with Poincare's inequality, gives

$$\int_{-1}^{-\varepsilon}\int_{B_1} |\widetilde{v}_j^{\pm} - M^{\pm}(t)|^2\,dxdt \le C_n \int_{-1}^{-\varepsilon}\int_{B_1} |\nabla \widetilde{v}_j^{\pm}|^2\,dxdt \le C(n,\varepsilon) I(1,\widetilde{v}_j^{\pm}\psi_j),$$

where $M_j^{\pm}(t)$ denotes the corresponding mean value of $\tilde{v}_j^{\pm}$ on the t-section.

Using this and (4.13) we will have

$$\left(\int_{-1}^{-\varepsilon}\int_{B_1} |\widetilde{v}_j^{+} - M_j^{+}(t)|^2 dxdt\right)\left(\int_{-1}^{-\varepsilon}\int_{B_1} |\widetilde{v}_j^{-} - M_j^{-}(t)|^2 dxdt\right) \le$$
$$C(n,\varepsilon)\,\Phi(1, v_j\psi) \le C(n,\varepsilon)\left(\frac{2^{-2k_j}}{S_{k_j+1}}\right)^4.$$

Using (4.6) and letting $j \to \infty$ (and then $\varepsilon \to 0$), we obtain

$$\int_{-1}^{0}\int_{B_1} |v^+ - M^+(t)|^2 \int_{-1}^{0}\int_{B_1} |v^- - M^-(t)|^2 = 0, \tag{4.14}$$

where $M^{\pm}(t)$ denotes the corresponding mean value of $v^{\pm}$ on t-sections over B_1. Obviously, (4.14) implies that either of $v^{\pm}$ is equivalent to $M^{\pm}(t)$ in Q_1^-, and thus independent of the spatial variables. Let us assume $v^- = M^-(t)$. Then $-\partial_t v^- = H(v^-) = 0$, i.e. M^- is constant in Q_1^-. Since $v(0,0) = 0$ we must have $M^- = 0$, i.e. $v \ge 0$ in Q_1^-). Hence by the minimum principle $v \equiv 0$ in Q_1^-. Since $v = \partial_e u_0$, and e is arbitrary direction we conclude that u_0 is constant in Q_1^-. Also $u_0(0,0) = 0$ implies that the constant must be zero, i.e $u_0 \equiv 0$ in Q_1^-. This contradicts (4.11) and the lemma is proved. □

In fact, for the proof of Theorem 4.1 we will need also the extension of Lemma 4.2 to the "upper half" as well.

Lemma 4.3. *Let $u \in \mathcal{P}_1(M)$. Then there exist a constant $C = C(n)$ such that*

$$\sup_{Q_r} |u| \le CMr^2 \tag{4.15}$$

for any $0 \le r \le 1$.

Proof. Define $w_1 = CM(|x|^2 + 2nt)$, where C as in Lemma 4.2. Then $H(w_1) = 0 \le H(u)$ in Q_1^+. Also, by (4.1), $w_1 \ge u$ on the parabolic boundary $\partial_p Q_1^+$. Hence by the comparison principle we will have $w_1 \ge u$ in Q_1^+.

Similarly we define $w_2 = -CM(|x|^2 + 2nt) - t$, which satisfies $H(w_2) = 1 \ge H(u)$ in Q_1^+. Also, by (4.1), on the parabolic boundary $\partial_p Q_1^+$ we have $w_2 \le u$. Hence by the comparison principle $w_2 \le u$ in Q_1^+. This completes the proof of the lemma. □

Now, we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. For $(x_0, t_0) \in \Omega \Subset Q_{1/4}^-$ let

$$d = d^-(x_0, t_0) = \sup\{r : Q_r^-(x_0, t_0) \subset \Omega \cap Q_1^-\},$$

the *parabolic distance* to the free boundary. Then Lemma 4.3 implies that

$$|u(x,t)| \le CMd^2 \quad \text{in } Q_d^-(x_0, t_0).$$

Now consider the function

$$v(x,t) = \frac{1}{d^2}\, u(d\,x + x_0, d^2 t + t_0) \quad \text{in } Q_1^-.$$

Then v satisfies $Hv = 1$, and $|v|$ is uniformly bounded in Q_1^-. Hence by standard parabolic estimates (see for instance [Fri64]) $\partial_{ij} v(0,0) = \partial_{ij} u(x_0,t_0)$, and $\partial_t v(0,0) = \partial_t u(x_0,t_0)$ are uniformly bounded (independent of x_0 and t_0), which is the desired result. The theorem is proved. □

The above theorem has the following obvious implication.

Corollary 4.4. *Let $u \in \mathcal{P}_\infty^-(M)$. Then*

$$|\partial_{ij} u(x,t)| + |\partial_t u(x,t)| \le C_0 M \quad \text{in } \Omega.$$

Proof. Let u_r be a scaling of u at the origin, i.e.

$$u_r(x,t) = \frac{1}{r^2}\, u(rx, r^2 t) \quad \text{in } Q_1^-.$$

Then $u \in \mathcal{P}_\infty^-(M)$ implies $u_r \in \mathcal{P}_1^-(3M)$ for $r \ge 1$. Hence by Theorem 4.1 we have

$$\sup_{\Omega_r \cap Q_{1/4}^-} \big(|\partial_{ij} u_r(x,t)| + |\partial_t u_r(x,t)| \big) \le C_0 M,$$

i.e.,

$$\sup_{\Omega \cap Q_{r/4}^-} \big(|\partial_{ij} u(x,t)| + |\partial_t u(x,t)| \big) \le C_0 M.$$

Letting $r \to \infty$ we will obtain the statement of the corollary. □

5. Nondegeneracy

5.1. **Nondegeneracy.** The reader may have wondered what happens if the function u_r under the blow-up process converges identically to zero (i.e. it degenerates). This happens if the function decays to zero faster than quadratically. This, however, does not happen if we blow-up at a free boundary point.

Lemma 5.1. *Let u be a solution of* (1.2) *and* $(x_0, t_0) \in \Gamma$. *Then there exists a universal constant $C_n > 0$ such that*

$$\sup_{Q_r^-(x_0,t_0)} u \ge C_n r^2 \tag{5.1}$$

for any $r > 0$ such that $Q_r^-(x_0,t_0) \subset D$. More generally, for any $(x_0,t_0) \in \Lambda$ we have that either (5.1) *holds or $u \equiv 0$ in $Q_{r/2}^-(x_0,t_0)$ for any $r > 0$ as above.*

Proof. Consider first $(x_1,t_1) \in \{u > 0\}$ and set

$$w(x,t) = u(x,t) - u(x_1,t_1) - \frac{1}{2n+1}\Big(|x - x_1|^2 - (t - t_1) \Big).$$

Then w is caloric in $\Omega \cap Q_r^-(x_1,t_1)$ and strictly negative on $\partial\Omega \cap Q_r^-(x_1,t_1)$. Since $w(x_1,t_1) = 0$, the maximum of w on the parabolic boundary of the cylinder $Q_r^-(x_1,t_1)$ is nonnegative. In particular we obtain

$$\sup_{\partial_p Q_r^-(x_1,t_1)} \left(u(x,t) - u(x_1,t_1) - \frac{r^2}{2n+1} \right) \ge 0.$$

Hence

$$\sup_{Q_r^-(x_1,t_1)} u \geq u(x_1,t_1) + \frac{r^2}{2n+1}. \tag{5.2}$$

Then a limiting argument shows that (5.1) holds if (x_0, t_0) is in the closure of $\{u > 0\}$ with $C_n = 1/(2n+1)$. Moreover, if $Q^-_{r/2}(x_0, t_0)$ contains a point (x_1, t_1) in $\{u > 0\}$, we still have

$$\sup_{Q_r^-(x_0,t_0)} u(x,t) \geq \sup_{Q_{r/2}^-(x_1,t_1)} u(x,t) \geq u(x_1,t_1) + \frac{(r/2)^2}{2n+1} \geq C_n r^2.$$

Finally, in the case when $u \leq 0$ in $Q^-_{r/2}(x_0, t_0)$, the maximum principle implies that $u \equiv 0$ in $Q^-_{r/2}(x_0, t_0)$, since $u(x_0, t_0) = 0$. Thus (x_0, t_0) is not a free boundary point. □

The next lemma shows that we have also a certain nondegeneracy at the points of $\partial\Omega \cap D$ even if they are not in Γ.

Lemma 5.2. *Let u be a solution of* (1.2) *and* $(x_0, t_0) \in \partial\Omega \cap D$. *Then there exists a constant $C_n > 0$ such that*

$$\sup_{Q_r(x_0,t_0)} |u| \geq C_n r^2. \tag{5.3}$$

for any $r > 0$ with $Q_r(x_0, t_0) \subset D$.

Proof. Consider two cases: (i) $Q_{r/2}(x_0, t_0)$ contains a point (x_1, t_1) in $\{u > 0\}$ and (ii) $u \leq 0$ in $Q_{r/2}$. As in the proof of the previous lemma, we obtain that in the first case

$$\sup_{Q_r(x_0,t_0)} u \geq C_n r^2$$

(and we are done) and in the second case that $u \equiv 0$ in $Q^-_{r/2}(x_0, t_0)$. Moreover, in the second case we claim that

$$\inf_{Q^+_{r/2}(x_0,t_0)} u \leq -C_n r^2.$$

Indeed, first observe that $u < 0$ in $Q_{r/2}(x_0, t_0) \cap \{t > t_0\}$, otherwise we would have $u \equiv 0$ in $B_{r/2}(x_0) \times (t_0 - r^2/4, t_1)$ for some $t_1 > t_0$, which contradicts to the assumption that $(x_0, t_0) \in \partial\Omega$. The parabolic scaling

$$v(x,t) = \frac{1}{r^2} u(rx + x_0, r^2 t + t_0)$$

satisfies

$$Hv = 1, \quad v < 0 \quad \text{in } Q^+_{1/2}.$$

But then

$$\inf_{Q^+_{1/2}} v \leq -C_n,$$

otherwise we would have a sequence of functions $-1/k \leq v_k \leq 0$ in $Q^+_{1/2}$ satisfying $Hv_k = 1$. This is impossible, since the limit function v_0, which is identically 0, should also satisfy $Hv_0 = 1$.

Scaling back, we complete the proof of the lemma. □

5.2. **Stability under the limit.** Let u_j be any converging sequence in the class $\mathcal{P}_1^-(M)$ and let $u_0 = \lim_{j\to\infty} u_j$. Then we claim $u_0 \in \mathcal{P}_1^-(M)$.

To prove this statement, we may assume that the convergence is in $C_x^{1,\alpha} \cap C_t^{0,\alpha}$. hence we have

$$\limsup_{j\to\infty} \Lambda(u_j) \subset \Lambda(u_0), \tag{5.4}$$

where $\limsup_{j\to\infty} E_j$ for the sequence of sets E_j is defined as the set of all limit points of sequences $(x_{j_k}, t_{j_k}) \in E_{j_k}$, $j_k \to \infty$. Then for any $(x,t) \in \Omega(u_0)$ there exists $\varepsilon > 0$ such that $Q_\varepsilon^-(x,t) \subset \Omega(u_j)$, thus implying that

$$Hu_0 = 1 \quad \text{in } \Omega(u_0).$$

Since also u_0 is $C_x^{1,1} \cap C_t^{0,1}$ regular, it follows that u_0 is a solution of (1.2).

Next, we claim that

$$\limsup_{j\to\infty} \Gamma(u_j) \subset \Gamma(u_0). \tag{5.5}$$

In particular, if $(0,0) \in \Gamma(u_j)$ then $(0,0) \in \Gamma(u_0)$. This follows immediately from the nondegeneracy Lemma 5.1.

In fact, we also have a similar inclusion for $\partial\Omega$:

$$\limsup_{j\to\infty} \partial\Omega(u_j) \subset \partial\Omega(u_0). \tag{5.6}$$

Indeed, (5.6) will follow once we show that if $u_0 = 0$ in $Q_r(x_0,t_0)$ then $u_j = 0$ in $Q_{r/2}(x_0,t_0)$ for sufficiently large j. Assume the contrary. Then we will have either $Q_{r/2}(x_0,t_0) \subset \Omega(u_j)$ or $Q_{r/2}(x_0,t_0) \cap \Omega(u_j) \neq \emptyset$ over infinitely many $j = j_k$. In the first case we will obtain that u_0 satisfies $H(u_0) = 1$ in $Q_{r/2}(x_0,t_0)$ and in the second that $\sup_{Q_r(x_0,t_0)} |u_0| \geq C_n r^2$, both of which are impossible for $u_0 = 0$ in $Q_r(x_0,t_0)$.

The same argument as above shows also that

$$\limsup_{j\to\infty} \Omega(u_j) \subset \overline{\Omega(u_0)}. \tag{5.7}$$

Generally, we cannot prove inclusions similar to (5.4)–(5.7) for t-levels of the sets Ω and Λ. The reason is the lack of the "elliptic version" of (5.3) on t-sections. However, for the Stefan problem when one assumes that $\partial_t u \geq 0$, one has $\Delta u = \chi_\Omega + \partial_t u \geq \chi_\Omega$, which allows to prove the elliptic version of (5.3).

5.3. **Lebesgue measure of $\partial\Omega$.** From the nondegeneracy and the $C_x^{1,1} \cap C_t^{0,1}$ regularity one can deduce that $\partial\Omega$ (hence also the free boundary Γ), has $(n+1)$-dimensional Lebesgue measure zero. It is enough to show that $\partial\Omega$ has Lebesgue density less than 1 at every its point.

Indeed, take any $(x_0,t_0) \in \partial\Omega$. Using (5.3), we can find $(x_1,t_1) \in Q_{r/4}(x_0,t_0)$ such that $|u(x_1,t_1)| \geq Cr^2$. On the other hand, by Theorem 4.1, we have $|u(x_1,t_1)| \leq C_1 d^2(x_1,t_1)$ (where d is the parabolic distance to Λ). Hence $d(x_1,t_1) \geq Cr$. In particular the set $\Omega \cap Q_r(x_0,t_0)$ contains a cylinder of a size, proportional to $Q_r(x_0,t_0)$.

In fact, we claim that for any parabolic cylinder $Q_r(x,t)$, not necessarily centered at a point on $\partial\Omega$, $Q_r(x,t) \setminus \partial\Omega$ contains a cylinder proportional to $Q_r(x,t)$. To prove it, consider the following two alternatives: either $Q_{r/2}(x,t)$ contains a point on $\partial\Omega$ or it doesn't. In the first case the claim follows from the arguments above and in the second case $Q_{r/2}(x,t)$ itself is contained in $Q_r(x,t) \setminus \partial\Omega$.

