THE BEHAVIOR OF THE FREE BOUNDARY NEAR THE FIXED BOUNDARY FOR A MINIMIZATION PROBLEM

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ABSTRACT. We show that the free boundary $\partial\{u > 0\}$, arising from the minimizer(s) u, of the functional

$$J(u) = \int_\Omega |\nabla u|^2 + \lambda_+^2 \chi_{\{u>0\}} + \lambda_-^2 \chi_{\{u<0\}},$$

approaches the (smooth) fixed boundary $\partial\Omega$ tangentially, at points where the Dirichlet data vanishes along with its gradient.

1. Introduction

1.1. **Problem Setting.** Our objective in this paper is to analyze the behavior of the free boundary arising from the minimization problem for the functional considered by H.W. Alt, L.A. Caffarelli, and A. Friedman [ACF1],

$$J(u) = \int_\Omega |\nabla u(x)|^2 + q^2(x)\lambda^2(u), \tag{1.1}$$

where $q(x)$ is a given smooth function, $q(x) \neq 0$ for all x, and

$$\lambda^2(u) = \begin{cases} \lambda_+^2 & \text{if } u > 0 \\ \lambda_-^2 & \text{if } u < 0 \end{cases}$$

and $\Lambda = \lambda_+^2 - \lambda_-^2 \neq 0, \lambda_+ > 0, \lambda_- > 0$. If $\Lambda > 0$ we define (following [ACF1]) $\lambda^2(0) = \lambda_-^2$, while if $\Lambda < 0$ we define $\lambda^2(0) = \lambda_+^2$. In the sequel we will consider for simplicity $q \equiv 1, \Lambda > 0$, the general case being similar. Here Ω is a smooth domain and the admissible class of minimization is

$$\mathcal{K}_f := \{u : \ u - f \in W_0^{1,2}(\Omega)\}$$

where f is smooth enough function. The main result (Theorem 5.1, see also Theorem 6.1) of this paper asserts:
At contact points between the fixed and the free boundary, where f and ∇f vanish simultaneously, the free boundary approaches the fixed one in a tangential fashion.

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Note that this complements the result of Gurevich [G], who roughly speaking, has shown that if $\nabla f = 0$ wherever $f = 0$, then $u \in Lip(\overline{\Omega})$. The latter denotes the class of Lipschitz function.

1.2. **Motivation.** In mathematical modeling of industrial processes such as shape opitmization, fluids flow in porous medium, crystalization, and many others, one encounters minimization of certain functionals (such as that presented in this paper), over an admissible class of functions, defined in a bounded or unbounded region.

In many situations, the interface, separating the active and non-active region (or two, different in nature, active regions), may come in contact with the fixed boundary $\partial\Omega$. The question that may be raised, then, is how does the interface (free boundary) meet the fixed one (the container of the physical process).

Obviously, the Dirichlet data prescribed on $\partial\Omega$ should play a crucial role on the behavior of the free boundary near the fixed one. E.g., in the so-called Dam problem of reservoir (see [AG]) the free boundary is locally a smooth graph near the fixed boundary (boundary of the reservoir), and the angle of contact depends on the pressure function (the Dirichlet data) given on the boundary of the reservoir.

In a recent work by the third author and Nina Uraltseva [SU], a similar analysis (for the case of an obstacle-like problem) has been carried out. See also [M], and [A] for extensions of the results in [SU].

This paper is an attempt to make a similar analysis to that of [SU] for the case of minimizers of the above functional.

1.3. **Plan of the paper.** Section 2 contains all definitions needed in this paper. In Section 3, we prove a technical theorem, which more or less takes care of the stability of solutions, under mild assumptions. In Section 4, we classify global solutions. First we take care of the homgeneous global solutions. Then, using Weiss' monotonicity lemma, we show that, under suitable conditions, global solutions are one dimensional linear functions. The main result, on uniform tangential touch, is stated and proven in Section 5. In Section 6, under more relaxed conditions, we prove a weaker (non-uniform) variant of the main result.

2. Definitions and Notation

2.1. **Notation.** We will use the following notations throughout the paper.

$C_0, C_n, \cdots$	generic constants
χ_D	the characteristic function of the set D, $(D \subset \mathbf{R}^n,\ n \geq 2)$
$\overline{D}$	the closure of D
∂D	the boundary of a set D
x, x'	$x = (x_1, \cdots, x_n), \quad x' = (0, x_2, \cdots, x_n)$
$\mathbf{R}^n_+, \mathbf{R}^n_-$	$\{x \in \mathbf{R}^n :\ x_1 > 0\};\ \{x \in \mathbf{R}^n :\ x_1 < 0\}$
$B_r(x)$,	$\{y \in \mathbf{R}^n : \|y - x\| < r\}$
$B_r^+(x)$	$B_r(x) \cap \mathbf{R}^n_+$
$B_r,\ B_r^+$	$B_r(0),\ B_r^+(0)$
$\lambda_\pm$	positive numbers
Λ	$\Lambda = \lambda_+^2 - \lambda_-^2 \neq 0$
Π	$\{x :\ x_1 = 0\}$
$\mathcal{P}_r, \cdots$	see Definitions 2.1, 2.3,
v^+, v^-	$\max(v, 0);\ \max(-v, 0)$.

2.2. **Preliminary definitions.** To start with we need to define the class of boundary values that we will work with.

Conditions on f. To fix the ideas we will consider the origin as a point of contact between the free and the fixed boundary. The key assumptions, throughout this paper, are the following: The function f is defined (for simplicity) over the entire $\mathbf{R}^n$ and

$$\|f\|_{C^1} \leq R, \qquad |f(x)| \leq R|x|\omega(|x|), \qquad \int_0^1 \frac{\omega(t)}{t} dt \leq R, \tag{2.1}$$

where R is a positive constant and ω is a modulus of continuity.

For a fixed domain $D \subset \mathbf{R}^n$, we define the functional J_D as

$$J(u) = J_D(u) = \int_{B_r^+} |\nabla u|^2 + \lambda_+^2 \chi_{\{u>0\}} + \lambda_-^2 \chi_{\{u\leq 0\}}, \tag{2.2}$$

where $\Lambda := \lambda_+^2 - \lambda_-^2 > 0$. Now we can define the main class of functions we will work with.

Definition 2.1. We define the class of functions $\mathcal{P}_r = \mathcal{P}_r(n, R, \lambda_-, \lambda_+)$ which are minimizers of $J_{B_r^+}$ over the set of functions

$$\mathcal{K}_f := \{u : u \in W^{1,2}(B_r^+),\ u - f \in W_0^{1,2}(B_r^+)\},$$

with f satisfying (2.1).

