GLOBAL SOLUTIONS OF THE OBSTACLE PROBLEM IN HALF-SPACES, AND THEIR IMPACT ON LOCAL STABILITY.

JOHN ANDERSSON AND HENRIK SHAHGHOLIAN

ABSTRACT. We show that there are an abundance of *non-homogeneous* global solutions to the obstacle problem, in the half-space,

$$\Delta u = \chi_{\{u>0\}}, \qquad u \geq 0 \qquad \text{in } \mathbb{R}^2_+,$$

with a (fixed) *homogeneous* boundary condition

$$u(0,x_2) = \lambda^2 (x_2^+)^2 \qquad (0 < \lambda < 1/\sqrt{2}).$$

As a consequence we obtain local instability of the free boundary under $C^{1,1}$ perturbation, of the Dirichlet data.

1. Introduction

1.1. **Background.** This paper has grown out of a recent attempt to classify global solutions to the obstacle problem in half spaces, having homogeneous boundary condition. To fix ideas, let us consider the global obstacle problem

$$\Delta u = \chi_{\{u>0\}}, \qquad u \geq 0 \qquad \text{in } \mathbb{R}^n \quad (n \geq 2). \tag{1}$$

Here χ_D is the characteristic function of D. A solution to this system is called a global solution. There are an abundance of functions solving (1). E.g. $u = (x_1^+)^2/2$, or any rotation and translation of this function. Any non-negative, second degree polynomial $p(x)$ with $\Delta p = 1$ is also a global solution. One can show (see [Sh]) that the interior of any ellipsoid is the zero set of such a solution, and the solution itself can be given in terms of the Newtonian potential. For appropriate constants A, B one can show that $A + B|x|^{2-n} + |x|^2/2n$ solves (1) outside the unit ball.

Let us now restrict the problem to the upper half space $\mathbb{R}^n_+ = \{x_1 > 0\}$, i.e., we consider equation (1) in $\mathbb{R}^n_+$. We need also to specify boundary condition $u(0, x') = f(x')$, where $x' = (x_2, \cdots, x_n)$. Natural choices for f should be restriction of global solutions to $\Pi := \{x_1 = 0\}$. Obviously a problem of such a generality would not be easy to approach. Therefore, in this paper we will only consider boundary data coming from the restriction of rotations of the one dimensional solution $(x_1^+)^2/2$. Hence

$$f = \left((\alpha' \cdot x')^+\right)^2 /2$$

where $\alpha' = (0, \alpha_2, \cdots)$ is a unit vector.

Having set the problem as above, one can now ask whether all solutions to the above problem, with boundary data coming from the one dimensional solution, are also one dimensional(?). This would naturally be an ideal case, since then one can

H. Shahgholian was supported in part by Swedish Research Council.
2000 *Mathematics Subject Classification.* Primary 35R35, 34D10.
Key words and phrases: Global solutions, instability, obstacle problem.

go on as in [SU] and prove local regularity results for the free boundary $\partial\{u>0\}$, close to the contact point with the fixed boundary.

In this paper we shall construct a non-homogeneous global solution in $\mathbb{R}^2_+$ with boundary data $f=\lambda^2(x_2^+)^2$. In particular our solution does not coincide with the natural solution in the whole space.

A consequence of our construction is a local instability of the free boundary close to the fixed one. Another consequence is that uniform regularity of the free boundary (as in [SU] and [A]) fails in general.

1.2. **The Obstacle Problem.** We will call the minimizer of the following functional

$$J(u)=\int_D \frac{1}{2}|\nabla u|^2+u \tag{2}$$

over the set

$$K=\{u\in H^1(D);\ u\geq 0,\ u=f \text{ on } \partial D\},$$

a solution to the obstacle problem. In other texts the obstacle problem is defined in a more general way, see for instance [C] and [F], but we will restrict ourselves to this model case. We will also assume, for simplicity that $D=B_1^+=\{|x|<1;\ x_1>0\}$, the upper half ball.