Further, that $\partial\Omega$ has Lebesgue density less than 1 at (x_0, t_0) will follow if we show that for every hyperbolic cylinder

$$C_r(x_0, t_0) := B_r(x_0) \times (t_0 - r, t_0 + r)$$

$C_r(x_0, t_0) \setminus \partial\Omega$ contains a set E of Lebesgue measure proportional to that of $C_r(x_0, t_0)$. Note, it is enough to show this statement for $r = 1/k$, $k = 1, 2, \dots$ Subdivide $C_{1/k}(x_0, t_0)$ into k parabolic cylinders

$$Q_i = Q_{1/k}(x_0, t_i), \quad t_i = t_0 + 1 - (2i - 1)/k, \quad i = 1, 2, \dots, k.$$

Then $Q_i \setminus \partial\Omega$ contains a cylinder $\widetilde{Q}_i$ proportional to Q_i and one can take

$$E = \bigcup_{i=1}^{k} \widetilde{Q}_i.$$

Thus, $\partial\Omega$ has $(n + 1)$-dimensional Lebesgue measure 0.

6. Homogeneous global solutions

Definition 6.1. We say that the solution $u(x, t)$ is *homogeneous* (with respect to the origin) if

$$\frac{1}{r^2}\, u(rx, r^2 t) = u(x, t)$$

for every $r > 0$.

Simple examples of homogeneous solutions are the polynomial solutions of the type

$$u(x, t) = mt + P(x),$$

where m is a constant and P is a homogeneous quadratic polynomial satisfying $\Delta P = m + 1$, and the halfspace solutions

$$u(x, t) = \frac{1}{2}\,(x \cdot e)_+^2$$

for spatial unit vectors e. As we will see below these are the only nonzero homogeneous solutions in $\mathbb{R}^n \times \mathbb{R}^-$.

As was already mentioned in Remark 3.5, solutions u, homogeneous in the past, have the property that their Weiss functional $W(r; u)$ is a constant (and vice versa.) We denote this constant by $W(u)$.

Lemma 6.2.

(i) *For every spatial direction e*

$$W\left(\frac{1}{2}\,(x \cdot e)_+^2\right) = W\left(\frac{1}{2}\,(x_1)_+^2\right) =: A;$$

(ii) *For every constant m and homogeneous quadratic polynomial $P(x)$ satisfying $\Delta P = m + 1$*

$$W\,(mt + P(x)) = W\left(\frac{1}{2}\,(x_1)^2\right) = 2A$$

Proof. Part (i) is obvious because of the rotational symmetry of the functional W. Part (ii) follows from the direct computations. Indeed, for given $t < 0$,

$$\int_{\mathbb{R}^n} \left(|\nabla u(x,t)|^2 + 2u(x,t) + \frac{u(x,t)^2}{t} \right) G(x,-t)\, dx \quad = $$
$$\int_{\mathbb{R}^n} u \left(-\Delta u - \frac{1}{2t}\, x \cdot \nabla u + \frac{u}{t} + 2 \right) G(x,-t)\, dx \quad = \quad \int_{\mathbb{R}^n} u(x,t)\, G(x,-t)\, dx = -t,$$

where we first integrated by parts the term $|\nabla u|^2\, G = \nabla u \cdot (G\, \nabla u)$ and then used that

$$\Delta u + \frac{1}{2t}\, x \cdot \nabla u - \frac{u}{t} - 1 = 0$$

for homogeneous solutions. This equation can be obtained, for instance, from $\Delta u - \partial_t u = 1$ and the homogeneity property $x \cdot \nabla u + 2t\, \partial_t u = 2u$ by eliminating $\partial_t u$.

Hence

$$W(u,r) = \frac{1}{r^4} \int_{-4r^2}^{-r^2} (-t) dt = \frac{15}{2}$$

and the lemma follows with $A = 15/4$ □

The importance of this lemma is emphasized by the following fact.

Lemma 6.3. *The only nonzero homogeneous solutions of* (1.2) *in* $D = \mathbb{R}^n \times \mathbb{R}^-$ *are of the type*

(i) $u(x,t) = \frac{1}{2}(x \cdot e)_+^2$ *for a certain spatial unit vector e;*

(ii) $u(x,t) = mt + P(x)$*, where m is a constant and* $P(x)$ *is a homogeneous quadratic polynomial satisfying* $\Delta P = m + 1$.

Proof. From the homogeneity of u it follows that the time sections $\Omega(t) = \{x : (x,t) \in \Omega\}$ satisfy

$$\Omega(r^2 t) = r\Omega(t)$$

for any $r > 0$. We consider two different cases.

Case 1. Ω^c has an empty interior. This will happen if and only if $\Omega(t)^c$ has an empty interior for one and thus for all t. Since both u and $|\nabla u|$ vanish on Ω^c, it follows that u satisfies $\Delta u - \partial_t u = 1$ in the whole $\mathbb{R}^n \times \mathbb{R}^-$. But then $\partial_t u$ is a bounded caloric function in $\mathbb{R}^n \times \mathbb{R}^-$, thus a constant by the Liouville theorem. Similarly, $\partial_{ij} u$ are constants. Therefore we obtain the representation

$$u(x,t) = mt + P(x)$$

where P is a homogeneous quadratic polynomial such that $\Delta P = m + 1$.

Case 2. Ω^c has nonempty interior. By homogeneity, for every unit spatial direction e,

$$\Phi(t; \partial_e u) \equiv const$$

where Φ is as in Theorem 3.1. However, this is possible only if the spatial supports of $(\partial_e u)_+$ and $(\partial_e u)_-$ are complementary halfspaces at almost all t, or if $\Phi(t; \partial_e u) \equiv 0$, see Remark 3.2. The former case cannot occur since $\Omega(t)^c$ is nonempty for all $t < 0$, and the latter case implies $\partial_e u \geq 0$ or $\partial_e u \leq 0$ in whole $\mathbb{R}^n \times \mathbb{R}^-$. Since this is true for all spatial directions e, it follows that $u(x,t)$ is one-dimensional, i.e. in suitable spatial coordinates $u(x,t) = u(x_1,t)$. We may assume therefore that in the rest of the proof that the spatial dimension $n = 1$. From homogeneity we also have the representation

$$u(x,t) = -t\, f\left(\frac{x}{\sqrt{-t}} \right),$$

where $f = u(\cdot, -1)$. The function f satisfies

$$f''(\xi) - \frac{\xi}{2} f'(\xi) + f(\xi) - 1 = 0 \quad \text{in } \Omega(-1)$$
$$f(\xi) = f'(\xi) = 0 \quad \text{on } \Omega(-1)^c$$

The general solution to the ordinary differential equation above is given by

$$f(\xi) = 1 + C_1 (\xi^2 - 2) + C_2 \left(-2\xi\, e^{\xi^2/4} + (\xi^2 - 2) \int_0^\xi e^{s^2/2} ds \right)$$

and if a is a finite endpoint of a connected component of $\Omega(-1)$, we have

$$C_2 = \frac{1}{4}\, a\, e^{-a^2/4}, \quad C_1 = \frac{1}{2} - \frac{1}{4}\, a\, e^{-a^2/4} \int_0^a e^{-s^2/4} ds$$

In particular, we see that there could not be two different values of a, hence each connected component of $\Omega(-1)$ is unbounded.

Next, on the unbounded interval we must have $C_2 = 0$, since f has at most quadratic growth at infinity. This implies $C_1 = 1/2$ and the only possible value of a is 0. Thus, $\Omega(-1)$ is either $(0, \infty)$ or $(-\infty, 0)$ and $f(\xi) = (1/2)\, \xi_+^2$ or $f(\xi) = (1/2)\, \xi_-^2$ respectively. □

Remark 6.4. As shows the example before Theorem I, Lemma 6.3 is valid only for solutions in lower-half space $\mathbb{R}^n \times \mathbb{R}^-$ but not for the solutions in the whole $\mathbb{R}^n \times \mathbb{R}$. In fact, if we take any homogeneous solution in $\mathbb{R}^n \times \mathbb{R}^-$ and continue it to $\mathbb{R}^n \times \mathbb{R}$ by solving the Cauchy problem for $Hu = 1$ in $\mathbb{R}^n \times \mathbb{R}^+$, then the resulting function will still be homogeneous, but will not have one of the forms in Lemma 6.3 in $\mathbb{R}^n \times \mathbb{R}^+$.

7. Balanced energy

Let $u \in \mathcal{P}_\infty^-(M)$ be a global solution. Then we define the *balanced energy*

$$\omega = \lim_{r \to 1} W(r; u) \tag{7.1}$$

which exists due to Weiss' monotonicity formula. Recall that the functional W has the scaling property

$$W(rs; u) = W(s; u_r), \tag{7.2}$$

where

$$u_r(x, t) = \frac{1}{r^2} u(rx, r^2 t).$$

Since the functions u_r are locally uniformly in class $C_x^{1,1} \cap C_t^{0,1}$ in $\mathbb{R}^n \times \mathbb{R}^-$, we can extract a converging subsequence u_{r_k} to a global solution u_0 in $\mathbb{R}^n \times \mathbb{R}^-$. Then passing to the limit in (7.2) we will obtain that

$$\omega = W(s, u_0)$$

for any $s > 0$. This implies that the blow-up u_0 is a homogeneous global solution. Moreover, from Lemmas 6.2 and 6.3 it follows that ω can take only three values: 0, A, or $2A$.

Similarly we define the balanced energy at any point $(x, t) \in \Lambda$ for a global solution $u \in \mathcal{P}_\infty^-(M)$ by

$$\omega(x, t) = \lim_{r \to 0} W(r, x, t; u).$$

Definition 7.1. We say that a point $(x,t) \in \partial\Omega$ is a *zero, low, or high energy point* of the global solution $u \in \mathcal{P}_\infty^-(M)$ if

$$\omega(x,t) = 0,\ A,\ 2A$$

respectively. Here A is as in Lemma 6.2.

Remark 7.2. The balanced energy function ω is upper semicontinuous, since

$$W(r,\cdot,\cdot;u) =: \omega_r \searrow \omega \quad \text{as } r \searrow 0$$

and functions ω_r are continuous on $\partial\Omega$. Hence, if

$$\mathcal{E}_j = \{\omega = jA\}, \quad \text{for } j = 0,1,2,$$

then $\mathcal{E}_0$, $\mathcal{E}_0 \cup \mathcal{E}_1$ are relatively open and $\mathcal{E}_2$ is closed.

7.1. **Zero energy points.** By definition, $(x_0,t_0) \in \partial\Omega$ is a zero energy point for u if there exists a blow-up u_0 of u at (x_0,t_0), such that $u_0 \equiv 0$ in $\mathbb{R}^n \times \mathbb{R}^-$. From Lemma 5.1 and we have that either

$$\sup_{Q_r^-(x_0,t_0)} u \geq C_n r^2,$$

for all $r > 0$, or $u \equiv 0$ in $Q_r^-(x_0,t_0)$ and $u < 0$ in $Q_r^+(x_0,t_0)$ for some $r > 0$. In the first case the point (x_0,t_0) is either of low or high energy, since no blow-up u_0 at (x_0,t_0) can vanish identically in $\mathbb{R}^n \times \mathbb{R}^-$. And only in the second case the point (x_0,t_0) is of zero energy. Thus, zero energy points are parabolically interior points of Λ.

Also, we obtain that the free boundary Γ cannot contain zero energy points and in fact

$$\Gamma = \partial\Omega \setminus \mathcal{E}_0 = \mathcal{E}_1 \cup \mathcal{E}_2.$$

In other words, the free boundary points consists of low and high energy points of $\partial\Omega$.

Finally, we remark that if $u \geq 0$, $\partial\Omega$ coincides with the free boundary Γ, since there could be no zero energy points.

7.2. **High energy points.**

Lemma 7.3. *Let $(x_0,t_0) \in \partial\Omega$ be a high energy point for a global solution u. Then*

$$u(x,t) = m(t-t_0) + P(x-x_0) \tag{7.3}$$

in $\mathbb{R}^n \times \mathbb{R}^-_{t_0}$, where m is a constant and P is a homogeneous quadratic polynomial.

Proof. Without loss of generality we may assume that $(x_0,t_0) = (0,0)$. Consider then the functional $W(r;u)$. It is nondecreasing in r and therefore

$$\omega = W(0+;u) \leq W(\infty;u) = W(u_\infty), \tag{7.4}$$

where u_∞ is a shrink-down limit over a subsequence of scaled functions u_r as $r \to \infty$. Since $(0,0)$ is a high energy point, $\omega = 2A$ and we obtain $W(u_\infty) \geq 2A$. The shrink-down function u_∞ is homogeneous and therefore from Lemmas 6.2 and 6.3 we have also $W(u_\infty) \leq 2A$. Hence $W(u_\infty) = 2A$, which is possible if and only if

$$W(r;u) = 2A \quad \text{for all } r > 0.$$

This implies that $u(x,t)$ is homogeneous with respect to $(0,0)$ in $\mathbb{R}^n \times \mathbb{R}^-$. Applying Lemma 6.3 to we finish the proof. □

Remark 7.4. A simple corollary from the lemma above is that all high energy points, if they exist, are on the same time level $t = t_0$, if $m \neq 0$ in the representation (7.3). Moreover x_0 must be on the hyperplane $\{\nabla u(\cdot, t_0) = 0\} = \{\nabla P(\cdot - x_0)\}$. Except the case when $P \equiv 0$ (equivalently $u(x,t) = -(t - t_0)$) $\{\nabla u(\cdot, t_0) = 0\}$ is a lower-dimensional hyperplane in $\mathbb{R}^n$.

If $m = 0$ in (7.3) then there exists a maximal $t_* \geq t_0$ (possibly infinite) such that $u(x,t) = P(x - x_0)$ for all x and $t \leq t_*$. If t_* is finite, then u has no high energy points (x,t) with $t > t_*$.

7.3. **Low energy points.**

Theorem 7.5. *Let $u \in \mathcal{P}^-_\infty(M)$ be a global solution and $(x_0, t_0) \in \partial\Omega$ be a low energy point. Then there exists $r = r(x_0, t_0) > 0$ such that $u \geq 0$ in $Q^-_r(x_0, t_0)$. Moreover, we can choose $r > 0$ such that $\partial\Omega \cap Q^-_r(x_0, t_0)$ is a Lipschitz (in space and time) surface.*

The proof is based on the following two useful lemmas.