Similary we define the following subclass of $\mathcal{P}_1$

$$\mathcal{P}_1(n, R, \lambda_+, \lambda_-, r_0, c) = \{v \in \mathcal{P}_1(n, R, \lambda_+, \lambda_-) :\ \frac{|B_r^+ \cap \{u > 0\}|}{|B_r^+|} \geq c\} \tag{2.3}$$

where the density property should hold for all $0 < r < r_0$.

A standard method in treating free boundary problems, from the regularity view point, is a scaling, and blow-up argument. The scaling also needs to preserve the minimizer. Therefore, for a sequence of functions $u_j \in \mathcal{P}_1(n, R, \lambda_+, \lambda_-)$, and a sequence of numbers r_j ($\to r_0$, with $r_0 \in \{0, \infty\}$), we define

$$v_j(x) = \frac{u_j(r_j x)}{r_j}. \tag{2.4}$$

A main argument in this paper will be to look at the limit function(s), as j tends to infinity, of the sequence v_j in (2.4).

Remark 2.2. (Linear growth of solutions) For $u \in \mathcal{P}_1(n, R, \lambda_+, \lambda_-)$ there holds

$$|u(x)| \leq CR|x|, \qquad x \in B_1^+. \tag{2.5}$$

Indeed, let w solve the Dirichlet problem

$$\begin{cases} \Delta w = 0 & \text{in } B_1^+ \\ w = f(x) & \text{on } \partial B_1^+. \end{cases}$$

Then standard estimates on Green's function for the half ball yields $w(x) \leq CR|x|$. Moreover, since u is subharmonic in B_1^+ (Theorem 2.3 in [ACF1]), then u^+ is also subharmonic in B_1^+. Also u being harmonic in $\{u \neq 0\}$ (Theorem 2.2 [ACF1]) implies that u^- is also subharmonic in B_1^+. (Recall $u^+ = max(u, 0)$, $u^- = max(-u, 0)$). Thus we may invoke the maximum principle to obtain (2.5).

Definition 2.3. (Global solutions) We say $u \in \mathcal{P}_\infty = \mathcal{P}_\infty(n, R, \lambda_+, \lambda_-)$ is a *global solution*, if

(i) $|u(x)| \leq C|x|$ for some $C > 0$,

(ii) u is a minimizer of J_D over

$$\{w \in W^{1,2}(D) : w = 0 \text{ on } \Pi, w - u \in W_0^{1,2}(D)\}$$

for each $D \subset\subset \mathbf{R}^n_+$.

Assumption (i) is justified by (2.5).

3. Technicalities

We now return to our scaled function v_j, as in (2.4) with $r_j \searrow 0$. Since $f(0) = 0$, one readily verifies that $v_j \in \mathcal{P}_{1/r_j}(n, CR, \lambda_+, \lambda_-)$.

Theorem 3.1. *Let v_j be as in (2.4), with $u_j \in \mathcal{P}_1(n, R, \lambda_+, \lambda_-)$. Then, after passing to a subsequence, there exists $v \in \mathcal{P}_\infty(n, \lambda_+, \lambda_-)$ so that*

(i) *$v_j \to v$ uniformly on compact subsets of $\mathbf{R}^n_+$ and in $C^{0,\alpha}(K), 0 < \alpha < 1$, for each $K \subset\subset \mathbf{R}^n_+$,*

(ii) *for each M, $v_j \rightharpoonup v$ weakly in $W^{1,2}(B_M^+)$,*

(iii) *for each M, $\chi\{v_j > 0\} \to \chi\{v > 0\}$ in $L^1(B_M^+)$,*

(iv) *$\nabla v_j(x) \to \nabla v(x)$ for a.e. x,*

(v) *For each* $\delta > 0$, $K \subset B_M^+$, $dist(K, \Pi) \geq \delta$, $0 < r < \delta/4$, *for* j *large*

$$\partial\{v_j > 0\} \cap K \subset \cup_{x \in \{v>0\} \cap K_{\delta/2}} B_r(x),$$

and

$$\partial\{v > 0\} \cap K \subset \cup_{x \in \{v_j>0\} \cap K_{\delta/2}} B_r(x),$$

where $K_{\delta/2}$ *is a* $\delta/2$*-neighborhood of* K*.*

Proof: The proof of this technical theorem follows, more or less, from [ACF1]. However, there can be some points in the proof of [ACF1], that might need modifications. Therefore, for the readers convenience we will mention all the steps that one needs to carry out in order to obtain the theorem. For some of the steps we also give the details.

Step 1: If $K \subset\subset B_{1/r_j}^+$, $\mathrm{dist}(K, \partial B_{1/r_j}^+) \geq \delta$, and $K \subset B_M^+$, then there is $C = C(R, M, \delta, n, \lambda_+, \lambda_-)$ s.t.

$$\sup_{x \in K} |\nabla v_j(x)| \leq C. \tag{3.1}$$

This follows from the proof of Theorem 5.3 in [ACF1].

Step 2: For $u_j \in \mathcal{P}_1(n, R, \lambda_+, \lambda_-)$, let f_j (as in (2.1)) be such that u_j is a minimizer for $J_{B_1^+}$ over K_{f_j}. Define $g_j(x) = \frac{f_j(r_j x)}{r_j}$ so that v_j is a minimizer for J_{B_{1/r_j}^+}, over K_{g_j}. We claim that

$$\int_{B_M^+} |\nabla v_j(x)|^2 \leq C_{M,R} \tag{3.2}$$

for each $M > 0$. To see this, note that, after change of variables we need to show that

$$\frac{1}{r_j^n} \int_{B_{r_j M}^+} |\nabla u_j(x)|^2 \leq C_{M,R}.$$

Let h_j be the solution to

$$\begin{cases} \Delta h_j = 0 & \text{in } B_{2r_j M}^+ \\ h_j = u_j & \text{on } \partial B_{2r_j M}^+ \end{cases}.$$