It is well known that the solution to the obstacle problem satisfies

$$\begin{array}{ll} \Delta u=\chi_{\{u>0\}} & \text{in } B_1^+ \\ u\geq 0 & \\ u=f & \text{on } \partial B_1^+. \end{array} \tag{3}$$

We will call the set $\partial\{u>0\}\cap B_1^+$ the free boundary and denote it Γ_u.

Global solutions, in this paper, will be minimizers of the above functional for every $D\subset\mathbb{R}^n_+$, with prescribed boundary values.

1.3. **Notation.** $\mathbb{R}^n_+$ is the upper half space: $\mathbb{R}^n\cap\{x_1>0\}$.
$B_r(x_0)$ is the open ball of radius r centered at x_0:
$\{x;|x-x_0|<r\}$.
$B_r^+(x_0)$ is the ball intersected with $\mathbb{R}^n_+$.
$\Omega_u=\{u>0\}$ is the non-coincidence set.
Γ_u denotes the free boundary of u, that is $\Gamma_u=\partial\Omega_u\cap\{x_1>0\}$.
χ_Ω is the characteristic function of Ω, i.e,

$$\chi_\Omega(x)=\begin{cases} 1 & x\in\Omega \\ 0 & x\notin\Omega. \end{cases}$$

It is noteworthy that one can show that global solutions have quadratic growth. Indeed, one can show, for universal positive constants c_0, C_0,

$$c_0r^2\leq \sup_{B_r(x^0)} u-u(x^0)\leq C_0r^2, \tag{4}$$

for any $r>0$, and $x^0\in\partial\overline{\{u>0\}}$. See e.g. [C].

The re-scaled function

$$u_r=\frac{u(rx)}{r^2}$$

will be a solution in $B^+_{1/r}$, with the re-scaled boundary values. An important technical tool will be to blow-up a solution u. By blow-up we mean

$$\lim_{r\to 0}\frac{u(rx)}{r^2},$$

through some converging subsequence (see Section 2 for details). We can also define the shrink down of a global quadratically bounded solution u, by taking $r\to\infty$. We denote the shrink down by u_∞ (if there is no ambiguity).
Π is the plane $\{x_1 = 0\}$.
Ω^c is the complement of Ω.
By x' we will mean the vector $(x_2, x_3, ..., x_n)$, and sometimes write $x = (x_1, x')$.
By spt$u$ we will mean the support of u, that is the closure of $\{u \neq 0\}$.

1.4. **Main Results.**

Theorem 1. *There exists non-homogeneous global, quadratically bounded, solutions to the obstacle problem with boundary values*

$$f(x') = \lambda^2 (x_2)_+^2,$$

for $0 < \lambda < \frac{1}{\sqrt{2}}$.

It is noteworthy to mention that the solutions we construct are a "half space phenomenon" and cannot be extended to the entire space. This follows easily by the classification theorem of [CKS], which states that global solutions with large zero set must be a half-space solutions (i.e. one dimensional).

The case $\lambda > \frac{1}{\sqrt{2}}$ is less interesting from our point of view since if $\lambda > \frac{1}{\sqrt{2}}$ then the origin can't be a contact point of the free boundary. Let's indicate how to prove this. If we make a blow-up at a contact point where $\lambda > \frac{1}{\sqrt{2}}$ we should get a homogeneous solution with the same boundary values, we must obviously prove that the blow-up converges but we leave the details to the reader. But by writing the Laplacian in polar coordinates it is easily seen that such a solution doesn't exist.

These non-homogeneous solutions have bearings on the regularity theory of the free boundary near contact points with the fixed boundary, see [A], and also on the stability of the free boundary as the following Corollary states. But first we need a definition.

Definition 1. ***(Stability)*** *We say the normal of the free boundary, Γ_u, of a solution u of the obstacle problem is stable near the origin under small $C^{1,1}$ perturbations, if for any sequence $g_j \in C^{1,1}(\partial B_1^+)$ such that*
(i) $\|g_j\|_{C^{1,1}} \to 0$,
(ii) w_j the solution to the obstacle problem with boundary data $u + g_j$ has the origin as a free boundary point,
we have, for each $x \in \mathbb{R}^2_+$,

$$\lim_{j\to\infty}\lim_{r\downarrow 0}\frac{w_j(rx)}{r^2} = \lim_{r\downarrow 0}\frac{u(rx)}{r^2}.$$

Remark: The condition that the origin is a free boundary point for the perturbated function w_j is essential, without that condition the left limit in the definition will not converge.