Lemma 7.6. *Let u be a bounded solution of* (1.2) *in Q^-_1 and h be caloric in $Q^-_1 \cap \Omega$. Suppose moreover that*

(i) *$h \geq 0$ on $\partial\Omega \cap Q^-_1$ and*
(ii) *$h - u \geq -\varepsilon_0$ in Q^-_1, for some $\varepsilon_0 > 0$.*

Then $h - u \geq 0$ in $Q^-_{1/2}$, provided $\varepsilon_0 = \varepsilon_0(n)$ is small enough.

Proof. Suppose the conclusion of the lemma fails. Then there are u and h satisfying the conditions of the lemma such that

$$h(x_0, t_0) - u(x_0, t_0) < 0 \tag{7.5}$$

for some $(x_0, t_0) \in Q^-_{1/2}$. Let

$$w(x,t) = h(x,t) - u(x,t) + \frac{1}{2n+1}(|x - x_0|^2 - (t - t_0)).$$

Then w is caloric in $\Omega \cap Q^-_{1/2}(x_0, t_0)$, $w(x_0, t_0) < 0$ by (7.5) and $w \geq 0$ on $\partial\Omega \cap Q^-_{1/2}(x_0, t_0)$. Hence by the minimum principle the negative infimum of w is attained on $\partial_p Q^-_{1/2}(x_0, t_0)$. We thus obtain

$$-\varepsilon_0 \leq \inf_{\partial Q^-_{1/2}(x_0,t_0)\cap\Omega} (h - u) \leq -\frac{1}{4(2n+1)},$$

which is a contradiction as soon as $\varepsilon_0 < 1/4(2n+1)$. This proves the lemma. □

Lemma 7.7. *Let $u \in \mathcal{P}^-_\infty(M)$ be a global solution and $(x_0, t_0) \in \partial\Omega$ be such that $\overline{Q}_\varepsilon(x_0, t_0) \cap \partial\Omega$ consists only of low energy points for some $\varepsilon > 0$. Then the time derivative $\partial_t u$ vanishes continuously at (x_0, t_0):*

$$\lim_{(x,t)\to(x_0,t_0)} \partial_t u(x,t) = 0.$$

Proof. Consider the family of continuous functions ω_r defined on $\overline{Q}_\varepsilon(x_0, t_0) \cap \partial\Omega$ for every $r > 0$ by

$$\omega_r(x,t) = W(r, x, t; u).$$

Functions ω_r are continuous and converge pointwise to the balanced energy function ω as $r \to 0$. Since $\overline{Q}_\varepsilon(x_0, t_0) \cap \partial\Omega$ consists only of low energy points, $\omega = A$ there. Hence

$$\omega_r \searrow A \quad \text{as } r \searrow 0 \text{ on } \overline{Q}_\varepsilon \cap \partial\Omega,$$

as it follows from Weiss' monotonicity formula. From Dini's monotone convergence theorem it follows that the convergence $\omega_r \to A$ is uniform. In particular, for any sequences $(y_j, s_j) \to (x_0, t_0)$ and $r_j \to 0$ we have

$$W(r_j, y_j, s_j; u) \to A.$$

Let now $(x_j, t_j) \to (x_0, t_0)$ be the maximizing sequence for $\partial_t u$ in the sense that

$$\partial_t u(x_j, t_j) \to m := \limsup_{(x,t)\to(x_0,t_0)} \partial_t u(x, t).$$

Let $d_j = d^-(x_j, t_j) = \sup\{r : Q_r^-(x_j, t_j) \subset \Omega\}$ and $(y_j, s_j) \in \partial_p Q_{d_j}^-(x_j, t_j) \cap \partial\Omega$. Without loss of generality we may assume that

$$\frac{1}{d_j^2} u(d_j x + x_j, d_j^2 t + t_j) =: u_j(x, t) \to u_0(x, t)$$

in $\mathbb{R}^n \times \mathbb{R}^-$ and

$$\Big((y_j - x_j)/d_j, (s_j - t_j)/d_j^2\Big) =: (\widetilde{y}_j, \widetilde{s}_j) \to (\widetilde{y}_0, \widetilde{s}_0) \in \partial Q_1^-.$$

Observe that since $Q_1^- \subset \Omega(u_j)$, we will have $Q_1^- \subset \Omega(u_0)$ and may assume that the convergence in Q_1^- is locally uniform in $C_x^2 \cap C_t^1$ norm. Thus

$$\partial_t u_0(0, 0) = \lim_{j\to\infty} \partial_t u_j(0, 0) = \lim_{j\to\infty} \partial_t u(x_j, t_j) = m$$

and

$$\partial_t u_0(x, t) = \lim_{j\to\infty} \partial_t u_j(x, t) = \lim_{j\to\infty} \partial_t u(d_j x + x_j, d_j^2 t + t_j) \le m$$

for any $(x, t) \in Q_1^-$. Since also $\partial_t u_0 = 0$ in Q_1^-, the maximum principle implies

$$\partial_t u_0 = m \quad \text{in } Q_1^-. \tag{7.6}$$

On the other hand

$$W(r, \widetilde{y}_0, \widetilde{s}_0; u_0) = \lim_{j\to\infty} W(r, \widetilde{y}_j, \widetilde{s}_j; \widetilde{u}_j) = \lim_{j\to\infty} W(d_j r, y_j, s_j; u) = A \tag{7.7}$$

for every $r > 0$. In particular, u_0 is homogeneous with respect to $(\widetilde{y}_0, \widetilde{s}_0) \in \partial\Omega(u_0)$ in $\mathbb{R}^n \times \mathbb{R}^-_{\widetilde{s}_0}$, and from Lemmas 6.2 and 6.3 we obtain that

$$u_0(x, t) = \frac{1}{2}\,((x - \widetilde{y}_0) \cdot e)_+^2 \quad \text{in } \mathbb{R}^n \times \mathbb{R}^-_{\widetilde{s}_0}, \tag{7.8}$$

for a spatial unit vector e.

We want to show now that (7.6) and (7.8) contradict each other, unless $m = 0$. Indeed, if $\widetilde{s}_0 > -1$, $Q_1^- \cap (\mathbb{R}^n \times \mathbb{R}^-_{\widetilde{s}_0})$ is nonempty and the contradiction is immediate. Next, if $\widetilde{s}_0 = -1$ and $B_1 \cap \{(x - \widetilde{y}_0) \cdot e > 0\} =: E$ is nonempty, we obtain that $\partial_t u_0$ is discontinuous on $E \times \{-1\} \subset \Omega(u_0)$, which is not possible since u_0 is caloric in $\Omega(u_0)$. Hence, the remaining case is when $\widetilde{s}_0 = -1$ and u_0 vanish on $B_1 \times \{-1\}$, which implies that $u_0(x, t) = m(t + 1)$ in Q_1^-. Now, using that $u_0(x, t)$ is analytic in variable x in $\Omega(u_0)$, we obtain that the whole strip $\mathbb{R}^n \times (0, 1)$ is contained in $\Omega(u_0)$ and that $u_0(x, t) = m(t + 1)$ there. This is a contradiction, since u_0 must be continuous across $\{(x - \widetilde{y}_0) \cdot e > 0\} \times \{-1\}$.

This shows that $\limsup \partial_t u(x, t) = 0$ as $(x, t) \to (x_0, t_0)$. Similarly one can prove that $\liminf \partial_t v(x, t) = 0$, which will conclude the proof of the lemma. □

Proof of Theorem 7.5.
Step 1. Without loss of generality we may assume $(x_0, t_0) = (0, 0)$. Consider then the rescaled functions u_r, which converge to a homogeneous global solution $u_0 \in \mathcal{P}^-_\infty(M)$ over a subsequence $r = r_j \to 0$. Since $(0, 0)$ is a low energy point, $u_0(x, t) = \frac{1}{2}(x \cdot e_0)^2_+$ for some spatial direction e_0. Choose now a spatial direction e such that $e \cdot e_0 > 1/2$. Then

$$\partial_e u_0 - u_0 \geq (e \cdot e_0)(x \cdot e_0) - \frac{1}{2}(x \cdot e_0)^2 \geq 0 \quad \text{in } Q^-_1.$$

Since the functions u_r converge uniformly in $C^{1,\alpha}_x \cap C^{0,\alpha}_t$-norm on Q^-_1, for $r = r_j$ sufficiently small we will have

$$\partial_e u_r - u_r \geq -\varepsilon_0 \quad \text{in } Q^-_1,$$

where ε_0 is the same as in Lemma 7.6. Applying Lemma 7.6 with $h = \partial_e u_r$, we obtain

$$\partial_e u_r - u_r \geq 0 \quad \text{in } Q^-_{1/2}. \tag{7.9}$$

Step 2. Next, we claim that for any $\varepsilon > 0$, and small $r < r(\varepsilon)$

$$u_r = 0 \quad \text{on } \{x \cdot e_0 \leq -\varepsilon\} \cap Q^-_{1/2}. \tag{7.10}$$

Indeed, this follows easily from the nondegeneracy Lemma 5.1.

Now, observe that (7.9) can be written as

$$\partial_e \left(e^{-(x \cdot e)} u_r \right) \geq 0. \tag{7.11}$$

So, integrating this along the direction e and using (7.10), we obtain that $u_r \geq 0$ in $Q^-_{1/2}$, which after scaling back translates into $u \geq 0$ in $Q^-_{r/2}$.

Moreover, (7.11) implies that for any point $x_0 \in \partial\Omega_r(t) \cap B_{1/4}$ and $-1/4 \leq t \leq 0$, the cone $x_0 + \mathcal{C}$, where $\mathcal{C} = \{-se : 0 \leq s \leq 1/4, e \cdot e_0 > 1/2\}$ is contained in $\Omega^c_r(t)$. Hence the time sections $\partial\Omega(t)$ are Lipschitz regular in $B_{r/4}$ for $-r^2/4 \leq t \leq 0$.

Step 3. We have proved now that $u \geq 0$ in $Q^-_{r/2}$. A simple consequence of this is that for some $r_1 < r$, the intersection $\partial\Omega \cap Q^-_{r_1}$ consists of low energy points. Indeed, first observe that there could be no zero energy points in $Q^-_{r_1}$, which follows from nonnegativity of u, see Subsection 7.1. Next, if there are high energy points, they all should be below some t-level, with $t = t_* < 0$. Hence, if we take $r_1 < \min(r/2, \sqrt{-t_*})$, the intersection $\partial\Omega \cap Q^-_{r_1}$ may consists only of low energy points.

Scaling parabolically with $r < r_1$, we see that $\partial\Omega_r \cap Q^-_1$ consists of low energy points of u_r. Applying Lemma 7.7 we obtain an important fact that the time derivative $\partial_t u_r$ vanishes continuously on $\partial\Omega_r \cap Q^-_1$. Consider then a caloric function

$$h = \partial_e u_r + \eta \partial_t u_r$$

in $Q^-_1 \cap \Omega_r$, where $|\eta| < \eta_0$ is a small constant. Observe that $h = 0$ continuously on $\partial\Omega_r \cap Q^-_1$. Since $\partial_t u_r$ are uniformly bounded, arguing as in Step 1 above, we obtain that

$$(\partial_e u_r + \eta \partial_t u_r) - u_r \geq 0 \quad \text{in } Q^-_{1/2}$$

for r sufficiently small. Then, as in Step 2, we obtain the existence of space-time cones at every point on $\partial\Omega$, which implies the joint space-time Lipschitz regularity of $\partial\Omega \cap Q_{r/2}$.

The theorem is proved. □

8. Positive global solutions

Theorem 8.1. *Let $u \in \mathcal{P}^-_\infty(M)$ be a global solution and assume that $u \ge 0$ in $\mathbb{R}^n \times \mathbb{R}^-$. Then $\partial_t u \le 0$ and $\partial_{ee} u \ge 0$ in $\Omega \cap (\mathbb{R}^n \times \mathbb{R}^-)$ for any spatial direction e. In particular, the time sections $\Lambda(t)$ are convex for any $t \le 0$.*

Proof.
Part 1. $\partial_t u \le 0$.

Indeed, assume the contrary, and let

$$m := \sup_{\Omega\cap(\mathbb{R}^n\times\mathbb{R}^-)} \partial_t u > 0.$$

Choose a maximizing sequence $(x_j, t_j) \in \Omega \cap (\mathbb{R}^n \times \mathbb{R}^-)$ for the value m, i.e.

$$\lim_{j\to\infty} \partial_t u(x_j, t_j) = m.$$

Let $d_j = d^-(x_j, t_j) = \sup\{r : Q^-_r(x_j, t_j) \subset \Omega\}$ and consider the scaled functions

$$u_j(x,t) = \frac{1}{d_j^2} u(d_j x + x_j, d_j^2 t + t_j). \tag{8.1}$$

Then functions u_j are uniformly $C^{1,1}_x \cap C^{0,1}_t$ regular in $\mathbb{R}^n \times \mathbb{R}^-$ and we can extract a converging subsequence to a global solution u_0. Since we assume $u \ge 0$, we have $u_0 \ge 0$. Therefore $\Omega(u_0) = \{u_0 > 0\}$. Next, observe that since $Q^-_1 \subset \Omega(u_j)$ by definition, we will have $Q^-_1 \subset \Omega(u_0)$. In particular, $H(u_0) = 1$ in Q^-_1 and the convergence of u_j to u_0 will be at least in $C^2_x \cap C^1_t$ norm on $Q^-_{1/2}$ and more generally on compact subsets of $\Omega(u_0)$. Hence

$$\partial_t u_0(0,0) = \lim_{j\to\infty} \partial_t u_j(0,0) = \lim_{j\to\infty} \partial_t u(x_j, t_j) = m.$$

On the other hand for every $(x,t) \in \Omega(u_0)$

$$\partial_t u_0(x,t) = \lim_{j\to\infty} \partial_t u_j(x,t) = \lim_{j\to\infty} \partial_t u(d_j x + x_j, d_j^2 t + t_j) \le m.$$

Since $\partial_t u_0$ is caloric in Q^-_1, from the maximum principle we immediately obtain that $\partial_t u_0 = m$ everywhere in Q^-_1 and therefore

$$u_0(x,t) = mt + f(x) \tag{8.2}$$

in Q^-_1. Moreover, (8.2) valid in the parabolically connected component of $\Omega(u_0)$ that contains the origin. It is easy to see that this implies the representation (8.2) for every $t \in (-f(x)/m, 0)$ with $x \in B_1$. Indeed, starting at $(x,0)$ and moving down along the vertical line $\{x\} \times \mathbb{R}^-$ as long as $u(x,t) > 0$, we can extend the equality $\partial_t u_0 = m$ (and thus (8.2)) from the point (x,t) to it's small neighborhood by applying the maximum principle.