The minimizer property gives $J_{B_{2r_j M}^+}(u_j) \leq J_{B_{2r_j M}^+}(h_j)$, and hence

$$\int_{B_{2r_j M}^+} |\nabla u_j|^2 - |\nabla h_j|^2 \leq C r_j^n.$$

Using this we'll have

$$\begin{aligned} \int_{B_{2r_j M}^+} |\nabla(u_j - h_j)|^2 &= \int_{B_{2r_j M}^+} \nabla(u_j - h_j)\nabla(u_j + h_j) - 2\int_{B_{2r_j M}^+} \nabla(u_j - h_j)\nabla h_j \\ &= \int_{B_{2r_j M}^+} |\nabla u_j|^2 - |\nabla h_j)|^2 - 2\int_{B_{2r_j M}^+} \nabla(u_j - h_j)\nabla h_j \\ &\leq C r_j^n, \end{aligned}$$

where we have used the fact that the last term vanishes by the choice of h_j. As a corollary of this, we obtain that $\int_{B^+_{r_jM}} |\nabla(u_j - h_j)|^2 \leq Cr_j^n$, and hence to establish (3.2) it suffices to show that

$$\int_{B^+_{r_jM}} |\nabla h_j|^2 \leq Cr_j^n.$$

In order to prove this, we rescale once more and consider

$$w_j(x) = \frac{h_j(r_jx)}{r_j},$$

which is harmonic in B^+_{2M}, has boundary value $\frac{u_j(r_jx)}{r_j}$ on the top part of ∂B^+_{2M} and boundary value $\frac{f_j(r_jx)}{r_j}$ on $\Pi = \{x_1 = 0\}$. We need to prove that

$$\int_{B^+_M} |\nabla w_j|^2 \leq C_{M,R}.$$

Note that because of (2.5)

$$\left|\frac{u_j(r_jx)}{r_j}\right| \leq CR, \qquad \forall\, x \in B^+_{2M}, \text{ and } j \text{ large}$$

and

$$\left|\frac{f_j(r_jx)}{r_j}\right| \leq CR\omega(r_j|x|) \leq CR, \qquad x \in B^+_{2M}, \quad r_j \text{ large.} \tag{3.3}$$

For simplicity let $\alpha_j(x) = \frac{f_j(r_jx)}{r_j}$, and φ_M be a cut-off function,

$$\varphi_M \equiv 1 \text{ on } B_M, \qquad \mathrm{supp}\varphi_M \subset B_{2M}, \qquad |\nabla\varphi_M| \leq \frac{c}{M}.$$

Consider $(w_j - \alpha_j)\varphi_M^2$, which is 0 on ∂B^+_{2M}, and now compute

$$\begin{aligned}
0 &= \int_{B^+_{2M}} \nabla w_j \nabla((w_j - \alpha_j)\varphi_M^2) \\
&= \int_{B^+_{2M}} \nabla w_j \nabla w_j \varphi_M^2 + 2\int_{B^+_{2M}} \nabla w_j w_j \varphi_M \nabla\varphi_M \\
&- \int_{B^+_{2M}} \nabla w_j \nabla\alpha_j \varphi_M^2 - 2\int_{B^+_{2M}} \nabla w_j \alpha_j \varphi_M \nabla\varphi_M.
\end{aligned}$$

Rearranging terms and using Cauchy-Schwarz inequality we have

$$\begin{aligned}
\int_{B^+_{2M}} |\nabla w_j|^2\varphi_M^2 &\leq C\int_{B^+_{2M}} w_j^2|\nabla\varphi_M|^2 \\
&+ C\int_{B^+_{2M}} |\nabla\alpha_j|^2\varphi_M^2 + C\int_{B^+_{2M}} \alpha_j^2|\nabla\varphi_M|^2.
\end{aligned}$$

The first term is bounded by CR^2M^{n-2}, because by the maximum principle $w_j \leq CR$ in B^+_{2M} (see (3.3)). Next $|\nabla\alpha_j(x)| = |(\nabla f_j)(r_jx)| \leq$

R, while

$$|\alpha_j(x)| = \left|\frac{f_j(r_j x)}{r_j}\right| \leq CR,$$

so our estimate follows.

Because of Steps 1 and 2 a subsequence converges in the appropiate sense, and the limit function has zero trace.

Step 3: Let $K \subset B_M^+$ for some M, $dist(K, \Pi) \geq \delta$. Then, $\nabla v_j \to \nabla v$ a.e. in K. Moreover we can prove that for $\delta > 0$, $r < \delta/4$, and j large we have

$$\partial\{v_j > 0\} \cap K \subset \bigcup_{x \in \partial\{v>0\} \cap K_{\delta/2}} B_r(x)$$

and

$$\partial\{v > 0\} \cap K \subset \bigcup_{x \in \partial\{v_j>0\} \cap K_{\delta/2}} B_r(x),$$

where $K_{\delta/2}$ is the $\delta/2$ neighborhood of K.

This is contained in Lemma 6.1 of [ACF1].

Step 4: There is c such that for any $x_0 \in \partial\{v > 0\} \cap K$, $r < \delta/4$, we have $\frac{1}{r} \fint_{\partial B_r(x_0)} v^+ \geq c$.

Use nondegenercy (Corollary 3.2 in [ACF1]), and Step 3.

Step 5: Using Step 4, we can show that there is an $\varepsilon = \varepsilon(K)$ such that, for any $x_0 \in \partial\{v > 0\} \cap K$, and all $0 < r < \delta/4$, we have

$$\varepsilon \leq \frac{|\{v > 0\} \cap B_r(x_0)|}{r^n}.$$

Step 6: For all $K \subset \mathbf{R}^n_+$

$$|\partial\{v > 0\} \cap K| = 0$$

Use a contradiction argument in conjunction with Step 5.

Step 7: For each K, $\chi_{v_j>0} \to \chi_{v>0}$ in $L^1(K)$.

Use Step 6.

Step 8: There holds

$$\chi_{\{v_j>0\}} \to \chi_{\{v>0\}} \qquad \text{in } L^1(B_M^+).$$

Step 9: The limit function v, is a global solution.
Proof of step 9: It is enough to check the minimizer condition on B_M^+ for each M. Thus let $w \in W^{1,2}(B_M^+)$, $w = 0$ on $\Pi, w - v \in W_0^{1,2}(B_M^+)$, and fix M.

Let $\eta \in C_0^\infty(B_M)$, $0 \leq \eta \leq 1$ be fixed. Choose also

$$\theta \in C_0^\infty(\mathbf{R}), \qquad \theta \equiv 1 \text{ for } |x_1| \leq 1/2, \qquad supp\theta \subset \{|x_1| < 1\},$$

and choose $d_j \to 0$ so that $\frac{\omega(r_j M)}{d_j^{1/2}} \to 0$. Recall from Step 2 that if $g_j(x) = \frac{f_j(r_j x)}{r_j}$, then v_j is a minimizer for $J_{B^+_{1/r_j}}$ over $\mathcal{K}_{g_j}$, and that f_j satisfies (2.1).

Set $\theta_j(x) = \theta(x_1/d_j)$ and define $w_j = w + (1-\eta)(v_j - v) + \theta_j \eta g_j$, so that $w_j = v_j$ on ∂B_M^+ and hence

$$J_{B_M^+}(v_j) \leq J_{B_M^+}(w_j).$$

Using the above steps to carry out some details, we can go to the limit with j ($j \to \infty$), and with $\eta \uparrow 1$, in order to arrive at

$$0 \geq 0 \int_{B_M^+} |\nabla v|^2 - |\nabla w|^2 + \Lambda(\chi_{\{v>0\}} - \chi_{\{w>0\}}),$$

which is the desired conclusion.

