QUADRATURE DOMAINS AND BROWNIAN MOTION (A HEURISTIC APPROACH)

TO HAROLD S. SHAPIRO ON THE OCCASION OF HIS 75TH BIRTHDAY

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Abstract. In this note we will make an attempt to link the theory of the so-called quadrature domains (QD) to stochastic analysis. We show that a QD, with the underlying measure μ, can be represented as the set of points x, for which the expectation value (average reward)

$$E^x\left(-\theta+\int_0^\theta \mu(X_t)\right),$$

is positive for some (bounded) stopping time θ. Here X_t denotes the Brownian motion starting at the point x, and E^x denotes the expectation with respect to the underlying probability measure P^x.

1. SETTING AND BACKGROUNDS

Our objective in this note is to find a stochastic interpretation of the so-called quadrature domains. To fix the idea, let D be a bounded domain in $\mathbf{R}^n$ $(n \geq 2)$, and

$$\mu = M\chi_D, \qquad (M > 1) \tag{1.1}$$

where χ_D is the characteristic function. One can consider a more general class of functions, or measures μ. However, for simplicity and clarity we stick to the case of multiples of characteristic functions.

A quadrature domain (QD) with respect to the function μ, and a class of functions H, is a bounded domain Ω with properties:

$$D \subset \Omega, \qquad \int_\Omega h dx \geq \int h d\mu \qquad \forall\ h \in H. \tag{1.2}$$

Actually the more restrictive condition $\overline{D} \subset \Omega$ is prefered, but this in general is much harder to achieve.

We adopt the notation

$$\Omega \in QD(\mu, H)$$

to denote that Ω is a QD w.r.t. μ and the class H. Throughout this paper we mainly consider the class of integrable subharmonic functions over Ω, denoted by $SL^1(\Omega)$.

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A PDE point of view of QD is the consideration of the function (also called the modified Schwarz potential (MSP))

$$u(x) := c_n \int_\Omega |y-x|^{2-n}\,(dy - d\mu_y) \qquad x \in \Omega. \tag{1.3}$$

Here we have assumed $n \geq 3$, and for $n = 2$, we replace $|y-x|^{2-n}$ with $-\log|y-x|$. Also, the constant c_n is a normalization factor.

Elementary PDE then tells us that u, the MSP, satisfies

$$\Delta u = \chi_\Omega - \mu, \qquad u \geq 0, \quad \text{in } \mathbf{R}^n, \qquad u = 0 \quad \text{in } \mathbf{R}^n \setminus \Omega, \tag{1.4}$$

where the Laplacian is taken in the sense of distributions.

Conversely if (1.4) holds for a triple (u, μ, Ω) then we can show, using Green's identity, that $\Omega \in QD(\mu, SL^1)$.

So we have

$$\Omega \in QD(\mu, SL^1) \qquad \Longleftrightarrow \qquad (u, \Omega, \mu) \quad \text{solves (1.4)}. \tag{1.5}$$

We refer the reader to the papers [Sak83], [Gus90], and [GS], for background and further results.

2. Brownian Motion and stopping times

In this section we will recall some definitions and facts about Brownian motion. Let $W_t = (W_t^1, \cdots, W_t^n)$ be a standard Brownian motion in $\mathbf{R}^n$, i.e. for each $j = 1, \cdots, n$, W_t^j is a real-valued, continuous stochastic process with independent and stationary increments. Being standard means

$$W_0 = 0, \qquad E(W_t^j) = 0, \qquad E((W_t^j)^2) = t, \qquad j = 1, \cdots, n.$$

Let us now fix a point in $\mathbf{R}^n$, and consider a Brownian motion X_t starting at x, i.e., $X_t = W_t + x$.

Let P^x denote the underlying probability measure, and E^x, the mathematical expectation w.r.t. P^x. Let also $\mathcal{F}_t$ denote the natural filtration of increasing family of σ-algebras generated by W_t.

A stopping time τ is a random variable with values in $\mathbf{R}^+ \cup \infty$, and such that

$$\{\tau \leq t\} \in \mathcal{F}_t.$$

The first exit time of a bounded domain D,

$$\tau_D := \inf\{t > 0,\ X_t \notin D\},$$

is a stopping time.