Corollary 1. *Let u be a solution to the obstacle problem in $\mathbb{R}^2$ with boundary values f satisfying*

$$\lim_{r\to 0}\frac{f(rx_2)}{r^2}=\lambda^2(x_2)_+^2$$

for any sequence $r\to 0$, assume also that $0\in\Gamma_u$. Then the normal of the free boundary is stable near the origin under small $C^{1,1}$-changes in f (that leaves the origin as a free boundary point) if u_0, the blow-up of u, is the smaller solution in Theorem 3 part 1.

The proof of this corollary will be given at the end of Section 3. It should also be pointed out that the normal of the free boundary isn't uniformly stable.

Finally we state a stability result in higher dimensions.

Theorem 2. *Let u be a solution to the obstacle problem in $\mathbb{R}^n$ whose blow-up is the homogeneous global solution with least Weiss energy Ψ (see the Weiss monotonicity formula below). Assume also that $0\in\Gamma_u$. Then the normal of Γ_u is stable, near the origin, under small $C^{1,1}$-perturbations that leave the origin as a free boundary point.*

Obviously Corollary 1 is included in Theorem 2. But the impact of non-homogeneous solutions on local stability is more explicitly formulated in the proof of Corollary 1, so we have chosen to formulate the two dimensional case separately to accentuate our ideas more clearly.

2. Technical Tools and Known Results.

2.1. **Technical Tools.** In this subsection we gather some known results and tools which we will use later on in this investigation. First and foremost we state the following Lemma essentially due to G.S. Weiss [W], see [A] for a proof in this case.

Weiss Monotonicity Function: *Let u solve the obstacle problem in B_R^+, with f homogeneous of degree two, then the function*

$$\Psi(r,u)\equiv r^{-n-2}\int_{B_r(0)\cap\mathbb{R}^n_+}(|\nabla u|^2+2u)-r^{-n-3}\int_{\partial B_r(0)\cap(R)^n_+}2u^2 \tag{5}$$

is nondecreasing in $r<R$.

Remarks: Besides the monotonicity the Weiss energy functional, Ψ, has some other important properties, which we would like to accentuate.

(1) Let u solve the obstacle problem and define $u_r(x)=\frac{u(rx)}{r^2}$, then

$$\Psi(r,u)=\Psi(1,u_r).$$

This is shown by a simple change of variables.

(2) By the previous paragraph we see that if u is homogeneous of second degree then $\Psi(\cdot,u)$ is constant. The converse implication is also true. That is, if $\Psi(r,u)$ is constant in r then u is homogeneous of second degree.

(3) $\Psi(r,u)$ is obviously continuous in C^1 changes in u.

By (4) we will have, by standard estimates, that a blow-up sequence will converge in $C^{1,\alpha}$ locally to a global solution.

2.2. **Known Results.** An important tool in our discussion is the following comparison principle. See for instance [B] for a proof.

Comparison Principle: *Let u and v be two solutions to the obstacle problem in D, and assume that $u \geq v$ on the boundary ∂D, then $u \geq v$ in D, in fact*

$$0 \leq u - v \leq \sup_{\partial D}(u - v).$$

We will also need the following uniqueness result for the blow-up of the obstacle problem, see [A] for proof.

Uniqueness of Blow-up. *Let u be a solution of the obstacle problem, with the blow-up of $f \in C^{1,1}$, $f_0 = \lambda^2(x_2)_+^2$ for $\lambda \leq \frac{1}{\sqrt{2}}$. Then the blow-up limit of u is unique.*

The author of [A] also gives the following classification of blow-ups and shrink downs.