This completes the proof of Theorem 3.1. This theorem justifies our interest in the class $\mathcal{P}_\infty$.

4. Global solutions

4.1. **Homogeneous global solutions.** Wishful thinking suggests that global solutions should be one dimensional and have no free boundary in the upper half space. This would be the ideal case, and indeed, this is mostly the case for our problem, as will be shown below.

In order to treat global solutions we'll need two monotonicity arguments (Lemmas 4.1, 4.7). The first one, classical by now, is the Alt-Caffarelli-Friedman monotoncity formula. A refined version of it reads as follows.

Lemma 4.1. *[ACF1] Let h_1, h_2 be two non-negative continuous subsolutions of $\Delta u = 0$ in $B(x^0, R)$ $(R > 0)$. Assume further that $h_1 h_2 = 0$ and that $h_1(x^0) = h_2(x^0) = 0$, and set (for $0 < r < R$)*

$$\varphi(r) = \varphi(r, h_1, h_2, x^0) = \frac{1}{r^4} \left(\int_{B(x^0,r)} \frac{|\nabla h_1|^2 \, dx}{|x - x^0|^{n-2}} \right) \left(\int_{B(x^0,r)} \frac{|\nabla h_2|^2 \, dx}{|x - x^0|^{n-2}} \right).$$

Then

$$\frac{d}{dr}\varphi(r) \geq \frac{2\varphi(r)}{r} A_r, \tag{4.1}$$

where $A_r > 0$ is given by (see [CKS] *Lemmas 2.2-2.3)*

$$\sqrt{A_r} = \frac{C_n}{r^{n-1}} Area\left(\partial B_r \setminus (supp\ h_1 \cup supp\ h_2)\right). \tag{4.2}$$

Using this lemma we can show that global solutions don't change sign, i.e., there exists oly one-phase global solutions.

Theorem 4.2. *Let $u \in \mathcal{P}_\infty(n, \lambda_+, \lambda_-)$. Then either $u \geq 0$, or $u \leq 0$.*

Proof. We apply the monotonicity formula of [ACF1], since both of u^+, u^- have linear growth and vanish on Π, and both are subharmonic, we extend them as 0 to the complement of the set $\{u^\pm > 0\}$. For r such that $\varphi(r, u^+, u^-) \neq 0$ we have $(\varphi(r) = \varphi(r, u^+, u^-))$

$$\frac{d}{dr}\varphi(r) \geq \frac{2\varphi(r)}{r} A_1$$

where $\sqrt{A}_1 \geq \frac{c_n}{2} Area(\partial(B_1))$, since $u^\pm \equiv 0$ on $\mathbf{R}^n_-$. If for some r_0, $\varphi(r_0) > 0$ integrating the ODE we get that, for $r > r_0$, $\varphi(r) \geq \varphi(r_0)\left(\frac{r}{r_0}\right)^{2A_1}$, contradicting that $\varphi(r) \leq C$ by linear growth of u. □

Lemma 4.3. *Let $u \in \mathcal{P}_\infty(n, \lambda_+, \lambda_-)$ and assume that $u \leq 0$. Then either $u \equiv 0$ or $u = -cx_1$ for some $c > 0$.*

Proof. Since u is subharmonic in B_1^+(Theorem 2.3 in [ACF1]), and $u \leq 0$, we can invoke strong maximum principle to conclude $u < 0$ or $u \equiv 0$. The latter case implies that u must be harmonic on $\mathbf{R}^n_+$. It also vanishes on Π, and has linear growth. Let

$$\widetilde{u} = \begin{cases} u(x) & \text{if } x \in \mathbf{R}^n_+ \\ -u(-x_1, x') & \text{if } x = (x_1, x') \in \mathbf{R}^n_- \end{cases}$$

Then $\widetilde{u}$ is harmonic on $\mathbf{R}^n$, has linear growth, vanishes on $x_1 = 0$, so by Liouville's theorem $\widetilde{u}(x) = -cx_1$. Since $u = \widetilde{u}$ on $\mathbf{R}^n_+$, $u \leq 0$, then $c \geq 0$. Since $u \not\equiv 0, c > 0$. □

We now concentrate on $u \geq 0$, $u \in \mathcal{P}_\infty(n, \lambda_+, \lambda_-)$. Let $Q^2 = (\lambda_+^2 - \lambda_-^2)$. Then u is a minimizer for

$$J_{D,Q}(u) = \int_D |\nabla u|^2 + Q^2 \chi_{\{u>0\}},$$

for all $D \subset \mathbf{R}^n_+$, over $\{w \in W^{1,2}(D) : w = 0 \text{ on } \Pi,\ w - u \in W_0^{1,2}(D)\}$.

Lemma 4.4. *Let $u \in \mathcal{P}_\infty(n, \lambda_+, \lambda_-), u \geq 0$, and assume, that u is homogeneous of degree one. Then either $u \equiv 0$ or $u = cx_1, c \geq Q$.*

Proof. Assume that $u \not\equiv 0$. Assume first that there exists $x_0 \in \partial\{u > 0\}$ in $\mathbf{R}^n_+$. Then by Lemma 3.7 of [ACF2], for small r we have $|B_r(x_0) \cap \{u > 0\}| \leq (1-c)|B_r|$, so that $|B_r(x_0) \cap \{u \equiv 0\}| \geq c|B_r|$. Here $|\{u \equiv 0\}| > 0$. By homogeneity

$$\frac{H^{n-1}\left(\partial B_r^+(0) \cap \mathbf{R}^n_+ \cap \{u \equiv 0\}\right)}{r^{n-1}} \geq c_0,$$

where c_0 is independent of r. Now let

$$u_+(x) = \begin{cases} u(x) & \text{if } x \in \mathbf{R}^n_+ \\ 0 & \text{if } x \in \mathbf{R}^n_- \end{cases}$$

$$u_-(x) = \begin{cases} 0 & \text{if } x \in \mathbf{R}^n_+ \\ u(-x_1, x') & \text{if } x \in \mathbf{R}^n_-. \end{cases}$$