Thus, the free boundary becomes the graph of a function $t = t(x) := -f(x)/m$. Since, $(x, t(x)) \in \Lambda(u_0)$ we must have $\nabla u_0(x,t) = 0$ at $t = t(x)$. But $\nabla u_0(x,t) = \nabla f(x)$ for $0 > t > -t(x)$ and since ∇u_0 is continuous we obtain

$$\nabla f(x) = 0$$

for every $x \in B_1$. Hence $f(x) = c_0$ is constant in B_1 and $u(x,t) = mt + c_0$ in Q^-_1. Then $H(u_0) = -m$ in Q^-_1, which means $m = -1$. This contradicts to the assumption that $m > 0$ and the first statement of Theorem 8.1 is proved.

Part 2. $\partial_{ee} u \ge 0$ for any spatial unit vector e.

The reasoning is very similar to the Part 1 above, so we will skip some details. Without loss of generality assume $e = e_n = (0, \dots, 0, 1)$. Let

$$-m = \inf_{\Omega \cap (\mathbb{R}^n \times \mathbb{R}^-)} \partial_{nn} u < 0$$

and $(x_j, t_j) \in \Omega \cap (\mathbb{R}^n \times \mathbb{R}^-)$ be the minimizing sequence for $-m$, i.e.

$$\lim_{j\to\infty} \partial_{nn} u(x_j, t_j) = -m.$$

Considering the rescaled functions u_j as in (8.1), extract a converging subsequence to a function $u_0 \geq 0$. As in Part 1, we have $Q_1^- \subset \Omega(u_0)$. From the locally $C_x^2 \cap C_t^1$ convergence of u_j to u_0 in $\Omega(u_0)$, we obtain that

$$\partial_{nn} u_0(0, 0) = -m$$

and

$$\partial_{nn} u_0(x, t) \geq -m$$

in $\Omega(u_0)$. Since $\partial_{nn} u_0$ is caloric in $\Omega(u_0)$, the minimum principle implies that $\partial_{nn} u_0 = -m$ in the parabolically connected component of $\Omega(u_0)$, in particular in Q_1^-. From there we obtain the representations

$$\partial_n u_0(x, t) = f(x', t) - m\, x_n \tag{8.3}$$

and

$$u_0(x, t) = g(x', t) + f(x', t)\, x_n - \frac{m}{2}\, x_n^2 \tag{8.4}$$

in Q_1^- where $x' = (x_1, \dots, x_{n-1})$. Now let chose a point $(x', 0, t) \in Q_1^-$ and start moving in the direction e_n, as long as we stay in $\Omega(u_0)$. By applying the minimum principle for $\partial_{nn} u_0$ while we move, we can prove $\partial_{nn} u_0 = -m$ and both of the representations (8.3) and (8.4) as long as we stay in $\Omega(u_0)$. Observe however, sooner or later we will hit $\Lambda(u_0)$, otherwise, if x_n becomes very large, (8.4) will imply $u_0 < 0$, which is impossible. Let therefore $x_n = \xi(x', t)$ be the first value of x_n for which we hit $\Lambda(u_0)$. Then $\partial_n u_0(x', x_n, t) = 0$ for $x_n = \xi(x', t)$, hence $\xi(x', t) = f(x', t)/m$. Since we also have the condition $u_0(x', x_n, t) = 0$ for $x_n = \xi(x', t)$, the representation (8.4) takes the form

$$u_0(x, t) = -\frac{m}{2}\, (x_n - \xi(x', t))^2$$

which is not possible, since $u_0 \geq 0$. This concludes the proof of the theorem. □

9. Classification of global solutions

In this section we classify global solutions in $\mathbb{R}^n \times \mathbb{R}^-$. First we make some observations, that follow from the previous sections.

For a given $t_0 \leq 0$ consider the set $\Lambda(t_0)$. We claim that for $x_0 \in \partial\Lambda(t_0)$, the corresponding point $(x_0, t_0) \in \partial\Lambda$ cannot be a zero energy point. Indeed, for zero energy point (x_0, t_0) there would exist r such that $u = 0$ in $Q_r^-(x_0, t_0)$ and in particular $B_r(x_0) \subset \Lambda(t_0)$, see Subsection 7.1.

Next, if (x_0, t_0) is a high energy point, necessarily $u(x, t) = m(t - t_0) + P(x - x_0)$ for $t \leq t_0$, $\Lambda(t_0)$ is a k-dimensional plane, $k = 0, \dots, n$ and all points on $\Lambda(t_0)$ are high energy points, see Lemma 7.3.

Hence, if there is a low energy point (x_0, t_0) then all points on $\partial\Lambda(t_0) \times \{t_0\}$ are low energy. Also, according to Theorem 7.5, in that case the boundary $\partial\Lambda(t_0)$ is a locally Lipschitz surface. In particular, the set $\Lambda(t_0)$ is a regular closed set, i.e. the closure of its interior. We can say even more. Theorem 7.5 implies, that for given $R > 0$ there exists

$\delta = \delta(u, R, t_0) > 0$ such that $u \geq 0$ in $U_{\delta,R} := \mathcal{N}_\delta(\Lambda(t_0) \cap B_R) \times (t_0 - \delta^2, t_0)$, where $\mathcal{N}_\delta(E)$ denotes the δ-neighborhood of the set E, and $\partial\Lambda \cap U_{\delta,R}$ is a Lipschitz in space and time surface. Another important fact is that $\partial_t u$ will be continuous up to $\partial\Lambda$ in $B_R \times (t_0 - \delta^2, t_0)$. The latter implies that $(\partial_t u)^\pm$ are subcaloric functions in $B_R \times (t_0 - \delta^2, t_0)$.

The main theorem of this section is as follows.

Theorem 9.1. *Let $u \in \mathcal{P}^-_\infty(M)$ be a global solution and suppose that $(0, 0)$ is a low energy point. Then $u \geq 0$ in $\mathbb{R}^n \times \mathbb{R}^-$.*

Before proving the theorem, it is convenient to introduce the *balanced energy at* ∞ of a global solution $u \in \mathcal{P}^-_\infty(M)$. We define it as

$$\omega_\infty = \lim_{r\to\infty} W(r; u).$$

In analogy with the construction from Section 7, consider the rescaled functions u_r and let $r \to \infty$. Then, over a sequence $r = r_k \to \infty$, u_r will converge to a function u_∞ in $\mathbb{R}^n \times \mathbb{R}^-$, which will be a solution of (1.2). Moreover, u_∞ will be a homogeneous solution and

$$\omega_\infty = W(u_\infty).$$

Thus ω_∞ can take only values 0, A, and $2A$ by Lemmas 6.3 and 6.2. Respectively we say that ∞ is a zero, low, and high energy point.

Lemma 9.2. *For $u \in \mathcal{P}_\infty(M)$*

(i) $\omega_\infty = 0$ *implies that $u = 0$ in $\mathbb{R}^n \times \mathbb{R}^-$;*

(ii) *if both $\omega = \omega_\infty = A$, then u is a stationary half-space solution of the form $\frac{1}{2}(x \cdot e)^2_+$.*

Proof. In both cases we obtain that $W(r; u)$ is constant, since $\omega \leq W(r; u) \leq \omega_\infty$. Hence u is a homogeneous solution with energy 0 or A and the lemma follows. □

When $u \in \mathcal{P}_\infty(M)$ and $\omega_\infty = 2A$, then the shrink-down u_∞ is a polynomial solution

$$u_\infty(x, t) = mt + P_\infty(x),$$

where $P_\infty(x)$ is a homogeneous quadratic polynomial. The next lemma shows what information we can extract, if $P_\infty(x)$ is degenerate in some direction.

Lemma 9.3. *Suppose $P_\infty(x)$ is degenerate in the direction e, i.e. $\partial_{ee} P_\infty = 0$. Then $\partial_e u$ has a sign in $\mathbb{R}^n \times \mathbb{R}^-$, i.e. $\partial_e u \geq 0$ or $\partial_e u \leq 0$ everywhere.*

Proof. Let ψ be a cut-off function with $\psi = 1$ on $B_{1/2}$, $\operatorname{supp} \psi \subset B_{3/4}$ and let $\psi_r(x) = \psi(x/r)$. Consider the scaled functions u_r in Q^-_1. Then $(\partial_e u_r)^\pm$ are subcaloric and vanish at $(0, 0)$. Hence we can apply [Caf93] monotonicity formula (Theorem 3.3) to obtain the estimate

$$\Phi(t; (\partial_e u_r)\psi) \leq C_0 \|(\partial_e u_r)\|^4_{L^2(Q^-_1)},$$

for any $0 < t < \tau_0$, where τ_0, C_0 do not depend on r. Observe now, that over a subsequence of $r = r_k \to \infty$ for which $u_r \to u_\infty$, $\partial_e u_r$ converges uniformly to $\partial_e u_\infty = 0$ in Q^-_1. Hence

$$\Phi(t; (\partial_e u_r)\psi) \to 0 \quad \text{as } r = r_k \to \infty$$

uniformly for $0 < t < \tau_0$. Scaling back to the function u, using that $\partial_e u_r(x, t) = (1/r)\partial_e u(rx, r^2 t)$, we obtain

$$\Phi(t; (\partial_e u)\psi_r) = \Phi(t/r^2; (\partial_e u_r)\psi) \to 0 \quad \text{as } r = r_k \to \infty$$

uniformly for $0 < t < \tau_0 r^2$. Therefore for any fixed $t > 0$,

$$\left(\frac{1}{t}\int_{-t}^{0}\int_{B_{r/2}} |\nabla(\partial_e u)^+|^2 G(x,-s)\,dx\,ds\right)\left(\frac{1}{t}\int_{-t}^{0}\int_{B_{r/2}} |\nabla(\partial_e u)^-|^2 G(x,-s)\,dx\,ds\right) \to 0$$

as $r = r_k \to \infty$, since $\psi_r = 1$ on $B_{r/2}$. Passing to the limit we obtain

$$\Phi(t;\partial_e u) = 0$$

for any $t > 0$. This is possible if and only if one of the functions $(\partial_e u)^\pm$ vanishes in $\mathbb{R}^n \times \mathbb{R}^-$.

The lemma is proved. □

We will need also the following modification of Lemma 7.7.

Lemma 9.4. *Let $u \in \mathcal{P}_\infty^-(M)$ be a global solution. Suppose that $(x_0, t_0) \in \partial\Omega$ and $\overline{Q}_\varepsilon(x_0, t_0) \cap \partial\Omega$ contains no high energy points for some $\varepsilon > 0$. Then*

$$m := \limsup_{(x,t)\to(x_0,t_0)} \partial_t u \le 0.$$

Proof. The proof will follow the lines of the proof of Lemma 7.7.

As there, consider the continuous functions

$$\omega_r(x,t) = W(r,x,t;u)$$

for small $r > 0$ and $(x,t) \in \overline{Q}_\varepsilon(x_0,t_0) \cap \partial\Omega$. Since there are no high energy points in a small neighborhood of (x_0, t_0), we have

$$\lim_{r\searrow 0} \omega_r(x,t) \le A.$$

Therefore, setting

$$\widetilde{\omega} = \max(\omega_r, A)$$

we obtain

$$\widetilde{\omega}_r(x,t) \searrow A \quad \text{as } r \searrow A$$

on $\overline{Q}_\varepsilon(x_0,t_0) \cap \partial\Omega$. Then, by Dini's theorem, the convergence $\widetilde{\omega}_r(x,t) \searrow A$ is uniform on $\overline{Q}_\varepsilon(x_0,t_0) \cap \partial\Omega$. Therefore, if $(y_j, s_j) \to (x_0, t_0)$ and $r_j \to 0$ are any sequences, we have

$$\lim_{j\to\infty} \widetilde{\omega}_{r_j}(y_j, s_j) = A.$$

This implies that

$$\limsup_{j\to\infty} W(r_j, y_j, s_j; u) \le A.$$

Now, having this, take a maximizing sequence $(x_j, t_j) \to (x_0, t_0)$ such that

$$\lim_{j\to\infty} \partial_t u(x_j, t_j) = m.$$

Assume $m > 0$. Then scaling u around (x_j, t_j) by the parabolic distance d_j, precisely as in in the proof of Lemma 7.7, we can extract a converging subsequence to a global solution u_0. Then, again, we can prove the identity (7.6). However, instead of equality (7.7) we will have inequality

$$W(r, \widetilde{y}_0, \widetilde{s}_0; u_0) = \lim_{j\to\infty} W(r, \widetilde{y}_j, \widetilde{s}_j; u_j) = \lim_{j\to\infty} W(r_j, y_j, s_j; u) \le A$$

for any $r > 0$. Applying Lemma 9.2, we see that either

$$u_0(x,t) = \frac{1}{2}((x - \widetilde{y}_0) \cdot e)_+^2 \quad \text{for } t \le \widetilde{s}_0$$

or

$$u_0(x,t) = 0 \quad \text{for } t \le \widetilde{s}_0.$$

In the first case, we finish the proof as in Lemma 7.7. In the second case, we have necessarily $\widetilde{s}_0 = -1$, since $H(u_0) = 1$ in Q_1^- and we obtain the representation

$$u_0(x,t) = m(t+1) \quad \text{in } Q_1^-.$$

Hence $H(u_0) = -m < 0$, while we must have $H(u_0) = 1$.

This concludes the proof of the lemma. □

Remark 9.5. Under the conditions of Lemma 9.4, the limit

$$\liminf_{(x,t)\to(x_0,t_0)} \partial_t u$$

can have only two possible values: 0 or -1. The same proof as above applies.

Proof of Theorem 9.1. First, we note that u cannot have zero balanced energy at ∞, and if it has low energy at ∞, the theorem readily follows from Lemma 9.2. So we need to consider only the case when $\omega_\infty = 2A$. Then let $u_\infty(x,t) = mt + P_\infty(x)$ be as above a shrink-down limit of rescaled functions u_r, as $r = r_k \to \infty$.