Theorem 3. *Let u solve the obstacle problem and let $f_0 = \lambda^2(x_2)_+^2$ be a blow-up of boundary data f.*

(1) *Then*

$$u_0 \equiv \lim_{r\to 0} \frac{u(rx)}{r^2} = \big(\pm \sqrt{\frac{1}{2} - \lambda^2} x_1 + \lambda x_2 \big)_+^2,$$

$$u_\infty \equiv \lim_{r\to \infty} \frac{u(rx)}{r^2} = \big(\pm \sqrt{\frac{1}{2} - \lambda^2} x_1 + \lambda x_2 \big)_+^2,$$

where the existence of the limit is assured by the preceding Lemma. Moreover $u_\infty \geq u_0$

(2) *If u is a homogeneous global solution then*

$$u = \big(\pm \sqrt{\frac{1}{2} - \lambda^2} x_1 + \lambda x_2 \big)_+^2.$$

3. Proof of the Main Theorems.

To prove Theorem 1 we only have to construct a non-homogeneous global solution for each λ. However for simplicity of notation, and definiteness, we will construct a non-homogeneous solution for $\lambda = \frac{1}{2}$, the argument is exactly the same for any other $0 < \lambda < \frac{1}{\sqrt{2}}$.

For $\lambda = \frac{1}{\sqrt{2}}$ there exists only homogeneous global solutions. This is an easy consequence of Theorem 3. By that theorem the blow-up and shrink-down must be the same function and thus homogeneous by Weiss monotonicity function.

Proof of Theorem 1: Consider $u = \frac{1}{4}(x_1 + x_2)_+^2$, which is a solution to the obstacle problem in B_1^+ with its restriction to the boundary as boundary values.

Let u_ϵ be the solution to the obstacle problem with $(1-\epsilon)u$ as boundary values. By the comparison principle we have

$$u_\epsilon \leq u. \tag{6}$$

If we blow up u_ϵ, we have only two possible limits by Theorem 3. By (6) we must have the solution with "-"-sign. This means that the blow-up of u_ϵ will be the smaller function, and this is true for every $\epsilon > 0$.

This means that Γ_{u_ϵ} will approach the x_2-axis in an angle of approximately $\pi/4$. We will also have $0 \in \Gamma_{u_\epsilon}$. The free boundary, Γ_{u_ϵ}, will also have a contact point

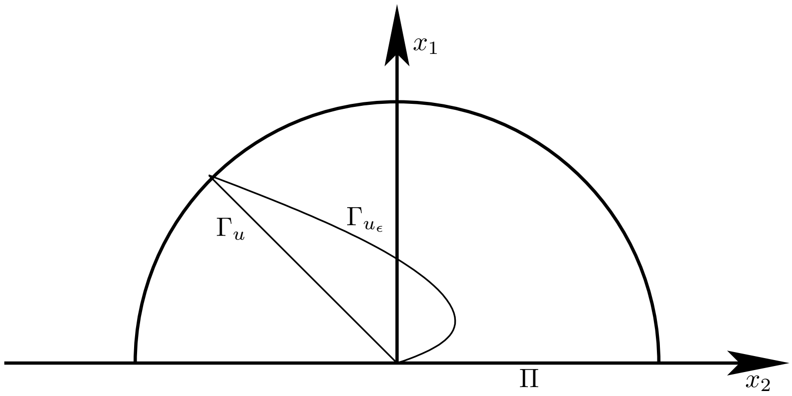

FIGURE 1

with the fixed boundary at $(1/\sqrt{2}, -1/\sqrt{2})$. So the free boundary of u_ϵ will have a point on the x_1-axis. Denote that point x_ϵ, and its norm $|x_\epsilon| = r_\epsilon$. By the comparison principle $u_\epsilon \to u$, and therefore $r_\epsilon \to 0$ by the non-degeneracy lemma.