We use the monotonicity formula to conclude that $u \equiv 0$. A contradiction. Thus there does not exist $x_0 \in \partial\{u > 0\}$ in $\mathbf{R}^n_+$ so that $u(x) > 0$ in $\mathbf{R}^n_+$, and hence it is harmonic. An argument as in Lemma 4.3 now shows that $u = cx_1, c > 0$. To bound c, fix M, choose $\eta \in C_0^\infty(B_M^+), 0 \le \eta \le 1$. Let for $\varepsilon > 0, u_\varepsilon = \eta c(x_1 - \varepsilon)_+ + (1-\eta)cx_1$ so that $u_\varepsilon = u$ on ∂B_M^+ and hence (with $u = cx_1$)

$$0 \le J(u_\varepsilon) - J(u).$$

Now

$$\begin{aligned}
\nabla u_\varepsilon &= c\nabla\eta(x_1-\varepsilon)_+ + c\eta\vec{e_1}\chi_{\{x_1>\varepsilon\}} - \nabla\eta c x_1 + c(1-\eta)\vec{e_1} \\
&= c(x_1-\varepsilon)\chi_{\{x_1>\varepsilon\}}\nabla\eta - c\nabla\eta x_1 + c\eta\vec{e_1}\chi_{\{x_1>\varepsilon\}} + c(1-\eta)\vec{e_1} \\
&= -c\varepsilon\chi_{\{x_1>\varepsilon\}}\nabla\eta - cx_1\nabla\eta\chi_{\{x_1\le\varepsilon\}} + c\vec{e_1}\chi_{\{x_1>\varepsilon\}} + c(1-\eta)\vec{e_1}\chi_{\{x_1\le\varepsilon\}}.
\end{aligned}$$

Thus,

$$\int_{B_M^+} |\nabla u_\varepsilon|^2 = -2c^2\varepsilon\int_{B_M^+}\nabla\eta\vec{e_1}\chi_{\{x_1>\varepsilon\}} + c^2\int_{B_M^+}(1-\eta)^2\chi_{\{x_1\le\varepsilon\}} + c^2\int_{B_M^+}\chi_{\{x_1>\varepsilon\}} + O(\varepsilon^2)$$

$$Q^2\chi_{\{u_\varepsilon>0\}} = Q^2\chi_{\{x_1>\varepsilon\}} + Q^2\chi_{\{\eta<1, x_1\le\varepsilon\}}, \qquad J(u) = c^2|B_M^+| + Q^2|B_M^+|,$$

and so

$$\begin{aligned}
J(u_\varepsilon) - J(u) &= O(\varepsilon^2) + c^2\int_{B_M^+}(1-\eta)^2\chi_{\{x_1\le\varepsilon\}} \\
&+ c^2\int_{B_M^+}\chi_{\{x_1>\varepsilon\}} - 2c^2\varepsilon\int_{B_M^+}\nabla\eta\vec{e_1}\chi_{\{x_1>\varepsilon\}} \\
&+ Q^2|B_M^+\cap\{x_1>\varepsilon\}| + Q^2|B_M^+\cap\{\eta<1, x_1\le\varepsilon\}| \\
&- c^2|B_M^+\cap\{x_1>\varepsilon\}| - c^2|B_M^+\cap\{x_1\le\varepsilon\}| \\
&- Q^2|B_M^+\cap\{x_1>\varepsilon\}| - Q^2|B_M^+\cap\{0\le x_1\le\varepsilon\}|
\end{aligned}$$

so

$$\begin{aligned}
0 \le \frac{J(u_\varepsilon)-J(u)}{\varepsilon} \to\ & c^2 \int_{\partial B_M^+\cap\Pi}(1-\eta)^2 dH^{n-1} - 2c^2\int_{B_M^+}\nabla\eta\vec{e_1} \\
&+ Q^2H^{n-1}(\partial B_M^+\cap\{\eta<1\}\cap\Pi) \\
&- c^2H^{n-1}(\partial B_M^+\cap\Pi) - Q^2H^{n-1}(\partial B_M^+\cap\Pi).
\end{aligned}$$

But

$$-2c^2\int_{B_M^+}\nabla\eta\vec{e_1} = 2c^2\int_{\partial B_M^+\cap\Pi}\eta$$

and hence

$$0 \le c^2H^{n-1}(\partial B_M^+\cap\Pi) - Q^2H^{n-1}(\partial B_M^+\cap\Pi)$$

if we make $\eta\uparrow 1$, so that $Q^2 \le c^2$.

4.2. **Further Properties of $\mathcal{P}_\infty$.**

Lemma 4.5. *Assume that $u \in \mathcal{P}_\infty$, $u \geq 0$, and $r > 0$. Then there exists C such that*

$$\frac{1}{r^n}\int_{B_r^+} |\nabla u|^2 \leq C.$$

Proof. By subharmonicity of u (Theorem 2.3 in [ACF1]) and that $u = 0$ on Π, we have

$$\frac{1}{r^n}\int_{B_r^+} |\nabla u|^2 \leq \frac{c}{r^{n+2}}\int_{B_{2r}^+} u^2 \leq C. \quad \square$$

Remark 4.6. Let $u \geq 0, u \in \mathcal{P}_\infty$. Then, by Remark 2.6 in [ACF2], $u \in Lip(\overline{B_M^+})$, for each $M > 0$. Note also that the proof of Remark 2.6 in [ACF2] and a simple scaling argument shows that, if $u \in P_\infty, |\nabla u(x)| \leq C, \forall x \in \mathbf{R}_+^n$,

Blow-up limits: Let $u \geq 0$, and $u \in \mathcal{P}_\infty$. Let $r_j \searrow 0$. Let $u_j(x) = \frac{u(r_j x)}{r_j}$. Then the conclusions of Theorem (3.1) apply to u_j. The limit u_0 (after passing to subsequence) will be called the blow-up limit. (Note that (2.5), (3.2) hold. This was the key in Theorem 3.1). Moreover, $\nabla u_j \to \nabla u_0$ in $L^2(B_M^+)$, for any M. This follows from (iv) in Theorem 3.1 and dominated convergence, in view of Remark 4.6.

Blow-down: Let $u \geq 0$, and $u \in \mathcal{P}_\infty$. Let $R_j \uparrow \infty$. Let $u_j(x) = \frac{u(r_j x)}{R_j}$. Then since $|u_j(x)| \leq C|x|, \int_{B_M^+} |\nabla u_j|^2 \leq C$, and u is global solution, the proof of Theorem 3.1 applies and the limit $u_\infty(x)$ will be called blow-down limit. Again $\nabla u_j \to \nabla u_\infty$ in $L^2(B_M^+)$ for any M.

4.3. **Weiss' Monotonicity formula.** Define

$$W(r,u) = \frac{1}{r^n}\int_{B_r^+(0)} \left(|\nabla u|^2 + Q^2\chi_{\{u>0\}}\right) - \frac{1}{r}\int_0^r \frac{1}{\rho^{n-1}}\int_{\partial B_\rho^+, top} (\nabla u \cdot \nu)^2 dH^{n-1} d\rho.$$

where ν is the outer unit normal to ∂B_ρ and $\partial B_{\rho,top}^+ = \partial B_\rho \cap \mathbf{R}_+^n$. Note that, by Remark 4.6, and the fact that $u \in \mathcal{P}_\infty$, we must have $W(r,u) \leq C$ for each r.