Step 1: Dimension reduction. Suppose that there exists a shrink-down $u_\infty(x,t)$ for which the homogeneous quadratic polynomial $P_\infty(x)$ is degenerate in the direction e. Then by Lemma 9.3, we may assume $\partial_e u \ge 0$ (otherwise we will have $\partial_{-e} u \ge 0$ and will just change the direction of e.) Also, without loss of generality, let $e = e_n = (0,\ldots,0,1)$. Since we assume that $(0,0)$ is a low energy point, from Theorem 7.5 it follows that $\partial\Lambda(0)$ is a Lipschitz surface in $\mathbb{R}^n$ and hence the interior of $\Lambda(0)$ is nonempty. Let $B_\delta(x_0) \subset \Lambda(0)$, $x_0 = (x_0', a)$. We claim now that

$$u(x,0) = 0 \quad \text{for } x = (x', x_n) \in B_\delta'(x_0') \times (-\infty, a). \tag{9.1}$$

Indeed, for $x' \in B_\delta'(x_0')$ define

$$b(x') = \inf\{b : u(x', s, 0) = 0 \text{ for all } s \in [b,a]\}.$$

Then obviously $b(x') \le a$ and $\xi := (x', b(x')) \in \partial\Lambda(0)$, if $b(x')$ is finite. Moreover, $(\xi, 0)$ can be only a low energy point, see the discussion preceding Theorem 9.1. Then, by Theorem 7.5, there is $r > 0$ such that $u \ge 0$ in $Q_r^-(\xi, 0)$, in particular $u(x,0) \ge 0$ in $B_r(\xi)$. On the other hand, since $u(\xi,0) = 0$ and $\partial_n u \ge 0$, $u(x', s, 0) \le 0$ for $s \in (b(x') - r, b(x'))$. Hence $u(x', s, 0) = 0$ for $s \in (b(x') - r, b(x'))$ and we arrive at the contradiction, if $b(x')$ is finite. Thus, (9.1) follows.

Now, for every $\tau \ge 0$ define the shifts of u in the direction e_n

$$v_\tau(x', x_n, t) = u(x', x_n - \tau, t).$$

Since $\partial_n u \ge 0$, the functions v_τ decrease as $\tau \to \infty$ and therefore there exist the limit

$$v = \lim_{\tau\to\infty} v_\tau.$$

Moreover, as it follows from (9.1), $v_\tau(x_0, 0) = 0$ for all $\tau \ge 0$ and we have the uniform estimate

$$|v_\tau(x,t)| \le C(M)(|x - x_0|^2 + |t|)$$

in $\mathbb{R}^n \times \mathbb{R}^-$. Hence v is finite everywhere in $\mathbb{R}^n \times \mathbb{R}^-$ and thus a solution of (1.2). Moreover, clearly, v is independent of the direction e_n. So we may think of $v = v(x', t)$ as a solution of (1.2) in $\mathbb{R}^{n-1} \times \mathbb{R}^-$. Observe also that

$$u(x', x_n, t) \geq v(x', t),$$

so if we prove that $v \geq 0$ in $\mathbb{R}^{n-1} \times \mathbb{R}^-$ we will be done.

Now, consider several cases. Suppose that for every $\varepsilon > 0$, v has a low energy point (x', t) with $-\varepsilon^2 \leq t \leq 0$. Taking such a point as the origin we arrive at the conditions of the theorem, but with the reduced dimension. So if the theorem is true for the dimension $n-1$, we conclude that $v \geq 0$ for $t \leq -\varepsilon^2$ and letting $\varepsilon \to 0$ we complete the proof.

Next case would be that there are no low energy points of v in $\mathbb{R}^{n-1} \times (-\varepsilon^2, 0]$ for some $\varepsilon > 0$. Observe that (9.1) implies $B'_\delta(x'_0) \subset \Lambda_v(0)$. Suppose for a moment that $(0, 0) \in \Gamma(v)$. Then it is a high energy point and v is a polynomial solution. Since $\Lambda_v(0)$ has nonempty interior, it will be possible only if $v(x, t) = -t > 0$ and we will be done. Thus, we may assume that for some small $0 < \eta < \varepsilon$, $Q'^-_\eta \subset \Lambda_v$. Since there are no low energy points for $-\eta^2 < t \leq 0$, $\partial\Lambda_v(t)$ must be empty, thus implying that $\Lambda_v(t) = \mathbb{R}^{n-1}$ for $-\eta^2 < t \leq 0$. The latter means that $v = 0$ in $\mathbb{R}^{n-1} \times (-\eta^2, 0]$.

Next, let $\eta = \eta_*$ be maximal (possibly infinite) with the property that $v = 0$ on $\mathbb{R}^{n-1} \times (-\eta^2, 0]$. If $\eta_* = \infty$, we will have that $v = 0$ identically. If η_* is finite, then the arguments above show that $v(x', t) = (\eta_*^2 - t)_+$ and we are done.

Thus, in all possible cases $v \geq 0$, which implies $u \geq 0$.

To complete the dimension reduction, we note that for $n = 0$ the statement of the theorem is trivial. Indeed, $\mathbb{R}^0 = \{0\}$, $u(0, 0) = 0$ and $H(u) = -\partial_t u \geq 0$ imply that $u(0, t) \geq 0$ for $t \leq 0$.

Step 2: Nondegenerate P_∞. The reasonings above allow us to reduce the problem to the case when $P_\infty(x)$ is nondegenerate for every shrink-down u_∞ over every sequence $r = r_k \to \infty$.

Lemma 9.6. *Suppose u has no high energy points in $\mathbb{R}^n \times \mathbb{R}^-$ and for every shrink-down $u_\infty(x, t) = mt + P_\infty(x)$ the polynomial $P_\infty(x)$ is nondegenerate. Then there exist a shrink-down with $m = 0$.*

Proof. First, suppose that $\Lambda \cap (\mathbb{R}^n \times \mathbb{R}^-)$ is bounded. Let (x_0, t_0) be a point with minimal t-coordinate. Then, obviously, (x_0, t_0) is a high energy point, contradicting the assumption. Therefore, there exists a sequence $(x_k, t_k) \in \Lambda$ such that

$$r_j := \max(|x_k|, \sqrt{-t_k}) \to \infty.$$

Consider then the scale functions u_{r_k}. Then

$$(x_k/r_k, t_k/r^2) \in \partial_p Q_1^- \cap \Lambda(u_{r_k}).$$

Hence if u_∞ is a shrink-down over a subsequence of $r = r_k \to \infty$, we will have

$$\partial_p Q_1^- \cap \Lambda(u_\infty) \neq \emptyset.$$

Since $P_\infty(x)$ is nondegenerate, this may happen only if $m = 0$. □

Lemma 9.6 has a consequence that only the following three cases are possible if for every shrink-down $u_\infty(x, t) = mt + P_\infty(x)$ the polynomial P_∞ is nondegenerate:

1. $m > 0$ and there exist a high energy point $(x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}^-$;
2. $m \leq 0$ and there exist a high energy point $(x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}^-$;

3. $m = 0$ for a shrink-down over a sequence $r = r_k \to \infty$ and there are no high energy points in $\mathbb{R}^n \times \mathbb{R}^-$.

We will threat these tree cases separately.

Case 1. $m > 0$ and there exists a high energy point $(x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}^-$.

Then $u(x,t) = c(t - t_0) + P(x - x_0)$ for $t \le t_0$ and $t_0 < 0$. Moreover, considering the shrink-down, we realize that $c = m$ and $P = P_\infty$. Hence

$$u(x,t) = m(t - t_0) + P_\infty(x - x_0) \quad \text{for } t \le t_0.$$

Next, since P_∞ is nondegenerate and $\Delta P_\infty = 1 + m > 0$, it follows that P_∞ is positive definite. Then $u(x, t_0) = P_\infty(x - x_0) \ge 0$. Consider now the function $w(x,t) = m(t - t_0) + P_\infty(x - x_0)$ in $\mathbb{R}^n \times (t_0, 0)$. We have $H(w) = 1$. On the other hand u satisfies $H(u) \le 1$ and $u(\cdot, t_0) = w(\cdot, t_0)$. Hence from the comparison principle

$$u(x,t) \ge w(x,t) = m(t - t_0) + P_\infty(x - x_0) > 0 \quad \text{in } \mathbb{R}^n \times (t_0, 0).$$

In particular (0, 0) can't be a free boundary point and we arrive at a contradiction. Therefore this case is not possible.

Before we proceed to consider the two remaining cases, we prove the following lemma.

Lemma 9.7. *Suppose in representation $u_\infty(x,t) = mt + P_\infty(x)$, the polynomial $P_\infty(x)$ is nondegenerate and $m \le 0$. Then $\partial_t u \le 0$ in $\mathbb{R}^n \times \mathbb{R}^-$.*

Proof. We subdivide the proof into two cases.

(i) There are no high energy points of u in $\mathbb{R}^n \times \mathbb{R}^-$.

Then Lemma 9.4 implies that $(\partial_t u)_+$ is continuous and therefore subcaloric in $\mathbb{R}^n \times \mathbb{R}^-$. Consider then the scaled functions $u_r \to u_\infty$ in Q_2^-. Since $\Lambda(u_\infty) = \{0\} \times \mathbb{R}^-$, the convergence will be at least $C_x^2 \cap C_t^1$ in $Q_1^- \setminus (B_\varepsilon \times [-1, 0])$ for any $\varepsilon > 0$. In particular, for $r = r_k$ very large,

$$\begin{aligned} &(\partial_t u_r)_+ \le \varepsilon && \text{on } \partial_p Q_1 \setminus (B_\varepsilon \times \{-1\}) \\ &(\partial_t u_r)_+ \le C(M) && \text{on } B_\varepsilon \times \{-1\}. \end{aligned}$$

Hence if v_ε is the solution of the Dirichlet problem for the heat equation with boundary data

$$\begin{aligned} &v = \varepsilon && \text{on } \partial_p Q_1 \setminus (B_\varepsilon \times \{-1\}) \\ &v = C(M) && \text{on } B_\varepsilon \times \{-1\}, \end{aligned}$$

we will have

$$(\partial_t u_r)_+ \le v_\varepsilon \quad \text{in } Q_1^-.$$

It is not hard to see that $v_\varepsilon \le c(\varepsilon) \to 0$ as $\varepsilon \to 0$ uniformly in $Q_{1/2}^-$, hence

$$(\partial_t u_r)_+ \le c(\varepsilon) \quad \text{in } Q_{1/2}^-.$$

Scaling back to u, we obtain

$$(\partial_t u)_+ \le c(\varepsilon) \quad \text{in } Q_{r/2}^-.$$

Letting $r = r_k \to \infty$ and then $\varepsilon \to 0$, we obtain the claim of the lemma.

(ii) There is a high energy point $(x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}^-$. Observe that x_0 is unique, since P_∞ is nondegenerate. Also t_0 is unique, unless $m = 0$. If the latter is the case, we will assume t_0 is the maximal value of t for which (x_0, t) is a high energy point.

If $t_0 = 0$, we are done. If $t_0 < 0$, Lemma 9.4 implies that $(\partial_t u)_+$ is continuous and thus subcaloric in $\mathbb{R}^n \times (t_0, 0)$. We want to show that it in fact vanishes there.

Considering as above the scaled functions u_r and there convergence to u_∞, as well as that $\partial_t u_\infty = m \le 0$, scaling back to u we obtain that

$$(\partial_t u)_+ \le \varepsilon \quad \text{on } \partial B_r \times (t_0, 0).$$

Moreover, since every point (x, t_0) is in Ω except (x_0, t_0), we will also have

$$\begin{aligned} &(\partial_t u)_+ = 0 && \text{on } (B_r \setminus B_\varepsilon(x_0)) \times \{t_0\} \\ &(\partial_t u)_+ \le C(M) && \text{on } (B_\varepsilon(x_0)) \times \{t_0\}. \end{aligned}$$

Hence if $w_{\varepsilon,r}$ is a solution to the Dirichlet problem for the heat equation in $B_r \times (t_0, 0)$ with the boundary values

$$\begin{aligned} &w = \varepsilon && \text{on } \partial_p B_r \times (t_0, 0) \setminus (B_\varepsilon(x_0) \times \{t_0\}) \\ &w = C(M) && \text{on } B_\varepsilon(x_0) \times \{t_0\} \end{aligned}$$

we will have

$$(\partial_t u)_+ \le w_{\varepsilon,r} \quad \text{in } B_r \times (t_0, 0).$$

It is easy to see that as $r \to \infty$ and $\varepsilon \to 0$, $w_{\varepsilon,r} \to 0$ uniformly on compact subsets of $\mathbb{R}^n \times (t_0, 0)$. Hence $\partial_t u \le 0$ in $\mathbb{R}^n \times (t_0, 0)$ as well and the proof is complete. □

Case 2. $m \le 0$ and there exists a high energy point $(x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}^-$.

Unless $m = 0$, (x_0, t_0) is unique. If $m = 0$, assume that t_0 is the maximal t such that there exists a high energy point at time $t = t_0$.

Then as in Case 1 above we obtain the representation

$$u(x, t) = m(t - t_0) + P_\infty(x - x_0) \quad \text{for } t \le t_0.$$

Next, since $m \le 0$, Lemma 9.7 implies that $\partial_t u \le 0$ in $\mathbb{R}^n \times \mathbb{R}^-$.

We claim now that $P_\infty(x)$ is positive definite. Since P_∞ is nondegenerate, the other possibility is that $P_\infty(x) \le 0$ everywhere, in particular $u(x, t_0) \le 0$. Then $\partial_t u \le 0$ implies $u \le 0$ in $\mathbb{R}^n \times (t_0, 0)$. Since u is subcaloric and $u(0, 0) = 0$, by the maximum principle $u = 0$ in $\mathbb{R}^n \times (t_0, 0)$. This is possible only if $P_\infty = 0$, which contradicts the assumption that P_∞ is nondegenerate. Hence P_∞ is positive definite.

Let now $c > 0$ be small enough such that

$$P_\infty(x) \ge c|x|^2.$$

Let also

$$\partial_t u \ge -C = -C(M).$$

Then

$$u(x, t) \ge c|x - x_0|^2 - C(t - t_0)$$

in $\mathbb{R}^n \times (t_0, 0)$. Consequently,

$$u(x, t) > 0 \quad \text{for } x \in \mathbb{R}^n \setminus B_{\kappa(t)}(x_0) \text{ and } t_0 < t < 0, \tag{9.2}$$

where $\kappa(t) = \sqrt{C(t - t_0)/c}$.