Now for a given bounded function μ and a (finite valued) stopping time θ, we consider the expected reward of stopping the process X_t at θ

$$U_\theta(x) = U_{\theta,\mu}(x) := E^x\left(-\theta + \int_0^\theta \mu(X_t)\,dt\right). \tag{2.1}$$

Observe that if $\theta \equiv 0$, then $U_\theta(x) = 0$.

Remark 2.1. For a game theoretic interpretation see the last section.

Our prime goal will be to show that if we take the supremum value of U_θ over all finite stopping times θ,

$$\sup_\theta U_\theta(x) \tag{2.2}$$

then the resulting function is (a multiple of) the MSP and the (interior of the) support of this function is a QD for the measure μ.

However, before moving on into the next section, and finding out about relations between Brownian motion and QD, we need to recall a couple of facts in Stochastic PDE.

We start with the infinitesimal generator of the Brownian motion, and recall from [Dyn] (see also [Oks]) that the operator

$$\frac{1}{2}\Delta$$

is the infinitesimal generator of the n-dimensional Brownian motion B_t. In other words,

$$\frac{1}{2}\Delta f(x) = \lim_{t\to 0} \frac{E^x(f(X_t)) - f(x)}{t} \qquad x \in \mathbf{R}^n,$$

which (by integration and using that $X_0 = x$) implies the Ito/Dynkin formula

$$E^x\left(\frac{1}{2}\int_0^\tau \Delta f(X_s)ds\right) = E^x(f(X_\tau)) - f(x), \tag{2.3}$$

for all functions f, with bounded Laplacian, and all bounded stopping times τ, with $E^x(\tau) < \infty$. A good source of reference to this formula is [Dyn], Chapter 5.

3. Connection between MSP and the expected reward

Before we start to show connections between QDs and expected reward, we should mention that there is a vast literature on the topic of variational inequalities and stochastic differential equation. A quite fresh reference is the book of B. Øksendal [Oks], and also nice (but unpublished) lecture notes by L.C. Evans [Ev]. We also bring the reader's attention to the (by now classical) book of A. Bensoussan and J.L. Lions [BL].

The variational formulation (in complementary form) of a QD, as we deduced above (1.4), is very well studied. The author of this note does NOT claim solving such problems for the first time. However, we believe that the (only) novelty of this paper is indeed the simple observation of the connection between QD and the Brownian motion, from a variational point of view.

Let us start posing a couple of questions, before formulating any results. After answering these questions, we will formulate a main theorem just for the sake of future references.

Problem 1. For each bounded stopping time θ set

$$\Omega_\theta := \{x \in \mathbf{R}^n :\ U_\theta(x) > 0\}, \qquad \Omega := \bigcup_\theta \Omega_\theta,$$

where U_θ is as in (2.1). Show that Ω is bounded.

Heuristically this seems obvious. First, one observes that for $U_\theta(x)$ to be positive, we need that the Brownian path enters the support of μ. Now if a point x is far from the support of μ, in this case $\overline{D}$, then the Brownian path will need much longer time to reach the set D. In other words, when x is far away from D, then the probability to reach D decreases. Moreover we need θ to be large to reach the set D. Since

$$U_\theta(x) = -E^x(\theta) + ME^x\left(\int_0^\theta \chi_D(X_t)dt\right),$$

intuitively, the negative part should dominate, for θ large. And one expects this to be negative if θ is large. See below for the rigorous proof.

Problem 2. Prove that $D \subset \Omega$.

Let μ be as in (1.1). Then for $x \in D$, and $\theta = \tau_D$ (first exit time from D) we have

$$U_{\tau_D}(x) \geq (M-1)E^x(\tau_D) > 0. \tag{3.1}$$

Hence $D \subset \Omega$.

Actually, if D is somewhat smooth then one can show that $\overline{D} \subset \Omega$. Also if M is large enough then one may get the same result. These statements follow from the work of M. Sakai [Sak83]. The probabilistic way of seeing this is that if D is smooth, then the Brownian path, starting at $x \in \partial D$, has good chances of entering into D, for short time intervals. Hence the contribution of the term $M\int_0^\theta \chi_D$ can be significant in relation to the the negative term $-\theta$.