Lemma 4.7. *(Weiss) If $0 < s < \rho$, then for $u \in \mathcal{P}_\infty$ there holds*

$$W(\rho,u) - W(s,u) \geq$$
$$\int_s^\rho t^{-3}\int_{\partial B_t^+}\left[t\int_0^t (\nabla u(r\xi)\cdot\xi)^2 dr - \left(\int_0^t \nabla u(r\xi)\cdot\xi dr\right)^2\right] dH^{n-1}(\xi) dt \geq 0\ .$$

Proof. The result is proved in [W] for the case of B_r. However, the argument works exactly the same way for the case of half ball B_r^+, since $u\lfloor_\Pi = 0$. In fact the only thing we need to verify is that the function $u_t := \frac{|x|}{t} u(t\frac{x}{|x|})$ satisfies $u_t = u$ on ∂B_t^+(see the proof of Theorrem 1.2 in [W]). This is the case for all u with $u(0,x')$ homogeneous of degree one.

4.4. **Classifications of Global solutions.**

Lemma 4.8. *Let $u \in \mathcal{P}_\infty, u \geq 0$ and let u_0, u_∞ be a blow-up and blow-down of u respectively. Then u_0 and u_∞ are homogeneous of degree 1 and thus $u_0(x) = c_0 x_1, u_\infty(x) = c_\infty x_1$, where $c_0 = 0$ or $c_0 \geq Q$ and $c_\infty = 0$ or $c_\infty \geq Q$.*

Proof. Once the homogenety is established the rest follows from Lemma 4.4. Let us prove it first for $u_0(x)$. Let again $u_j(x) = \frac{u(r_j x)}{r_j}$. We first claim that $W(r, u_0) = \lim_{j\to\infty} W(r, u_j)$. This is clear for

$$\frac{1}{r^n}\int_{B_r^+(0)} \left(|\nabla u|^2 + Q^2 \chi_{\{u>0\}}\right)$$

in view of

$$\nabla u_j \to \nabla u_0 \quad \text{in } L^2(B_r^+), \qquad \chi_{\{u_j>0\}} \to \chi_{\{u>0\}} \quad \text{in } L^1(B_r^+).$$

For

$$\frac{1}{r}\int_0^r \frac{1}{\rho^{n-1}}\int_{\partial B_\rho^+, top} (\nabla u_j \cdot \nu)^2 dH^{n-1} d\rho$$

just use dominated convergence and the fact that $|\nabla u_j| \leq C$ uniformly in $j, \nabla u_j \to \nabla u_0$ a.e. Thus, $W(r, u_0) = \lim_{j\to\infty} W(r, u_j)$, but $W(r, u_j) = W(rr_j, u)$. Note that $W(r, u)$ is a monotone increasing function, by Weiss' monotonicity formula $W_0 = \lim_{s\to 0} W(s, u)$ exists (note that $W(r, u) \leq C$). Thus $\lim_{j\to\infty} W(r, u_j) = W_0$. Hence, $W(r, u_0) \equiv W_0$. We now use Weiss' monotonicity formula again to conclude u_0 is homogeneous of degree one. The argument for u_∞ is similar.

Theorem 4.9. *Let $u \in \mathcal{P}_\infty, u \geq 0$ and assume that u_0, a blow-up of u, is not identicaly zero. Then $u = cx_1, c \geq Q$*

Proof. Let us first compute $W(r, cx_1)$ for $c > 0$. We get, for the first two terms, and with $\omega_n = |B_1|$, $\frac{\omega_n}{2}(c^2 + Q^2)$. For the other terms, we need to compute

$$\frac{c^2}{r}\int_0^r \frac{1}{\rho^{n-1}}\int_{\partial B^+_{\rho,top}} (\nu_1)^2 dH^{n-1} d\rho = c^2 \int_{\partial B^+_{1,top}} (\nu_1)^2 dH^{n-1}.$$

Now by the symmetry

$$\int_{\partial B^+_{1,top}} \nu_1^2 dH^{n-1} = \frac{1}{2}\int_{\partial B_1} \nu_1^2 dH^{n-1}$$

and

$$\int_{\partial B_1} \nu_1^2 dH^{n-1} = \int_{\partial B_1} \nu_j^2 dH^{n-1}$$

for any j, and hence

$$\int_{\partial B_1} (\nu_1)^2 dH^{n-1} = \frac{Area(\partial B_1)}{n}.$$

We thus get $\frac{c^2}{2n}Area(\partial B_1)$. But $\omega_n = \frac{Area(\partial B_1)}{n}$, and so we get $\frac{\omega_n}{2}Q^2$. Let now $u_r(x) = \frac{u(rx)}{r}$, and notice that $W(sr,u) = W(s,u_r)$. Consider now $r_j \downarrow 0, R_j \uparrow \infty$ and consider corresponding u_0, u_∞. We have

$$W(r,u) = W(\frac{r}{r_j}, u_{r_j}) \geq W(1, u_{r_j})$$

for any j large, since $\frac{r}{r_j} \geq 1$. Now $\lim_{j\to\infty} W(1,u_{r_j}) = W(1,u_0)$, as we saw. Moreover $W(1,u_0) = Q^2\frac{\omega_n}{2}$, since $u_0 \not\equiv 0$, by Lemma 4.8 and the first computation. Thus, $Q^2\frac{\omega_n}{2} \leq W(r,u)$

$$W(r,u) = W(\frac{r}{R_j}, u_{R_j}) \leq W(1, u_{R_j})$$

for j large ($\frac{r}{R_j} \leq 1$). $W(1,u_{R_j}) \to W(1,u_\infty)$. We then have $Q^2\frac{\omega_n}{2} \leq W(r,u) \leq W(1,u_\infty)$. In particular u_∞ cannot be identically 0. Hence

$$W(1,u_\infty) = Q^2\frac{\omega_n}{2}$$

and thus $W(r,u) \equiv Q^2\frac{\omega_n}{2}$. Lemma 4.7 applies again, to give u is homogeneous of degree 1, non-zero and the conclusion follows.

Remark 4.10. The solution $u(x) = Q(x_1 - 1)_+$ shows that the assumption on u_0 is needed.