Consider now the set $\Omega_- = \{u < 0\} \subset \mathbb{R}^n \times (t_0, 0)$ and suppose it is nonempty. Then

$$\Omega_-(t) \subset B_{\kappa(t)}$$

for $t_0 < t < 0$, by (9.2). In particular, Ω_- is bounded. Hence there exists a point $(x_1, t_1) \in \overline{\Omega_-}$ with minimal t-coordinate. Then $t_1 \ge t_0$ and $u(x, t_1) \ge 0$ for all x. Since also $u(x_1, t_1) = 0$, we obtain that $\nabla u(x_1, t_1) = 0$. Hence

$$(x_1, t_1) \in \Lambda \cap \overline{\Omega_-}.$$

We show below, that this is impossible.

Indeed, consider now the sets $\Lambda(t)$. Then, again, (9.2) implies

$$\Lambda(t) \subset B_{\kappa(t)}$$

for $t_0 < t < 0$, so the sets $\Lambda(t)$ are bounded. Also, since there are no high energy points for $t_0 < t < 0$, $\partial\Lambda(t)$ are Lipschitz surfaces and the interiors of $\Lambda(t)$ are nonempty, provided $\Lambda(t)$ are nonempty. Let now $U(0)$ be a connected component of the interior of $\Lambda(0)$. Then Theorem 7.5 implies that there exist an open set W such that $U(0) \subset\subset W$ and $u \geq 0$ in $W \times (-\delta, 0)$ for a small $\delta > 0$. Then $\partial_t u \leq 0$ implies $u \geq 0$ in $W \times \mathbb{R}^-$. Moreover $\partial_t u \leq 0$ implies also that

$$U(t) := W \cap \operatorname{Int}(\Lambda(t)) \searrow \text{ as } t \searrow .$$

Since $U(t_0) = \emptyset$, there exist $t_* \in [t_0, 0)$ such that $U(t) = \emptyset$ for $t < t_*$ and $U(t) \neq \emptyset$ for $t_* < t < 0$. Then the intersection

$$K_* = \bigcap_{t_* < t < 0} \overline{U(t)},$$

is nonempty. Choose now any $x_* \in K_*$. Then obviously $(x_*, t_*) \in \partial\Lambda$, and we claim that (x_*, t_*) is a high energy point. Clearly, it is not a zero energy point. Also, it's not a low energy by Theorem 7.5. Hence (x_*, t_*) is a high energy point. Since P_∞ is nondegenerate, necessarily $x_* = x_0$. (We also have $t_* = t_0$.) In particular,

$$x_0 \in U(0).$$

This implies immediately that $U(0)$ is the only connected component of the interior of $\Lambda(0)$. Starting at any time $t \in (t_0, 0)$, we can prove a similar statement for $\Lambda(t)$. Thus

$$\Lambda(t) = \overline{U(t)}.$$

In particular, $u \geq 0$ in a neighborhood $W \times (t_0, 0)$ of $\Lambda \cap (\mathbb{R}^n \times (t_0, 0))$. But we constructed $(x_1, t_1) \in \overline{\Omega_-} \cap \Lambda$, which is a contradiction. Thus Ω_- is empty, implying that $u \geq 0$.

Case 3. $m = 0$ and there are no high energy points in $\mathbb{R}^n \times \mathbb{R}^-$.

Then Lemma 9.7 implies that $\partial_t u \leq 0$ in $\mathbb{R}^n \times \mathbb{R}^-$.

We start with the claim that $\Lambda(t)$ are bounded sets for $-t$ sufficiently large. More specifically, we claim

$$\Lambda(t) \subset B_{\sqrt{-t}} \tag{9.3}$$

for $t \leq t_0$. Indeed, assume the contrary. Then there is a sequence $t = t_k \to -\infty$ such that (9.3) does not hold, and therefore we can find $x_k \in \Lambda(t_k)$ with

$$|x_k| \geq \sqrt{-t_k}.$$

Let now $r = r_k = |x_k|$ and consider the scaled functions u_r. Then

$$(\widetilde{x}_k, \widetilde{t}_k) := (x_k/r_k, t_k/r_k^2) \in \Lambda(u_r) \cap (\partial B_1 \times [-1, 0]).$$

Hence passing to the limit over a subsequence of $r = r_k \to \infty$, we obtain that $\Lambda(u_\infty) \cap (\partial B_1 \times [-1, 0])$ is nonempty. However, this is impossible if P_∞ is nondegenerate. Hence (9.3) should hold for $t \leq t_0$.

Next, suppose $\Lambda(t)$ is empty for all $t \leq t_0$. Then u will satisfy $H(u) = 1$ everywhere in $\mathbb{R}^n \times \mathbb{R}^-_{t_0}$ and thus will have the form $u(x, t) = ct + P(x)$ for $t \leq t_0$. Considering the shrink-down (recall we assume $m = 0$), we find that

$$u(x, t) = P_\infty(x - x_1)$$

for $t \leq t_0$, where $x_1 \in \mathbb{R}^n$ is some point. But then (x_1, t_0) is a high energy point, and we assume there are none.

Hence, without loss of generality we may assume that $\Lambda(t_0)$ is nonempty. Since there are no high energy points for $t \le 0$, the sets $\Lambda(t)$ have nonempty interiors and $\partial\Lambda(t)$ are Lipschitz surfaces, provided $\Lambda(t)$ themselves are nonempty.

Let $U(t_0)$ be a connected component of the interior of $\Lambda(t_0)$. Then we make a construction similar to the one in Case 2 above. There exists an open set $W \subset \mathbb{R}^n$ such that $U(t_0) \subset\subset W$ and $u \ge 0$ in $W \times (t_0 - \delta, t_0)$ for a small $\delta > 0$. Then from $\partial_t u \le 0$ we obtain that in fact $u \ge 0$ in $\mathbb{R}^n \times \mathbb{R}^-_{t_0}$. Moreover $u > 0$ in $W \setminus \overline{U(t_0)} \times \mathbb{R}^-_{t_0}$. Also, $\partial_t u \le 0$ implies that

$$U(t) := W \cap \operatorname{Int}(\Lambda(t)) \searrow \text{ as } t \searrow .$$

Consider then the intersection

$$K = \bigcap_{t \le t_0} \overline{U(t)}.$$

Since the sets $\overline{U(t)}$ are compact, K is empty if and only if $\overline{U(t)} = W \cap \Lambda(t)$ is empty for some $t \le t_0$. If this is so, let t_* be such that $W \cap \Lambda(t) = \emptyset$ for $t < t_*$ and $W \cap \Lambda(t_*)$ is nonempty. Take any $x_* \in \Lambda(t_*)$. Since $W \cap \Lambda(t) = \emptyset$ for $t < t_*$, $u > 0$ in $W \times (-\infty, t_*)$ and in particular (x_*, t_*) is a high energy point, and we assume there are none.

Thus K is nonempty and we can choose $x_0 \in K$. Then $(x_0, t) \in \Lambda$ for all $t \le t_0$ and we obtain the estimate

$$u(x,t) \le C(M)(|x - x_0|^2)$$

for $x \in \mathbb{R}^n$ and $t \le t_0$. Consider now the time shifts

$$v_\tau(x,t) = u(x, t - \tau)$$

defined in $\mathbb{R}^n \times \mathbb{R}^-_{t_0}$. Then from the estimate above and the monotonicity of $u(x,t)$ in t the limit

$$v_\infty(x,t) = \lim_{\tau\to\infty} v_\tau(x,t)$$

exists and is finite everywhere in $\mathbb{R}^n \times \mathbb{R}^-_{t_0}$. Thus v_∞ is also a solution of (1.2). Moreover, it is easy to see that v_∞ is independent of t, so $v_\infty = v_\infty(x)$ is a stationary global solution of (1.2).

As it follows from [CKS00], the stationary global solutions solutions are either polynomial, or nonnegative. Observe now that $x_0 \in \Lambda(v_\infty)$ and $v \ge 0$ in the neighborhood W of x_0. Hence if v_∞ is a polynomial solution, the polynomial must be positive semidefinite. Therefore in any case we have $v_\infty \ge 0$.

Now, for the positive global solutions it is known that the set $\Lambda(v_\infty)$ is convex, hence connected. Since $x_0 \in \Lambda(v_\infty)$, $x_0 \in \overline{U(t_0)} \subset W$ and $v_\infty > 0$ in $W \setminus \overline{U(t_0)}$, the only possibility is that

$$\Lambda(v_\infty) \subset \overline{U(t_0)}.$$

A simple consequences of this is that the interior of $\Lambda(t_0)$ consists only of one connected component. Now, if we made our construction starting at any $t \le t_0$, we would come to the conclusion that the interior of $\Lambda(t)$ has at most one component. Hence

$$\Lambda(t) = \overline{U(t)}.$$

Also, we obtain that $u \ge 0$ in the neighborhood $W \times \mathbb{R}^-_{t_0}$ of $\Lambda \cap (\mathbb{R}^n \times \mathbb{R}^-_{t_0})$. Then all the free boundary points in $\mathbb{R}^n \times \mathbb{R}^-_{t_0}$ are low energy and Lemma 7.7 implies that $\partial_t u = 0$ continuously on $\partial\Lambda \cap (\mathbb{R}^n \times \mathbb{R}^-_{t_0})$ and therefore $\partial_t u$ is supercaloric in $\mathbb{R}^n \times \mathbb{R}^-_{t_0}$. In fact, we claim that

$$\partial_t u = 0 \quad \text{in } \mathbb{R}^n \times \mathbb{R}^-_{t_0}. \tag{9.4}$$

Consider the scaled functions $u_r \to u_\infty$ in Q_2^-. Since $\Lambda(u_\infty) = \{0\}\times\mathbb{R}^-$, the convergence will be at least $C_x^2 \cap C_t^1$ in $Q_1^- \setminus (B_\varepsilon \times [-1, 0])$ for any $\varepsilon > 0$. In particular, for $r = r_k$ very large,

$$\begin{aligned} -\varepsilon \le \partial_t u_r \le 0 \quad &\text{on } \partial_p Q_1 \setminus (B_\varepsilon \times \{-1\}) \\ -C(M) \le \partial_t u_r \le 0 \quad &\text{on } B_\varepsilon \times \{-1\}. \end{aligned}$$

Moreover, $\partial_t u_r$ is supercaloric in $B_1^- \times (-1, t_0/r^2)$, so if v_ε is the solution of the Dirichlet problem for the heat equation with boundary data

$$\begin{aligned} v = \varepsilon \quad &\text{on } \partial_p Q_1 \setminus (B_\varepsilon \times \{-1\}) \\ v = C(M) \quad &\text{on } B_\varepsilon \times \{-1\} \end{aligned}$$

we will have

$$-v_\varepsilon \le \partial_t u_r \le 0 \quad \text{in } B_1 \times (-1, t_0/r^2).$$

It is not hard to see that $v_\varepsilon \le c(\varepsilon) \to 0$ as $\varepsilon \to 0$ uniformly in $Q_{1/2}^-$, hence

$$-c(\varepsilon) \le \partial_t u_r \le 0 \quad \text{in } B_{1/2} \times (-1/4, t_0/r^2).$$

Scaling back to u, we obtain

$$-c(\varepsilon) \le \partial_t u \le 0 \quad \text{in } B_{r/2} \times (r/4, t_0).$$

Letting $r = r_k \to \infty$ and then $\varepsilon \to 0$, we obtain (9.4). Thus,

$$u(x, t) = v_\infty(x) \quad \text{for any } x \in \mathbb{R}^n \text{ and } t \le t_0. \tag{9.5}$$

It remains to prove that $u \ge 0$ in $\mathbb{R}^n \times (t_0, 0)$, since we know that $v_\infty \ge 0$. In fact, we claim

$$v_\infty(x) \ge c|x|^2 \tag{9.6}$$

for some fixed $c > 0$ small and $|x| > R$ large. If this fails, we could easily construct a shrink-down of v_∞ (which is always a polynomial) that vanishes at a point on ∂B_1. Hence this polynomial is degenerate. But from (9.5) we see that any shrink-down of v_∞ corresponds to the one of u, for which we assume that P_∞ is nondegenerate, a contradiction. Hence the estimate (9.6) holds. Consequently, $u(x, t) > 0$ for $|x| > R_1$ and $t \in [t_0, 0]$. Hence $\Lambda(t)$ is bounded for all $t \le 0$. Also $\Lambda(0)$ is nonempty. So we could take $t_0 = 0$ in all the arguments above. Thus, u is stationary and

$$u(x, t) = v_\infty(x) \ge 0 \quad \text{for any } x \in \mathbb{R}^n \text{ and } t \in \mathbb{R}^-.$$

The theorem is proved. □

10. Proof of Theorem I

For a global solution u let us define

$$\begin{aligned} T_1 &= \sup\{t : (x, t) \in \partial\Omega \text{ is a high energy point}\} \\ T_* &= \sup\{t : (x, t) \in \partial\Omega \text{ is a low or high energy point}\} \\ T_2 &= \sup\{t : (x, t) \in \partial\Omega\}. \end{aligned}$$

Then

$$-\infty \le T_1 \le T_* \le T_2 \le a.$$

We claim that Theorem I holds with the values of T_1 and T_2 defined above. The parts (i) and (iii) of the theorem are easily verified, so we need to show only that (ii) holds.

Thus, if $T_1 = T_2$ we are done.

Suppose now, that $T_1 < T_2$. If it happens that $T_* = T_2$, then we will find a sequence (x_k, t_k) of low energy points with $t_k \nearrow T_2$ and applying Theorem 9.1 we will obtain that $u \geq 0$ in $\mathbb{R}^n \times (-\infty, T_2]$ and (ii) will follow from Theorem 8.1.

Suppose therefore that $T_* < T_2$. We claim now that

$$u = 0 \quad \text{in } \mathbb{R}^n \times (T_*, T_2).$$

By the very construction, the only points of $\partial\Omega$ in $\mathbb{R}^n \times (T_*, T_2)$ are of zero energy. A consequence is that for $t \in (T_*, T_2)$, $\partial\Lambda(t)$ is empty, implying that either $\Lambda(t)$ is empty itself or is the whole space $\mathbb{R}^n$. Moreover, if $\Lambda(t_0) = \mathbb{R}^n$ for some $t_0 \in (T_*, T_2]$, then $\Lambda(t) = \mathbb{R}^n$ for any $t \in (t_0 - \varepsilon^2, t_0]$, where ε is sufficiently small. Indeed, consider a point (x_0, t_0). Then it is either an interior point of Λ or a zero energy point. In both cases there is $\varepsilon > 0$ such that $u = 0$ on $Q_\varepsilon^-(x_0, t_0)$. Then $\Lambda(t)$ is nonempty for $t \in (t_0 - \varepsilon^2, t_0]$, thus implying $\Lambda(t) = \mathbb{R}^n$. In fact, by using a continuation argument we obtain that $\Lambda(t_0) = \mathbb{R}^n$ implies that $u = 0$ on $\mathbb{R}^n \times (T_*, t_0]$.

