THE STRUCTURE OF THE SINGULAR SET OF A FREE BOUNDARY IN POTENTIAL THEORY

LUIS A. CAFFARELLI AND HENRIK SHAHGHOLIAN

Abstract. In this paper we characterize the structure of the singular set in the following free boundary problem

$$(\Delta u - f)u = 0, \qquad \text{in} \quad B = B(0,1),$$

where f is Lipschitz, and $u \in W^{2,p}(B)$, $p > n$. The free boundary $\partial\Omega$, represented by $\partial\{\Delta u = f\}$, appears in certain problems in geophysics and inverse problems in potential theory.

1. Introduction

Let Ω be a domain in $\mathbf{R}^n$ $(n \geq 2)$, and f a Lipschitz function in $B = B(0,1)$ with $f(0) > 0$. Suppose there exists a function $u \in W^{2,p}(B)$ such that

$$\Delta u = f\chi_\Omega \quad \text{in } B, \qquad u = 0 \quad \text{in } B \setminus \Omega, \qquad 0 \in \partial\Omega. \tag{1.1}$$

Then, we are interested in the regularity of the free boundary $\partial\Omega$. In a recent work [CKS] the authors and L. Karp proved that there exists a modulus of continuity $\sigma(r)$ ($\sigma(0^+) = 0$ and it depends on the supremum-norm of u) such that if for some $r < 1$ the set

$$\{u = |\nabla u| = 0\}$$

(after suitable rotation) has points outside the strip

$$\{-r\sigma(r) < x_1 < r\sigma(r)\}$$

then, locally near the origin, the free boundary in (1.1) (with $f \equiv 1$) is the graph of a C^1-function. From this the real analyticity of the free boundary, near the origin, follows by classical results ([KN], [I]).

The free boundary obviously develops singularity at points where this condition fails. For convenience we refer to these points as singular points, and hence the singular set, here denoted by S_u, is the union of all such points. It is our objective here to study the singular set S_u in (1.1), at least in the half ball $B_{1/2}(0)$; this is tacitly understood throughout the paper. Before presenting our main result we give an example of free boundaries where cusp points are developed.

Example. ([KN], [Sc], [Sa1]; cf. also [Sa2]) Without deepening into details we recall from [KN page 387–390] that in two space dimensions one can give examples

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of the free boundary in (1.1), where cusps appear. These cusps are represented by the curves

$$x_2 = \pm x_1^{\mu/2}, \qquad 0 \le x_1 \le 1,$$

where $\mu = 4k + 1$, $(k = 1, 2, \cdots)$ gives non-negative solutions and $\mu = 4k + 3$, $(k = 0, 1, \cdots)$ gives solutions that become negative on the negative x_1-axis and near the origin. The solution is defined locally by

$$u(x) = x_2^2 - \frac{2}{1+\mu/2}\rho^{1+\mu/2}\sin(1+\mu/2)\theta + \cdots, \qquad x \in \Omega,\ |x| < \varepsilon,$$

for ε small. Here we've used both real and complex notation

$$x = (x_1, x_2), \qquad z = \rho e^{i\theta}, \quad 0 \le \theta \le 2\pi.$$

Also the domain Ω is the image of the set

$$\{z : \ |z| < 1,\ \mathrm{Im}\ z > 0\}$$

under the conformal mapping $f(z) = z^2 + iz^\mu$.

Let us now introduce some definitions.

Minimal diameter. The minimum diameter of a bounded set D, denoted $\mathrm{MD}(D)$, is the infimum of distances between pairs of parallel planes such that D is contained in the strip determined by the planes. We also define the density function

$$\delta_r(u) = \frac{\mathrm{MD}(\Lambda \cap B(0,r))}{r},$$

where $\Lambda := \{u = |\nabla u| = 0\}$.

Let now f be a Lipschitz function in $B(z,r)$ with Lipschitz norm $|f(x) - f(y)| \le C_1|x-y|$ in $B(z,r)$. In the sequel we'll assume $C_1 = 1$.

Local Solutions. We say a function u belongs to the class $P_r(z, C_0)$ if u satisfies (in the sense of distributions):

- $\Delta u = f\chi_\Omega$ in $B(z,r)$, $\big(\|f\|_{Lip,B(z,r)} \le 1, \text{ and } f(z) = 1\big)$,
- $u = |\nabla u| = 0$ in $B(z,r) \setminus \Omega$,
- $\|u\|_{\infty,B(z,r)} \le C_0$,
- $z \in \partial\Omega$.