5. Main Result

Theorem 5.1. *There exists a constant ρ_0, and a modulus of continuity σ such that, if*

$$u \in \mathcal{P}_1(n, R, \lambda_+, \lambda_-, r_0, c)$$

then

$$\partial\{u > 0\} \cap B^+_{\rho_0} \subset \{x : x_1 \leq \sigma(|x|)|x|\}$$

Proof. We will show that, given ε, there is a ρ_ε such that if $u \in \mathcal{P}_1(n, R, \lambda_+, \lambda_-, r_0, c)$, then

$$\partial\{u > 0\} \cap B^+_{\rho_\varepsilon} \subset B^+_{\rho_\varepsilon} \setminus K_\varepsilon$$

where $K_\varepsilon = \{x : x_1 > \varepsilon\sqrt{x_2^2 + \cdots + x_n^2}\}$. This clearly suffices. We argue by contradiction. If not there are $u_j \in \mathcal{P}_1(n, R, \lambda_+, \lambda_-, r_0, c)$ and $x_j \in \partial\{u_j > 0\} \cap B_1^+$ with $|x_j| \to 0$, and such that $x_j \in K_\varepsilon$. Let now $r_j = |x_j|$ and let $v_j(x) = \frac{u_j(r_jx)}{r_j}$. By Theorem 3.1 after passing to a subsequence, we can find $v \in \mathcal{P}_\infty$ such that $v_j \to v$ uniformly on compact subsets of $\mathbf{R}^n_+$. Note that $v_j(\frac{x_j}{|x_j|}) = 0$ and $\frac{x_j}{|x_j|} \in \partial B^+_{1,top} \cap K_\varepsilon$. Thus after passing to further subsequence, there exists $x_0 \in \partial B^+_{1,top} \cap K_\varepsilon$ such that $v(x_0) = 0$. Next, note that $\chi_{\{v_j>0\}} \to \chi_{\{v>0\}}$ in $L^1(B_R^+)$ for each R by Theorem 3.1. Then

$$\begin{aligned}\frac{1}{\frac{\omega_n}{2}R^n}\int_{B_R^+}\chi_{\{v>0\}} &= \lim_{j\to\infty}\frac{1}{\frac{\omega_n}{2}R^n}\int_{B_R^+}\chi_{\{v_j>0\}} = \lim_{j\to\infty}\frac{1}{\frac{\omega_n r_j^n}{2}R^n}\int_{B^+_{r_jR}}\chi_{\{u_j>0\}} \\ &= \lim_{j\to\infty}\frac{1}{|B^+_{r_jR}|}|\{u_j>0\}\cap B^+_{r_jR}| \geq c\end{aligned}$$

since $u_j \in \mathcal{P}_1(n, R, \lambda_+, \lambda_-, r_0, c)$. But then

$$\frac{B_R^+ \cap \{v > 0\}}{\frac{\omega_n}{2} R^n} \geq c$$

for each R. Thus $v \not\equiv 0$, and $v_0 \not\equiv 0$ by a similar argument, where v_0 is a blow-up of v. Because of Theorem 4.2 and Lemma 4.3 $v \geq 0$. Also Theorem 4.9 gives $v = cx_1, c \geq Q$. But then $v(x_0) > 0$, a contradiction.

Remark 5.2. If we consider $u_j(x) = Q(x_1 - r_j)_+$, with $r_j \downarrow 0$, we see that without (2.3) the conclusion of theorem 5.1 fails.

Remark 5.3. If there esits a $\delta, r_0 > 0$ such that for all $0 < r < r_0, B_r^+ \setminus \{0 < x_1 < \delta r\} \cap \partial\{u > 0\} \neq \emptyset$, then there is $c > 0$ such that $u \in \mathcal{P}_1(n, R, \lambda_+, \lambda_-, r_0, c)$, once $u \in \mathcal{P}_1(n, R, \lambda_+, \lambda_-)$. In fact, if $x_0 \in B_r^+ \setminus \{0 < x_1 < \delta r\} \cap \partial\{u > 0\}$ by Theorem 3.1 [ACF1] (nondegeneracy), $\frac{1}{s} \fint_{\partial B_s(x_0)} u^+ \geq C$, for $0 < s < \delta r$, and hence $|\{u > 0\} \cap B_{\delta r}(x_0)| \geq cr^n$ and thus $|\{u > 0\} \cap B_r^+| \geq cr^n$. The same is true if $B_r^+ \setminus \{0 < x_1 < \delta r\} \cap \{u > 0\} \neq \emptyset$.

Remark 5.4. Suppose that $u \in \mathcal{P}_1(n, R, \lambda_+, \lambda_-)$, and there exists $c > 0, r_0$ such that for $0 < r < r_0, \frac{1}{r}\fint_{\partial B_r^+} u^+ \geq c$. Then, $u \in \mathcal{P}_1(n, R, \lambda_+, \lambda_-, r_0, c)$ because on a substantial portion of $B_r^+ \setminus B_{r/2}^+$, we have $u^+ \geq cr$.

6. Non-uniform results

We now turn to the analog of Theorem 5.1 for the class $\mathcal{P}_1(n, R, \lambda_+, \lambda_-)$. Because of Remark 5.2 this cannot hold uniformly, but it does hold for each $u \in \mathcal{P}_1(n, R, \lambda_+, \lambda_-)$.

Theorem 6.1. *Given $u \in \mathcal{P}_1(n, R, \lambda_+, \lambda_-)$, there exists a modulus of continuity σ, depending on f and u, and a ρ_0 with the same dependnece, such that*

$$\partial\{u > 0\} \cap B_{\rho_0}^+ \subset \{x : x_1 \leq \sigma(x)|x|\}.$$

As before if suffices to show the following.

Lemma 6.2. *If $u \in \mathcal{P}_1(n, R, \lambda_+, \lambda_-)$, then given $\varepsilon > 0, \exists \rho_\varepsilon$ such that $\partial\{u > 0\} \cap B_{\rho_\varepsilon}^+ \subset B_{\rho_\varepsilon} \setminus K_\varepsilon$*

Before giving the proof of Lemma 6.2 we need a preliminary lemma

Lemma 6.3. *Let $u \in \mathcal{P}_1(n, \lambda_+, \lambda_-)$ be given and let $\alpha > 0$ be given. Then there exist $r_0, \delta > 0$, such that, if for some $0 < r < r_0$,*

$$\frac{1}{r} \fint_{B_r^+ \setminus B_{r/2}^+} u^+ \leq \delta,$$

then, $u(x) \leq \alpha|x|$, for $|x| < r/2$.