Problem 3. Define

$$u(x) := \frac{1}{2}\sup_\theta U_\theta(x),$$

where supremum is taken over all bounded stopping times θ. Show that u is a solution to (1.4). This, in view of (1.5), will show that $\Omega \in QD(\mu, SL^1)$.

Observe also that $u(x) \geq 0$, since we can always choose $\theta \equiv 0$.

Problem 4. Conversely, show that if $\Omega \in QD(\mu, SL^1)$ then u, the MSP of Ω as defined in (1.3), has the representation

$$u(x) = \frac{1}{2}\sup_\theta U_\theta(x).$$

Problem 5. Is there any stopping time θ, for which the supremum value, $\sup U_\theta$, is attained?

All the above problems are answered by the following theorem.

Theorem 3.1. *Let $\mu = M\chi_D$, with $M > 1$, and D a bounded set in $\mathbf{R}^n$. Then the maximal expected reward*

$$\frac{1}{2}\sup_\theta U_{\theta,\mu}$$

is the unique solution to the complementary problem (1.4). Moreover, the domain $\Omega := \{x : \sup_\theta U_{\theta,\mu} > 0\}$ is a (bounded) QD w.r.t. μ, and the supremum above is attained for the first exit time from Ω.

Proof. Let us start with small restriction of the class of stopping times. We first consider a fixed ball $B_R = B(0,R)$ with R large enough, and such that $D \subset B_R$. Then we consider all stopping times of the form

$$\theta_R = \min(\theta, \tau_R),$$

where τ_R is the first exit time from the ball B_R. Now we set

$$u_R(x) := \frac{1}{2}\sup_{\theta_R} U_{\theta_R}(x), \qquad \Omega_R = \{u_R > 0\}.$$

It is apparent from the definition of u_R, that if we define

$$2v_R := \sup_{\theta_R} E^x \left(-\theta_R + M \int_0^{\theta_R} \chi_{B_r}(X_t)dt \right),$$

with $D \subset B_r \subset B_R$, then $u_R \leq v_R$. Hence to show that Ω_R is uniformly bounded (independent of R) it suffices to do so for $\{v_R > 0\}$.

Define now $\tilde{v}_R$ according to

$$\tilde{v}_R = \frac{1}{2n}|x|^2 - \frac{M}{2n}|x|^2 + \frac{r^2}{2(n-2)}\left(M - M^{2/n}\right) \qquad \text{for } |x| \leq r$$

$$\tilde{v}_R = \frac{1}{2n}|x|^2 + \frac{Mr^2}{n(n-2)}|x|^{2-n} - \frac{r^2}{2(n-2)}M^{2/n} \qquad \text{for } r < |x| \leq \rho := rM^{1/n},$$

$$\tilde{v}_R = 0 \quad \text{in } B_R \setminus B_\rho,$$

where we have assumed $R > \rho$. Observe that also $\tilde{v}_R > 0$ in B_ρ, and $\tilde{v}_R$ is independent of R. Moreover, we have

$$\Delta \tilde{v}_R = \chi_{B_\rho} - M\chi_{B_r}. \tag{3.2}$$

According to standard theory of stochastic PDE one can show

$$v_R = \tilde{v}_R. \tag{3.3}$$

From here it follows that u_R ($\leq v_R$) has compact support, independent of R.

To prove (3.3), we see that by Ito's formula (2.3), and (3.2) we have (observe that $\theta_R = \tau_\rho$, the exit time from B_ρ)

$$E^x \left(-\tau_\rho + M \int_0^{\tau_\rho} \chi_{B_r}(X_t)dt \right) = E^x \left(\int_0^{\tau_\rho} -\Delta \tilde{v}_R(X_s)ds \right) = 2E^x(-\tilde{v}_R(X_{\tau_\rho})) + 2\tilde{v}_R(x).$$

Since $X_{\tau_\rho} \in \partial B_\rho$, and $\tilde{v}_R = 0$ on ∂B_ρ we are left with

$$E^x \left(-\tau_\rho + M \int_0^{\tau_\rho} \chi_{B_r}(X_t)dt \right) = 2\tilde{v}_R(x). \tag{3.4}$$

Hence $v_R \geq \tilde{v}_R$. Next by Ito's formula, (3.2), and $\tilde{v}_R \geq 0$, we have

$$2\tilde{v}_R(x) = E^x \left(\int_0^{\theta_R} -\Delta \tilde{v}_R(X_s)ds \right) + 2E^x(\tilde{v}_R(X_{\theta_R}))$$

$$\geq E^x \left(\int_0^{\theta_R} \left(\chi_{B_\rho} - M\chi_{B_r}\right)(X_t)dt \right),$$

for all θ_R. This gives the desired result.