Now make the following blow-up,

$$\tilde{u}_\epsilon(x) = \frac{u_\epsilon(r_\epsilon x)}{r_\epsilon^2},$$

and note that $x_\epsilon / r_\epsilon = (1,0) \in \Gamma_{\tilde{u}_\epsilon}$. A subsequence of $\tilde{u}_\epsilon$ will converge to a global solution u_0, with $(1,0) \in \Gamma_{u_0}$. By Theorem 3 there is no homogeneous solution with $(1,0) \in \Gamma_u$. So we can conclude that u_0 is a global non-homogeneous solution. □

Instability of larger solutions: We would like to point out that u_ϵ is a small perturbation of u in B_1^+, but still the change in the normal derivative of Γ_u will be uniformly large for each ϵ, that is Γ_u is not stable near the origin when $u = \frac{1}{4}(x_1 + x_2)_+^2$.

We will use Weiss monotonicity formula to prove the corollary, but first we need a lemma.

Lemma 1. *Let u be the homogeneous solution of the obstacle problem in B_1^+ with least Weiss energy, and prescribed boundary data $f = \lambda^2 (x_2)_+^2$ on Π. Then the normal of Γ_u is stable near the origin under small $C^{1,1}$ changes in the boundary values on ∂B_1^+, that leave the origin as a free boundary point.*

Proof: If we make a small $C^{1,1}$ perturbation of the boundary values and find a solution u_ϵ corresponding to the perturbed boundary values. Then by the stability of the Weiss energy under C^1 changes in u we will get $|\psi(1/2, u) - \psi(1/2, u_\epsilon)|$ is small. Then by the monotonicity of the Weiss functional we can deduce that the blow-up of u_ϵ is a function with Weiss energy close to the Weiss energy of u. But this will exclude that the blow-up of u_ϵ is the larger solution in Theorem 3. □

Proof of Corollary 1: If we have a solution of the obstacle problem whose blow-up is the smaller solution in Theorem 3 with a free boundary whose normal isn't stable under small $C^{1,1}$-changes in the boundary values. Then we can find a sequence of

$C^{1,1}(\partial B^1_+)$-functions g_j with $\|g_j\|_{C^{1,1}} \leq \frac{1}{j^3}$ such that the change in the normal of the free boundary at the origin is larger than ϵ under g_j-perturbations of the boundary values.

By making a blow-up $u_j(x) = u(jx)/j^2$ and a similar blow-up of the perturbed functions we see that this contradicts the preceding lemma. Therefore Γ_u must be stable. □

4. Stability Results in Higher Dimensions.

In the previous section we gained some understanding of the stability behavior of the free boundary near the origin. We will show in this section that we can deduce the same results even for higher dimensions.

The major problem is that there is no classification of global homogeneous solutions in higher dimensions so we have to deduce our results directly from the Weiss monotonicity formula.

Proof of Theorem 2: As in the proof of Corollary 1 it is enough to show that the global homogeneous solutions with least Weiss energy are stable near the origin.

So let u be a homogeneous solution, with least Weiss energy $\Psi(1,\cdot)$, among all functions with the same boundary values on Π. Make a small $C^{1,1}$-perturbation in the boundary values, and call the perturbed function u_ϵ. Since $\Psi(1,\cdot)$ is continuous in C^1-changes in its second argument it means that $|\Psi(1,u)-\Psi(1,u_\epsilon)| < \rho(\epsilon)$, where $\rho(\cdot)$ is the modulus of continuity of $\Psi(1,\cdot)$.

The proof is now very simple, since $u_\epsilon \to u$ in $C^{1,\alpha}$ when $\epsilon \to 0$. We know that $\Psi(1,u_\epsilon) \to \Psi(1,u)$. But the monotonicity in $\Psi(\cdot,u_\epsilon)$ we know that blow-ups of u_ϵ, call them $(u_\epsilon)_j$ must converge to a function $(u_\epsilon)_0$ with less Weiss energy $\Psi(1,(u_\epsilon)_0)$ in the unit ball. But since $\Psi(1,u)$ is the least possible energy to have in the unit ball with the same boundary values on Π as u we can deduce that the blow-ups $(u_\epsilon)_0 \to u$ as $\epsilon \to 0$. □

5. Examples in $\mathbb{R}^3_+$

In this section we will give some examples of global homogeneous solutions in $\mathbb{R}^3_+$. Let us begin with a trivial example.