Proof. Fix η small, and consider

$$K_\eta \cap \partial B^+_{\frac{3}{4}r, top} = \{x_1 > \eta\sqrt{x_2^2 + \cdots + x_n^2}\} \cap \partial B^+_{\frac{3}{4}r, top}.$$

Note that, for η small, for each x in this set, $B_{\eta r/2}(x) \subset B_r^+ \setminus B_{r/2}^+$. Hence,

$$\frac{1}{r} \fint_{B_{\eta r/2}(x)\setminus B_{\eta r/4}(x)} u^+ \leq C_\eta \delta,$$

and so, for some $\frac{\eta}{4}r < s < \frac{\eta}{2}r$ we have $\frac{1}{s} \fint_{\partial B_s(x)} u^+ \leq \widetilde{C_\eta}\delta$. Choose now δ so small, depending on η, so that $\widetilde{C_\eta}\delta \leq C$, where C is as in Theorem 3.1 in [ACF1], so that $u^+ \equiv 0$ in $B_{s/2}(x)$. With this choice of δ, we see that $u^+ \equiv 0$ on $\partial B^+_{\frac{3}{4}r,top} \cap K_\eta$. Recall also that $|u(x)| \leq C|x|$ in B_1^+, (see (2.5)). Consider now w_1 in $B^+_{\frac{3}{4}r}$, given by

$$\begin{cases} \Delta w_1 = 0 & \text{in } B^+_{\frac{3}{4}r} \\ w_1 = 0 & \text{on } \partial B^+_{\frac{3}{4}r,top} \cap K_\eta \\ w_1 = C|x| & \text{on } \partial B^+_{\frac{3}{4}r,top} \setminus K_\eta \\ w_1 = 0 & \text{on } \Pi. \end{cases}$$

We claim that, given $\alpha > 0$ and C as in above, we can choose an η so that

$$0 \leq w_1(x) \leq \frac{\alpha}{2}|x| \qquad \text{in } B^+_{r/2}.$$

Indeed, by $C^{1,\beta}(\overline{B^+_{r/2}})$ regularity we have $w_1(x) \leq \frac{Ax_1}{r} w_1(\frac{3}{8}r, 0)$, where $x \in B^+_{r/2}$, and A is a dimensional constant. But a scaling argument shows that we can choose η small so that $w_1(\frac{3}{8}r,0) \leq \frac{\alpha}{2A}r$, since the harmonic measure at the point $(\frac{3}{8},0)$ for ∂B_1^+ of the set $\partial B_1^+ \setminus (K_\eta \cup \Pi) \to 0$ as $\eta \to 0$. Let now $w_2(x)$ solve

$$\begin{cases} \Delta w_2 = 0 & \text{in } B^+_{\frac{3}{4}r} \\ w_2 = 0 & \text{on } \partial B^+_{\frac{3}{4}r,top} \\ w_2 = f(x) & \text{on } \Pi. \end{cases}$$

We claim that, given $\alpha > 0$, we can choose $r_0 > 0$ so small that

$$|w_2(x)| \leq \frac{\alpha}{2}|x| \qquad \text{in} B^+_{r/2}.$$

In fact, let $v_2(y) = w_2(\frac{3}{4}ry)$ for $y \in B_1^+$. Then

$$\begin{cases} \Delta v_2 = 0 & \text{in } B_1^+ \\ v_2 = 0 & \text{on } \partial B^+_{1,top} \\ v_2 = g_r(y) & \text{on } \Pi \end{cases}$$

where $g_r(y) = f(\frac{3}{4}ry)$. Now

$$|g_r(y)| \leq \frac{3}{4}rR|y|\omega(\frac{3}{4}r|y|).$$

Moreover

$$\int_0^1 \omega(\frac{3}{4}rt)\frac{dt}{t} = \int_0^{\frac{3}{4}r} \omega(t)\frac{dt}{t}$$

which is small if $r < r_0, r_0$ is small. Thus, we can choose r_0 so small that $|v_2(y)| \leq \frac{3}{4}ArR\frac{\alpha}{ArR}|y|$, and hence, $|w_2(x)| \leq \frac{\alpha}{2}|x|$. Now, since u is subharmonic, and $u \leq w_1 + w_2$ on $\partial B^+_{3/4r}$, the lemma follows.

Corollary 6.4. *Let $u \in \mathcal{P}_1(n, \lambda_+, \lambda_-)$ be given. Then, there exists r_0, δ such that, if for some $0 < r < r_0$, $\frac{1}{r}\fint_{B^+_r \setminus B^+_{r/2}} u^+ \leq \delta$, and $r_j \downarrow 0$, $u_j(x) = \frac{u(r_j x)}{r_j}$ and $v = \lim_{j\to\infty} u_j$ is as in Theorem 3.1, then $v \leq 0$.*

Proof. Since $v \in \mathcal{P}_\infty$, by Theorem 4.2 $v \leq 0$ or $v \geq 0$. Assume that $v \geq 0$. Let α be the constant as in Lemma 2.5 in [ACF2] (with $k = 1/2$; see also Remark 2.6 in [ACF2] and observe that $v = 0$ on Π), so that if $\frac{1}{R}\fint_{\partial B^+_R} v \leq \alpha$, then $v \equiv 0$ in $B^+_{R/2}$. Choose now δ, r_0 as in Lemma 6.3. We claim that

$$\frac{1}{R}\fint_{\partial B^+_R} v \leq \alpha.$$

Indeed

$$\frac{1}{R}\fint_{\partial B^+_R} v = \lim_{j\to\infty}\frac{1}{R}\fint_{\partial B^+_R} u_j = \lim_{j\to\infty}\frac{1}{Rr_j}\fint_{\partial B^+_{Rr_j}} u \leq \alpha,$$

since $u(x) \leq \alpha|x|, |x| \leq r/2$. Hence $v \equiv 0$

Proof of Lemma 6.2 Let r_0, δ be as in Corollary 6.4. Assume first that, for all $0 < r < r_0$, $\frac{1}{r}\fint_{B^+_r \setminus B^+_{r/2}} u^+ \geq \delta$. Then for all such r,

$$\frac{|\{u > 0\} \cap B^+_r|}{|B^+_r|} \geq c_\delta$$

and hence the conclusion follows from Theorem (5.1). Assume then, that there exists $0 < r < r_0$ such that

$$\frac{1}{r}\fint_{B^+_r \setminus B^+_{r/2}} u^+ \leq \delta.$$

If the conclusion does not hold, there exist $x_j \in \partial\{u > 0\} \cap B^+_1$ with $r_j = |x_j| \to 0$ and $x_j \in K_\varepsilon$ for some fixed $\varepsilon > 0$. Let $u_j(x) = \frac{u(r_j x)}{r_j}$, and $v = \lim_{j\to\infty} u_j$, as in Thoerem 3.1. Recall that, after passing to a subsequence, we can assume that $\frac{x_j}{|x_j|} \to x_0 \in \partial B^+_{1,top} \cap K_\varepsilon$, and hence $v(x_0) = 0$. Also by Corollary 3.2 [ACF1], $\frac{1}{r_j}\fint_{\partial B_{r_j/2}(x_j)} u^+ \geq c$, $c > 0$, and since $x_j \in K_\varepsilon$, it is easy to see that $v \not\equiv 0$. But by Corollary 6.4 $v \leq 0$, and hence, since $v \not\equiv 0$, $v(x) = -cx_1$, $c > 0$, by Lemma 4.3, which contradicts $v(x_0) = 0$.

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