Since the class $P_r(z, C_0)$ is translation invariant (only r changes), (1.1) may be considered in a neighborhood of any given point of the free boundary, and the results of [CKS], discussed earlier, can then be applied to every boundary point.

The following definition will be useful in declaring our main result.

Definition 1.1. We define the class of $n \times n$ matrices $\mathcal{M}$ as

$$\mathcal{M} = \{M_{n\times n} = (a_{ij});\ \mathrm{trace}(M) = 1,\ M = M^t\}.$$

In order to study the singular points we need to distinguish between singular points with different blow-ups. In other words if x^0 is a singular point of $\partial\Omega(u)$, then (as it will be clear later) any blow up of u at x^0 will be a polynomial $Q(x) = (x^t M x)/2$ with $M \in \mathcal{M}$. Now we want to classify the singular points in terms of the matrix M we obtain; more exactly in terms of the kernel of M. We give an exact definition of the singular points S_u introduced earlier.

Definition 1.2. For $u \in P_1(0, C_0)$, we say $x^0 \in S_u(a,k)$ if $x^0 \in \partial\Omega$, and there is a sequence $\{r_j\}$ such that functions $\{u_j\}$, where

$$u_j(x) = \frac{u(r_j x + x^0)}{r_j^2},$$

have a convergence subsequence to a polynomial $Q(x) = x^t A x$ with

$$A \in \mathcal{M} \qquad \dim(\mathrm{Ker}(A)) \leq k,$$

where we also assume that the eigenvalues are arranged as

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_{n-k}| \geq a, \qquad \text{and} \qquad |\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|.$$

As the definition goes x^0 may belong to different classes of singular sets, depending on different blow-ups. However, we'll show that this is not the case. Indeed, we prove that if $x^0 \in S_u(a_i, k_i)$ for $i = 1, 2$, then necessarily $k_1 = k_2$ and we may take $a_1 = a_2$.

Remark. The set $D := \{u = 0\} \setminus \{|\nabla u| = 0\}$, where u is a solution to (1.1), may be excluded from our analysis. Indeed, this set can be shown to lie (locally) in a C^1-manifold, as follows. First from the non-degeneracy (see [CKS; General Remarks]) it follows that $\partial\Omega$ has Lebesgue measure zero. A consequence of this is that $\Delta u = f$ in a neighborhood of D, so that u is $C^{2+\alpha}$. Therefore in considering the free boundary $\partial\Omega$, we may only look at points where also the gradient is zero, i.e., we need to consider only the set $\partial\Omega \cap \partial\overline{\Omega}$, since the set $\partial\Omega \setminus \partial\overline{\Omega}$ lies now in a C^2-manifold (see [Ref]).

The reader may consider, as an example, the function $u = x_1^2 - x_2^2/2$ in $\mathbf{R}^3$, which solves (1.1). Here one can consider $\Omega = \mathbf{R}^3$ and $\partial\Omega = \emptyset$, or $\Omega = \mathbf{R}^3 \setminus \{x_1 = x_2 = 0\}$ and $\partial\Omega$ is the x_3-axis. Also the function takes both positive and negative values near the free boundary.

Before stating our main result we'll recall two recent results in this filed that are very much pertinent to our analysis in this paper. The first result is about the regularity of the free boundary in (1.1).

Theorem 1.3. *([CKS]) Let u solve (1.1) with $|u| \leq C_0$, and $f \equiv 1$. Then $u \in C^{1,1}(B(0,1/2))$ and there exists a modulus of continuity σ ($\sigma(0^+) = 0$ and it depends on C_0) such that if $\delta_{r_0} > \sigma(r_0)$ for some $r_0 < 1/2$, then $\partial\Omega$ is the graph of a C^1 function in $B(0, c_0 r_0^2)$, where c_0 is some universal constant.*

The second result is about singular points of the free boundary in (1.1), with the extra condition that $u \geq 0$. This is due to the first author. We also refer to [CR] for some results in this direction.