Example 1: If the boundary value on Π is $\frac{x_2^2}{6} + \frac{x_3^2}{6}$. We have the polynomial solution $\frac{x_1^2}{6} + \frac{x_2^2}{6} + \frac{x_3^2}{6}$.

Example 2: In the case described we can also find another solution simply by writing the Laplacian in polar coordinates (r,ϕ,θ) and assume rotational symmetry, $u(x_1,x_2,x_3) = u(r,\theta)$. If u is homogeneous of second degree then $\Delta u = \chi_{\Omega_u}$ reduces to an ordinary differential equation in polar coordinates. The homogeneous solutions to the ODE are

$$\begin{aligned} &u_1(\theta) = (1 - 3\cos^2(\theta)) \\ &u_2(\theta) = \tfrac{1}{8}(1 - 3\cos^2(\theta)) \ln\left(\left|\cos(\theta) - 1\right|\right) - \tfrac{3}{4}\cos(\theta). \end{aligned}$$

Now for any $\theta_0 \in (0,\pi/2)$ we can choose a and b in such way that

$$u(r,\theta) = \begin{cases} \frac{r^2}{6} + ar^2 u_1(\theta) + br^2 u_2(\theta) & \text{if } \theta > \theta_0 \\ 0 & \text{if } \theta \leq \theta_0 \end{cases}$$

is a solution to the obstacle problem. The zero set of this function is a cone with opening angle θ_0, vertex in the origin and the central axis is the x_1-axis.

Example 3: Let us now turn to an example of a solution which is unstable. By the previous example we can find rotational symmetric solutions with zero set as a cone.

If we for instance take $a \approx 0.0775$ and $b \approx 0.207$ in the above equation we will get a solution to the obstacle problem with zero set $\Omega_u^c = \{\theta \in [-0.5, 0.5]\}$.

This solution is in fact a solution of the obstacle problem in any domain excluding the negative x_1-axis along which it has a singularity. In particular we can rotate the solution around the x_2-axis until the free boundary contains the positive x_3-axis. Restrict this rotated function to the upper half space and call it v.

This solution v will only have $C^{1,1}$ boundary data, but the boundary data is a strict super solution to $\Delta u = 1$ (here we have to use the particular values for a and b). It follows from the comparison principle that there exists another homogeneous solution to the obstacle problem with those boundary data with the x_1-axis in its zero set. So we have at least two global homogeneous solutions with those boundary data. The one with most Weiss energy will be unstable.

References

[A] J. Andersson, *On the regularity of a free boundary near contact points with a fixed boundary.* Submitted.

[B] I. Blank, *Sharp Results for the Regularity and Stability of the Free Boundary in the Obstacle Problem.* Indiana Univ. Math. J. 50 (2001), no. 3, 1077–1112.

[C] L.A. Caffarelli, *The Obstacle Problem Revisited.* J. Fourier Anal. Appl. 4 (1998), no. 4-5 383-402.

[CKS] L.A. Caffarelli, L. Karp, and H. Shahgholian, *Regularity of a free boundary with application to the Pompeiu problem* . Ann. of Math. (2) 151 (2000), no. 1, 269-292.

[F] A. Friedman, *Variational principles and free-boundary problems.* Robert E. Krieger publishing company, Malabar Florida, 1988 .

[Sh] H. Shahgholian, *On quadrature domains and the Schwarz potential.* J. Math. Anal. Appl. 171 (1992), no. 1, 61–78.

[SU] H. Shahgholian, N. Uraltseva, *Regularity properties of a free boundary near contact points with the fixed boundary.* Duke Math. J. 116 (2003), no. 1, 1-34

[W] G.S. Weiss, A homogeneity property improvement approach to the obstacle problem. Invent. Math. 138 (1999), no. 1, 23-50.

Department of Mathematics, Royal Institute of Technology, 100 44 Stockholm, Sweden

E-mail address: johnan@math.kth.se

E-mail address: henriksh@math.kth.se