Next, by the definition of T_2 there is a sequence of points $(x_k, t_k) \in \partial\Omega$ with $t_k \nearrow T_2$. Then $\Lambda(t_k) = \mathbb{R}^n$ by argument above and we obtain that $u = 0$ in $\mathbb{R}^n \times (T_*, T_2)$.

To complete the proof we consider the following two cases: $T_1 < T_*$ and $T_1 = T_*$. In the first case we finish the proof by applying Theorems 9.1 and 8.1. In the second case, there are two possibilities. If $T_1 = T_* = -\infty$, we obtain that $u = 0$ in $\mathbb{R}^n \times (-\infty, T_2)$, and if $T_1 = T_* > -\infty$, from the representation $u(x, t) = P(x) + m\,t$ we obtain that $u(x, t) = T_1 - t$, since u vanishes for $t = T_1$. Thus, in all cases (ii) holds and the proof is complete. □

11. Lipschitz regularity: global solutions

Theorem 11.1. *Let $u \in \mathcal{P}_\infty^-(M)$ be such that $(0, 0) \in \Gamma$ and suppose that Λ contains a cylinder $B \times [-1, 0]$, where $B = B_\rho(-se_n)$ for some $0 \leq s \leq 1$. Set $K(\delta, s, h) = \{|x'| < \delta, -s \leq x_n \leq h\}$ for any $\delta, h > 0$. Then*

(i) *$u \geq 0$ in $\mathbb{R}^n \times \mathbb{R}^-$;*

(ii) *For any spatial unit vector e with $|e - e_n| < \rho/8$ we have*

$$\partial_e u \geq 0 \quad \text{in } K(\rho/8, s, 1) \times [-1/2, 0];$$

(iii) *Moreover, there exists $C_0 = C_0(n, M, \rho) > 0$ such that*

$$C_0\, \partial_e u - u \geq 0 \quad \text{in } K(\rho/16, s, 1/2) \times [-1/2, 0];$$

(iv) *The free boundary $\partial\Omega \cap (K(\rho/32, s, 1/4) \times [-1/4, 0])$ is a space-time Lipschitz graph*

$$x_n = f(x', t),$$

where f is concave in x' and

$$|\nabla_{x'} f| \leq \frac{C}{\rho}, \quad |\partial_t f| \leq C(n, M, \rho).$$

Proof. The origin is either a low or high energy point. The existence of the cylinder $B \times [-1, 0]$ in Λ excludes the possibility of high energy. Moreover, by the same reason, u have no high energy point in $\mathbb{R}^n \times [-1, 0]$. Hence (i) follows from Theorem 9.1.

Next, applying Theorem 8.1, we obtain from (i) that $\partial_{ee} u \geq 0$ and $\partial_t u \leq 0$ in $\mathbb{R}^n \times \mathbb{R}^-$. Now suppose that $|e - e_n| < \rho/8$. Since $\partial_e u = 0$ on B and every halfline in the direction $-e$ originating at a point in $K(\rho/8, s, 1)$ intersects B, the convexity of u implies (ii).

Further, since $0 \in \partial\Lambda(0)$ we obtain that the cone $\mathcal{C} = \{x_n > (8/\rho)|x'|\}$ is contained in $\Omega(0)$, and thus in every $\Omega(t)$ for $-1 \leq t \leq 0$. Then, together with (ii) we find the representation $x_n = f(x', t)$ in $K(\rho/8, s, 1)$, with the spatial Lipschitz estimate $|\nabla_{x'} f| \leq C/\rho$. This proves the first part of (iv). To prove the estimate on $\partial_t f$ in (iv), as well as the estimate (iii) we need an additional lemma.

Lemma 11.2. *Let u be as in Theorem 11.1. Then there exist $r_0 = r_0(n, \rho) > 0$ and $\varepsilon_0 = \varepsilon_0(n, \rho) > 0$ such that*

$$W(r, x, t; u) \leq 2A - \varepsilon_0$$

for any $(x, t) \in \partial\Omega \cap Q^-_{1/2}$ and $0 < r \leq r_0$.

Proof. Assume the contrary. Then there exist a sequence of functions u_k satisfying the assumptions of the lemma and $(x_k, t_k) \in \partial\Omega(u_k) \cap Q^-_{1/2}$ such that

$$W(1/k, x_k, t_k; u_k) \geq 2A - 1/k.$$

From the uniform estimates on u_k we can extract a subsequence such that the functions

$$v_k(x, t) = u_k(x + x_k, t + t_k)$$

converge to a global global solution v_0 with $(0, 0) \in \Lambda(v_0)$. Then for every $r > 0$ we have

$$W(r; v_0) = \lim_{k\to\infty} W(r, x_k, t_k; u_k) \geq \limsup_{k\to\infty} W(1/k, x_k, t_k; u_k) \geq 2A.$$

Since also $W(r; v_0) \leq 2A$, we obtain that $W(r; v_0) = 2A$ for every $r > 0$. Then $(0, 0)$ is a high energy point and therefore

$$v_0(x, t) = ct + P(x)$$

where c is a constant and P is a homogeneous quadratic polynomial. On the other hand, since u_k vanishes on the cylinders $B_\rho(-s_k e_n) \times (-1, 0)$, v_0 vanishes on a cylinder $B \times (-1/2, 0)$, where B is a certain ball of radius ρ. But this is impossible unless v_0 is identically 0, a contradiction.

The lemma is proved. □

We continue the proof of Theorem 11.1. To show (iii) we assume the contrary. Then there exist a sequence of functions u_k satisfying the assumptions of the theorem and points $(x_k, t_k) \in \Omega(u_k) \cap (K(\rho/16, s, 1/2) \times (-1/2, 0))$

$$k\, \partial_e u_k(x_k, t_k) - u_k(x_k, t_k) \leq 0, \quad e = e(k). \tag{11.1}$$

Let now

$$\widetilde{x}_k = \big(x'_k, f_k(x'_k)\big) \in \partial\Omega_k(t_k), \quad h_k = (x_k)_n - f_k(x'_k)$$

and consider

$$v_k(x, t) = \frac{1}{h_k^2}\, u(h_k\, x + \widetilde{x}_k, h_k^2\, t + t_k).$$

Then from (11.1) we have

$$\partial_e v_k(e_n, 0) \leq \frac{h_k}{k}\, v_k(e_n, 0). \tag{11.2}$$

Functions v_k are locally uniformly bounded in $\mathbb{R}^n \times \mathbb{R}^-$, hence over a subsequence v_k converge to a global solution v. If we also assume that $e(k) \to e$, we will have

$$\partial_e v(e_n, 0) = 0.$$

Next, since $h_k \leq 2$, each of the sets $\partial\Omega_{v_k}(0) \cap \{|x'| < \rho/32\}$ is a graph of a concave Lipschitz function, containing 0 and with the Lipschitz constant $L \leq C/\rho$. Since also $\Omega_{v_k}(t)$ expand as t decreases we obtain that $D_k \times (-1/4, 0) \subset \Omega_{v_k}$, where $D_k = \Omega_{v_k}(0) \cap K(\rho/32, s_k/h_k, 1/h_k)$. In particular $H(\partial_{e_k} v_k) = 0$ in $D_k \times (-1/4, 0)$. Moreover, (ii) implies also that $\partial_{e_k} v_k \geq 0$ there. Passing to the limit, we can assume that D_k converge to a set D, having similar properties. Then

$$H(\partial_e v) = 0, \quad \partial_e v \geq 0 \quad \text{in } D \times (-1/4, 0).$$

Since $e_n \in D$, the maximum principle implies that $\partial_e v = 0$ in $D \times (-1/4, 0)$. Then we also obtain that $v(x, 0) = 0$ in $D \times \{|x'| < \rho/64\}$ and as a consequence that $v(x, 0)$ vanishes in a neighborhood of the origin.

The stability property implies that $(0, 0) \in \Gamma(v)$. Moreover, it cannot be low energy, since then $v(x, 0)$ wouldn't vanish in a neighborhood of the origin. So, the only possibility, is that $(0, 0)$ is a high energy point of v. The latter is possible only if

$$v(x, t) = -t \quad \text{in } \mathbb{R}^n \times \mathbb{R}^-.$$

To exclude this possibility, we apply Lemma 11.2. Indeed, we have

$$W(r, v) = \lim_{k\to\infty} W(r; v_k) = \lim_{k\to\infty} W(h_k r, \widetilde{x}_k, t_k; u_k) \leq 2A - \varepsilon_0,$$

provided $r < r_0/2$ (recall $h_k \leq 2$.) Hence $(0, 0)$ cannot be a high energy point. This proves (iii).

Finally, to prove the estimate on $\partial_t f$ in (iv), we apply the following generalization of Lemma 7.6.

Lemma 11.3. *Let u be a bounded solution of* (1.2) *in*

$$\mathcal{N}_\delta^-(E) = \bigcup \{Q_\delta^-(x, t) : (x, t) \in E\},$$

for a set E in $R^n \times \mathbb{R}^-$ and h be caloric in $\mathcal{N}_\delta^-(E) \cap \Omega$. Suppose moreover that

(i) *$h \geq 0$ on $\mathcal{N}_\delta^-(E) \cap \partial\Omega$ and*
(ii) *$h - u \geq -\varepsilon_0$ in $\mathcal{N}_\delta^-(E)$, for some $\varepsilon_0 > 0$.*

Then $h - u \geq 0$ in $\mathcal{N}_{\delta/2}^-(E)$, provided $\varepsilon_0 = \varepsilon_0(\delta, n)$ is small enough.

Proof. Consider h and u in every $Q_\delta^-(x, t)$ with $(x, t) \in E$, parabolically scale to functions in Q_1^- and apply Lemma 7.6. □

Now, for small $|\eta| < \eta_0(\rho, n, M)$ we obtain from (iii) in Theorem 11.1 that

$$(C_0 \partial_e u + \eta \partial_t u) - u \geq -\varepsilon_0$$

in $K(\rho/16, s, 1/2) \times [-1/2, 0]$. From Lemma 11.3 we have

$$(C_0 \partial_e u + \eta \partial_t u) - u \geq 0$$

in $K(\rho/32, s, 1/4) \times [-1/4, 0]$. Note, Lemma 11.3 is applicable with $h = C_0 \partial_e u + \eta \partial_t u$, since both $\partial_e u$ and $\partial_t u$ vanish on $\partial\Omega$. The latter follows from Lemma 7.7.

Then, as in the proof of Theorem 7.5, we obtain the existence of space-time cones with uniform openings at any point on $\partial\Omega$ in $K(\rho/32, s, 1/4) \times [-1/4, 0]$ and this proves the estimate on $\partial_t f$ in (iv).

The proof of the theorem is complete. □

12. Balanced energy: local solutions

In this short section we discuss how one can generalize the balanced energy that we defined for global solutions (see Section 7) for local solutions.

Let u be a solution of (1.2) in Q_1^- and $\psi(x) \geq 0$ be a C^∞ cut-off function with supp $\psi \subset B_1$ and $\psi|_{B_{3/4}} = 1$. Then for $w = u\,\psi$ and any $(x_0, t_0) \in \partial\Omega \cap Q_{1/2}^-$ the functional

$$W(r, x_0, t_0; w) + F_n(r)$$

is nondecreasing by the local form of Weiss' monotonicity theorem (Theorem 3.6). Hence there exists a limit

$$\omega(x_0, t_0) = \lim_{r\to 0} W(r, x_0, t_0; w),$$

since $F_n(r) \to 0$ as $r \to 0$. Moreover, if u_0 is a blow-up limit of parabolic scalings $u_r(x, t) = (1/r^2)u(rx + x_0, r^2 t + t_0)$, then u_0 is also a limit of corresponding scalings w_r of w, since $\psi = 1$ on $B_{1/4}(x_0)$ and we obtain that

$$W(s; u_0) = \lim_{r\to 0} W(s; w_r) = \lim_{r\to 0} W(sr, x_0, t_0; w) = \omega(x_0, t_0).$$

In particular, u_0 is a homogeneous global solution and $\omega(x_0, t_0)$ does not depend on the choice of the cut-off function ψ. The quantity $\omega(x_0, t_0)$ will be called the *balanced energy* of u at (x_0, t_0). Note, when u is a global solution, this definition coincides with the one from Section 7.

As in the global case, since $\omega(x_0, t_0) = W(u_0)$ and u_0 is homogeneous, we have only three possible values for the balanced energy: 0, A, and $2A$. Respectively, we classify the point $(x_0, t_0) \in \partial\Omega$ as of zero, low or high energy.

13. Lipschitz regularity: local solutions

Theorem 13.1. *For every $\sigma > 0$ there exists $R_0 = R_0(\sigma, n, M)$ such that if $u \in \mathcal{P}_R^-(MR^2)$ for $R \geq R_0$, $(0, 0) \in \Gamma$ and $\delta_1^-(u) \geq \sigma$ then $\partial\Omega \cap Q_{1/2}^-$ is space-time Lipschitz regular with Lipschitz constant $L \leq L(\sigma, n, M)$.*

We use the following approximation lemma and then apply the results from Section 11.