From now on one can consider the problem in B_ρ, and we may replace θ_R by θ_ρ.

Next, we see that by (3.1) we have $D \subset \Omega$.

Finally, in order to prove that u solves the complementary problem (1.4), we need to show that there exists a unique solution $\tilde{u}$ to the complementary problem and that it can be represented by

$$u(x) = \frac{1}{2}E^x \left(-\tau_G + \int_0^{\tau_G} \mu(X_t)\ dt \right) = \frac{1}{2}\sup_{\theta_\rho} U_{\theta_\rho}(x),$$

with τ_G being the first exit time from $G := \{\tilde{u} > 0\}$.

The existence and uniqueness of a solution to (1.4) has been shown earlier by [Sak83]. Cf. also [Gus90], [GS]. In general one can consider either a penalized version or a variational form of the problem at hand.

Now, having a unique solution to (1.4), we can make a similar analysis as above to obtain $u_\rho = \tilde{u}$. □

Remark 3.2. We give a direct heuristic prove of the statement that u solves (1.4). Let $\delta > 0$ and suppose that the system runs at least until time δ. Then the new state at time δ is X_δ, and the optimal cost after this time is $u(X_\delta)$. Hence

$$2u(x) = \sup_{\theta_R} E^x \left(-\theta_R + M \int_0^{\theta_R} \chi_D(X_s) ds \right) \geq E^x \left(-\delta + M \int_0^{\delta} \chi_D(X_s) ds \right) + 2u(X_\delta).$$

Now using Ito's formula for $u(x)$, we obtain

$$E^x \left(\int_0^\delta -\Delta u(X_s) ds \right) + 2E^x(u(X_\delta)) = 2u(x) \geq E^x \left(-\delta + M \int_0^\delta \chi_D(X_s) ds + 2u(X_\delta) \right),$$

i.e.,

$$E^x \left(\int_0^\delta -\Delta u(X_s) ds \right) \geq E^x \left(-\delta + M \int_0^\delta \chi_D(X_s) ds \right).$$

Dividing by δ and letting δ tend to zero we end up with

$$E^x \left(-\Delta u(X_0) \right) \geq E^x \left(-1 + M\chi_D(X_0) \right),$$

i.e.,

$$-\Delta u(x) \geq -1 + M\chi_D(x).$$

Observe that the pointwise Laplacian $\Delta u(x)$ may not exist, and the above argument needs to be carried out in the weak sense.

Remark 3.3. It is also noteworthy that if ∂D satisfies an interior sphere condition, then the explicit form of v_R can be used to show that $\overline{D} \subset \Omega$. To see this, let $B(y, s) \subset D$ be any ball touching ∂D at some point(s). Then $u_R \geq w_R$, where

$$2w_R := \sup_{\theta_R} E^x \left(-\theta_R + \int_0^{\theta_R} \chi_{B(y,s)}(X_t) dt \right).$$

Moreover, w_R is given (as above) by

$$w_R = \frac{s^2}{r^2} \tilde{v}_R\left(\frac{r(x-y)}{s}\right).$$

Hence $B(y, M^{1/n}s) \subset \Omega$.

4. Game theoretic interpretation

In this section we try to make a game theoretic interpretation of (2.2). However, at this time, we have no particular game in mind, played and run, by the rules U_θ. But the possibility always exists.

Suppose there is a game offered (nowadays on the internet) and the rules for playing it are governed by (2.2). Suppose further that the state of affairs offered by the game are $X_t \in \mathbf{R}^n$. This can mean almost anything, depending on the game.