Lemma 13.2. *Fix $\sigma > 0$ and $\varepsilon > 0$. Then there exists $R_0 = R_0(\varepsilon, \sigma, n, M)$ such that if $u \in \mathcal{P}_R^-(MR^2)$ for $R \geq R_0$, $(0, 0) \in \Gamma$ and $\delta_1^-(u) \geq \sigma$, then we can find a global solution $v \in \mathcal{P}_\infty^-(C_n M)$, $(0, 0) \in \Gamma(v)$, with the properties*

(i) $\|u - v\|_{C_x^1 \cap C_t^0(Q_1^-)} \leq \varepsilon$;

(ii) *there exists a ball $B = B_\rho(x) \subset B_1$ of radius $\rho = \sigma/(4n)$ such that v vanishes on $B \times [-1, 0]$.*

(iii) *u vanishes on $B_{\rho/2}(x) \times [-1/2, 0]$.*

Proof. The proof is by compactness. Assume the contrary. Then for every $k > 0$ we can find a solution $u_k \in \mathcal{P}_k^-(Mk^2)$ with $(0, 0) \in \Gamma(u_k)$ and $\delta_1^-(u_k) \geq \sigma$ such that for any global solution $v \in \mathcal{P}_\infty(C_n M)$ such that $(0, 0) \in \Gamma(v)$ and conditions (ii) and (iii) are satisfied, we have

$$\|u_k - v\|_{C_x^1 \cap C_t^0(Q_1^-)} \geq \varepsilon. \tag{13.1}$$

Solutions u_k are locally uniformly bounded, so we can extract a subsequence converging to a global solution u_0 in $C_x^{1,\alpha} \cap C_t^{0,\alpha}$-norm on compact subsets of $\mathbb{R}^n \times \mathbb{R}^-$. We claim now that $\Lambda_{u_0}(-1) \cap B_1$ contains a ball of radius $\rho = \sigma/(4n)$. Indeed, first note that $\delta_1^-(u_0) \geq \sigma/2$, otherwise we would have $\delta_1^-(u_k) < \sigma$ for large k. Next, from the stability note that $(0, 0)$

is not a zero energy point of u_0, since it is not for any of solutions u_k. Also, it is not a high energy point of u_0 since $\delta_1^-(u_0) \geq \sigma/2$. Hence the only possibility is that $(0,0)$ is a low energy point of u_0. Then Theorem 9.1 implies that $u_0 \geq 0$ in $\mathbb{R}^n \times \mathbb{R}^-$ and hence the set $\Lambda_{u_0}(-1) \cap B_1$ is convex by Theorem 8.1. Invoking F. John's lemma we obtain the existence of a ball $B = B_\rho(x)$ of radius $\rho = \sigma/(4n)$ in $\Lambda_{u_0}(-1) \cap B_1$. Moreover, $u_0 \geq 0$ implies that $\partial_t u_0 \leq 0$ and that the sets $\Lambda_{u_0}(t)$ shrink as t decrease. Hence $B \times [-1, 0]$ is contained in $\Lambda(u_0)$. Since $u_k \to u_0$, from the stability (see Subsection 5.2) we obtain that u_k vanishes on $B_{\rho/2}(x) \times [-1/2, 0]$ for large k.

So, conditions (ii) and (iii) are satisfied for the global solution $v = u_0$ and $u = u_k$ for large k. But also we have $\|u_k - u_0\|_{C^1_x \cap C^0_t(Q_1^-)} \to 0$, which contradicts (13.1). Hence the lemma follows. □

Proof of Theorem 13.1. Let $\varepsilon = \varepsilon(\sigma, n, M) > 0$ be small (to be specified later) and $R_0 = R_0(\varepsilon, \sigma)$ be as in Lemma 13.2 and suppose that $R \geq R_0$. Let also for $u \in \mathcal{P}_R^-(MR^2)$ with $\delta_1^-(u) \geq \sigma$ the global solution v and the ball $B \subset B_1$ be as in the conclusion of Lemma 13.2.

Rotating the spatial coordinate axes, we may assume that $B = B_\rho(-se_n)$ for $0 \leq s \leq 1$, $\rho = \sigma/(4n)$. Then by estimate (iii) in Theorem 11.1 applied to the global solution v we have

$$C_0 \partial_e v - v \geq 0 \quad \text{in } K(\rho/8, s, 1/2) \times [-1/2, 0]$$

for any spatial unit vector e with $|e - e_n| \leq \rho/8$. Since $|u - v| \leq \varepsilon$, $|\nabla u - \nabla v| \leq \varepsilon$ and $|\partial_t u| \leq C_n M$ in Q_1^-, we obtain automatically that

$$(C_0 \partial_e u - \eta \partial_t u) - u \geq -C_0 \varepsilon - \varepsilon - C_n M \eta_0 \geq -\varepsilon_0$$

if $\varepsilon = \varepsilon(\sigma, n, M)$ and $|\eta| \leq \eta_0(n, M)$ are small, where $-\varepsilon_0$ as in Lemma 11.3. Next, we claim that

$$(C_0 \partial_e u - \eta \partial_t u) - u \geq 0 \quad \text{in } K(\rho/16, s, 1/4) \times [-1/4, 0]. \tag{13.2}$$

This will follow from Lemma 11.3 with $h = C_0 \partial_e u - \eta \partial_t u$ if we know that $h \geq 0$ on $\partial\Omega$. We show next that this is indeed so.

Lemma 13.3. *Let u be as in Lemma 13.2 with $R \geq R_0$. Let also $\psi(x) \geq 0$ be a C^∞ cut-off function with* supp $\psi \subset B_1$ *and $\psi|_{B_{3/4}} = 1$. Then for $w = u\psi$ and any $(x_0, t_0) \in \partial\Omega \cap Q_{1/2}^-$ we have*

(i) *$W(r, x, t; w) \leq 2A - \varepsilon_0$ for $\varepsilon_0 = \varepsilon_0(\sigma, n, M) > 0$ and $r \leq r_0(\sigma, n, M)$;*
(ii) *$\partial_t u$ vanishes continuously at (x_0, t_0): $\lim_{(x,t)\to(x_0,t_0)} \partial_t u(x, t) = 0$.*

Proof. (i) is a generalization of Lemma 11.2. The proof is basically the same, only instead of Weiss' monotonicity theorem (Theorem 3.4) one have to use its local form (Theorem 3.6.)

(ii) is a generalization of Lemma 7.7. We note that u has no high energy points by (i) above. Then the proof is the same as of Lemma 7.7, with application of the local form of Weiss' theorem instead of global. □

The lemma above implies that we indeed have (13.2). In particular, we obtain that $\partial\Omega \cap (K(\rho/16, s, 1/4) \times [-1/4, 0])$ is Lipschitz in space and in time with a Lipschitz constant $L(\sigma, n, M)$.

To finish the proof of the theorem, we observe that we will come to the same conclusion as above (perhaps with different constants) if instead of $\delta_1^-(u) \geq \sigma$ we assume, say,

$$\delta_1^*(u) := \sup_{-1\leq t\leq -1/2} \operatorname{md}(\Lambda(t)\cap B_2) \geq \sigma.$$

This gives a little bit more flexibility. Now let $(x_0, t_0) \in \partial\Omega \cap Q_{1/2}^-$ and consider the function $u^*(x, t) = u(x+x_0, t+t_0)$. We will have $\delta_1^*(u^*) \geq \sigma$, thus after appropriate choice of coordinate axes we will find that $\partial\Omega$ is $L(\sigma, n, M)$-Lipschitz in a parabolic neighborhood of (x_0, t_0). This finishes the proof of the theorem. □

14. $C^{1,\alpha}$ REGULARITY

Theorem 14.1. *Under the conditions of Theorem 13.1, $\partial\Omega \cap Q_{1/4}^-$ is space-time $C^{1,\alpha}$ regular with the norm $C \leq C(\sigma, n, M)$.*

Proof. We are going to apply the result of [ACS96], Corollary 1, on mutual boundary regularity of positive caloric functions in Lipschitz domains.

We assume that $R \geq R_0$ and that the ball $B = B_\rho(-se_n)$ is as in Lemma 13.2, so that u vanishes on $B_{\rho/2}(-se_n) \times [-1/2, 0]$. As it follows from the proof of Theorem 13.1, we have

$$(C_0\partial_e u + \eta\partial_t u) - u \geq 0 \quad \text{in } K(\rho/16, s, 1/4) \times [-1/4, 0]$$

for any spatial unit vector e with $|e - e_n| < \rho/8$ and $|\eta|$ sufficiently small. In particular, we have that

$$\partial_e u + \varepsilon\partial_t u \geq 0,$$

where $\varepsilon = \eta/C_0$. Consider now two functions of the type above

$$\begin{aligned} u_1 &= \partial_n u \\ u_2 &= \partial_e u + \varepsilon\partial_t u \end{aligned}$$

with e sufficiently close to e_n and ε small. Then [ACS96], Corollary 1, implies that the ratio

$$\frac{u_2}{u_1}$$

is C^α regular (both in x and in t) in $\Omega \cap (K(\rho/32, s, 1/8) \times [-(\rho/32)^2, 0])$ up to $\partial\Omega$, with $0 < \alpha < 1$ and C^α norm depending on ρ, n, M, the Lipschitz norm of $\partial\Omega$, as well as on the bound from below on

$$m_i = u_i\left(A^-\right), \quad A^- = \left((3/16)\, e_n, -(\rho/16)^2\right).$$

We claim that

$$m_i \geq c_0(\rho, n, M) > 0.$$

It is enough to prove the bound only for m_1, since m_2 can be made as close to m_1 as we wish. Thus, we have to show that

$$\partial_n u \geq c_0 > 0$$

at A^-. Indeed, if it weren't so, by compactness we would easily construct a function u as above with $\partial_n u = 0$ at A^-. Then by the minimum principle $\partial_n u$ and consequently u would vanish in $K(\rho/32, s, 1/8) \times [-1/4, -(\rho/16)^2]$, a contradiction.

Hence,

$$\frac{\partial_e u + \varepsilon\partial_t u}{\partial_n u}$$

is C^α up to $\partial\Omega$ in $\Omega\cap Q^-_{\rho/32}$. Then varying e and ε we obtain that the ratios

$$\frac{\partial_i u}{\partial_n u},\quad i=1,\dots,n-1,\quad \frac{\partial_t u}{\partial_n u}$$

are C^α. This implies that $\partial\Omega\cap(K(\rho/32,s,1/8)\times[-(\rho/32)^2,0])$ is the graph $x_n = f(x',t)$ with

$$\|f\|_{C^{1,\alpha}}\le C(\rho,n,M)$$

since

$$\partial_i f=\frac{\partial_i u}{\partial_n u},\quad i=1,\dots,n-1,\quad \partial_t f=\frac{\partial_t u}{\partial_n u}.$$

Arguing as in the end of the proof of Theorem 13.1, we obtain that in fact $\partial\Omega\cap Q^-_{1/4}$ is $C^{1,\alpha}$ regular.

The proof is complete. □

15. Higher regularity

Theorem 15.1. *Under the conditions of Theorem 13.1, $\partial\Omega\cap Q^-_{1/8}$ is space-time C^∞ regular.*

Proof. First, we prove the higher regularity for u.

As it follows from the proof of Theorems 13.1 and 14.1, we have that

$$\partial_n u+\varepsilon\partial_t u\ge 0\quad \text{in } K(\rho/16,s,1/4)\times[-1/4,0]$$

for small $|\varepsilon|\le\varepsilon(\rho,n,M)$. Then for large $C=C(\rho,n,M)>0$

$$-C\partial_n u(x,t)\le\partial_t u(x,t)\le C\partial_n u(x,t).$$

Thus, $\partial_t u$ will grow linearly in $\Omega\cap Q^-_{1/4}$ away from $\partial\Omega$, implying that $|\nabla\partial_t u|$ will be uniformly bounded in $\Omega\cap Q^-_{1/8}$.

Next, we claim that u is $C^{2,\alpha}_x$ in $Q^-_{1/8}$. In fact, something stronger is true: if w is any partial derivative (in space or in time) of u, then w is $C^{1,\alpha}_x\cap C^{0,\alpha}_t$ regular in $\Omega\cap Q^-_{1/8}$ up to $\partial\Omega$. Indeed w satisfies the heat equation in $\Omega\cap Q^-_{1/4}$, it is uniformly bounded there, and vanishes continuously on the $C^{1,\alpha}$-graph $\partial\Omega$. Then the classical boundary regularity implies that w is $C^{1,\alpha}_x\cap C^{0,\alpha}_t$ regular.

Now we have enough regularity to apply the Kinderlehrer-Nirenberg technique [KN77].

Without loss of generality assume that e_1 is the (spatial) exterior normal to $\partial\Omega(0)$ at 0. Since $|\nabla u|=0$ on $\partial\Omega$, all spatial second order derivatives vanish at $(0,0)$, except $\partial_{11}u(0,0)$. Since also $\partial_t u$ vanishes at $(0,0)$ we obtain that

$$\partial_{11}u(0,0)=1.$$

Hence, in $\Omega\cap Q^-_{2r}$ for $r=r(\rho,n,M)>0$ small we will have that

$$\partial_{11}u\ge\frac12.$$

Then consider there the partial hodograph transform

$$(x,t)\mapsto(y,t)=(-\partial_1 u,x_2,\dots,x_n,t),$$

which is C^1 and has a nonsingular Jacobian, and the associated Legendre transform

$$v=x_1y_1+u=-x_1\partial_1 u+u.$$

Then $\partial\Omega$ transforms to a portion of $\{y_1 = 0\}$ and the equation for v takes the form

$$Lv := -\frac{1}{\partial_{11}v} - \frac{1}{\partial_{11}v}\sum_{i=2}^{n}(\partial_{i1}v)^2 + \sum_{i=2}^{n}\partial_{ii}v - \partial_t v = 1.$$

As can be shown, Lv is a uniformly parabolic equation on the image of $\Omega \cap Q_{2r}^-$ under the hodograph transform. Moreover v vanishes on the image of $\partial\Omega \cap Q_{2r}^-$, which is a subset of $\{y_1 = 0\}$. Hence v is C^∞ regular on the image of $\partial\Omega \cap Q_r^-$ and considering the inverse transformation

$$(y,t) \mapsto (x,t) = (\partial_1 v, y_2, \dots, y_n, t)$$

we find that $\partial\Omega \cap Q_r^-$, as well as u, are C^∞ regular. For details we refer to [KN77].

To finish the proof, we note that by similar reasoning one can show that $\partial\Omega \cap Q_r^-(x,t)$ is C^∞ regular near every point $(x,t) \in \partial\Omega \cap Q_{1/8}^-$. Hence the theorem follows. □

Remark 15.2. In fact, one can show that $\partial\Omega \cap Q_{1/8}^-$ is not only C^∞ but also analytic in the space variables and in the second Gevrey class with respect to the time variable, see [KN78].

16. Proof of Theorem II

For the solution $u \in \mathscr{P}_1^-(M)$ satisfying the assumptions of the theorem consider the parabolic scaling

$$u_{r_0}(x,t) = \frac{1}{r_0^2}\, u(r_0 x, r_0^2 t).$$

Then we arrive at the conditions of Theorem 13.1. Thus Theorems 14.1 and 15.1 are also applied. Scaling back, we conclude the proof of the theorem. □

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