Now a gambler enters into the game (at time $t = 0$) with state of affairs $X_0 = x$ (the player bets on this state). Now if the gambler chooses not to start the process, and pulls out, there will be neither loss nor win; $U_0(x) = 0$. If the gambler lets the game start and continues for a certain time length $t = \delta$, then the amount the gambler gains or looses (depending on the sign of U_δ) is $|U_\delta(x)|$.

Several questions arise:

Q1) *Can the gambler ever win?*

Q2) *What is the best time, for the gambler, to quit the game?*

Q3) *What state of affairs x should the gambler choose, for the best outcome?*

Q4) *Is it possible that the gambler never wins?*

These questions are important to both the gambler and those who offer the game. E.g., if the gambler chooses a state $x \in D = \text{Interior}(\ \text{supp}\mu)$ then obviously he wins for short time intervals $t = \delta$, provided

$$X_t \in D \qquad \forall\ 0 < t < \delta.$$

Indeed, $U_\delta(x) = (-1 + M)\delta$. Hence he can always choose to stop the process before the exit time from D, τ_D.

On the other hand, if the gambler starts at points far from D then $U_\delta(x) = -\delta$, as long as X_t has not entered into D, for $t \leq \delta$.

A third situation is that if the gambler starts at ∂D, then the outcome is not so clear anymore.

Naturally the company offering the game has to make sure that no arbitrage, i.e. risk-free profit, takes place. So the state of affairs $x \in D$ should be out of question. The same should go for affairs $x \in \overline{D}^c$, since there is an immediate loss for the gambler, and no guarantee of future gains.

There can be given many variants of combinations of state of affairs such that the loss/gain of short time intervals is not obvious to see. E.g. if the state $x \in \partial D$ as suggested above then it is not apparent what will happen after the game starts. One can also consider a strict rule from the company that for any chosen affair $x \in D$ one should chose a second affair y a certain distance d_x, away from D.

Let us now see what kind of strategy should the gambler choose, once he has started at a point $X_0 = x$. According to what we know from last section, if $x \notin \Omega$, then the gambler will never win. So the best is that he does not enter the game at such states. If on the other hand $x \in \Omega$ (observe that the location of Ω is not known in advance) then the best he can do is to continue until $t = \tau_\Omega$, a value unknown to the gambler. At this time the reward is the highest possible.

The set Ω (here called a QD) is called the continuation region and the complement of it the stopping region. Also at time t the gambler has access to all information on the past and present state but not on the future state. Mathematically, this is defined in terms of the σ-algebra $\mathcal{F}_t$. Also the decision/control variable θ is then a stopping time.

The problem of finding the value τ_Ω is naturally of great importance to both parties in the game.

References

[BL] A.Bensoussan, J.L.Lions, *Applications of variational inequalities in stochastic control.* Translated from the French. Studies in Mathematics and its Applications, 12. North-Holland Publishing Co., Amsterdam-New York, 1982.

[Dyn] E.B.Dynkin, *Markov processes. Vols. I, II.* Translated with the authorization and assistance of the author by J. Fabius, V. Greenberg, A. Maitra, G. Majone. Die Grundlehren der Mathematischen Wi ssenschaften, Bnde 121, 122 Academic Press Inc., Publishers, New York; Springer-Verlag, Berlin-Göttingen-Heidelberg 1965 Vol. I: xii+365 pp.; Vol. II: viii+274 pp.

[Ev] L.C.Evans *An introduction to stochastic differential equations.* Lecture notes (Version 1.2) for an advanced undergraduate course. Can be found at: http://math.berkeley.edu/ evans/SDE.course.pdf

[Gus90] B.Gustafsson *On quadrature domains and an inverse problem in potential theory.* J. d'Analyse Math. vol 55, 1990, 172–216.

[GS] B.Gustafsson, H.Shahgholian *Existence and geometric properties of solutions of a free boundary problem in potential theory.* J. Reine Angew. Math. 473 (1996), 137–179.

[Oks] B.Øksendal, *Stochastic differential equations. An introduction with applications.* Fifth edition. Universitext. Springer-Verlag, Berlin, 1998.

[Sak83] M.Sakai *Application of variational inequalities to the existence theorem on quadrature domains.* Trans. Amer. Math. Soc. 1983, vol. 276, pp 267–279.

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