Theorem 1.4. *(See [Ca3].) Let $u \geq 0$ be a solution to (1.1) with $f \equiv 1$. Suppose $x^0 \in S_u \cap B(0,1/2)$ and $|u| \leq C_0$. Then the following hold:*

a) *There exists a unique quadratic polynomial (and a unique matrix $M^{x^0} \in \mathcal{M}$)*

$$Q_u^{x^0} = \frac{1}{2}(x - x^0)^t M^{x^0} (x - x^0)$$

such that in some neighborhood of x^0

$$\sup_{B(x^0,r)} |u - Q_u^{x^0}| \leq r^2 \sigma(r).$$

Here σ is a universal modulus of continuity, depending on n, and C_0 only.
b) *M^{x^0} is continuous in x^0, and the kernel of M^{x^0} changes continuously in x^0. Moreover, the modulus of continuity of M^{x^0} is $\sigma(r)$, which appears in part a).*
c) *If dim(Ker(M^{x^0})) $= k$, then there exists a k-dimensional C^1-manifold $\Gamma_{x^0,u}$ such that*

$$S_u \cap B(x^0, r) \subset \Gamma_{x^0,u},$$

for some $r > 0$, depending on the singular point, and the smallest eigenvalue of M^{x^0}.

The dependents of the neighborhood on the smallest eigenvalue can be given by the following simple example in 3-dimensions

$$u(x) = \frac{1}{2}x_1^2 + (x_3 - \cos(1/x_2))^2 \, x_2^4.$$

Here , the singular set with kernel of dimension one, meanders into the singular set with kernel of dimension two, as the smallest eigenvalue degenerates to zero.

Most of the proof of Theorem 1.4 works perfectly in our case. There is only one, and a very crucial, point where it breaks, and we could not amend it; see [Ca3; proof of Lemma 14]. We will prove a slightly different and weaker version of this theorem.

Theorem 1.5. *(**MAIN**) For $u \in P_1(0, C_0)$ the following hold.*

(I) *Theorem 1.3 above holds true with Lipschitz f. The quantities, in general, depend also on the Lipschitz norm of f.*

(II) *For $x^0 \in S_u$ there exists a $(n-1)$-dimensional C^1-manifold $\Gamma_{x^0,u}$ such that*

$$S_u \cap B(x^0, r) \subset \Gamma_{x^0,u},$$

for some $r > 0$, depending on the constants n, C_0.

(III) *For $x^0 \in S_u(a, k)$ there exists a k-dimensional C^1-manifold $\Gamma_{x^0,u}$ such that*

$$S_u(a/2, k) \cap B(x^0, r) \subset \Gamma_{x^0,u},$$

for some $r > 0$, depending on the constants a, k, n, C_0.

(IV) *If $x^0 \in S_u$ and Q_1, Q_2 are two different blow-ups of u at x^0 then necessarily $Q_1 = Q_2$.*

(V) *In $B_{1/2}$, $\lim_{\Omega \ni x \to \partial\Omega} D_{ij}u$ exists for regular points of the free boundary, and it exists non-tangentially for singular points of the free boundary.*

It should be remarked that part (V) in Theorem 1.5 is the best result when $n \geq 3$. Indeed, it can be easily seen that if the origin is a singular point such that u_0 is a polynomial of at least two variables, and if $n \geq 3$ then the second derivatives of u are not necessarily continuous up to the origin. To see this, heuristically, let us consider a free boundary solutions where for any $r > 0$, the set $B_r \setminus \Omega$ has interior. Since the free boundary has zero Lebesgue measure we may assume that for any $r > 0$, the set $B_r \setminus \Omega$ contains a ball. This implies, in particular, that the part of free boundaries that can be touched by this balls, from $B_r \setminus \Omega$, are regular. Now take a sequence of such regular free boundary points $x^j \in \Omega$. Let us also for simplicity assume that the free boundary lies along the third coordinate axis x_3, so that the blow-up is $u_0(x) = a_1 x_1^2 + a_2 x_2^2$ and $\{u_0 = |\nabla u_0| = 0\}$ is the x_3-axis. Now it is easy to see that the normal vector ν_j and the tangent vector τ_j at x^j to $\partial\Omega$ will converge to ν_0 and τ_0, which are independent of x_3.

Now the point x^j being regular gives that $D_{\nu_j \nu_j} u = 1$ and $D_{\tau_j \tau_j} u = 0$. Next, having the blow-up $u_0(x) = a_1 x_1^2 + a_2 x_2^2$ and (supposedly) tangential continuity of the second derivatives we must have

$$0 = \lim_{x \to 0} D_{\tau_0 \tau_0} u(x) = 2a_1 \tau_0^1 + 2a_2 \tau_0^2 = 2|\tau_0|^2, \qquad \text{(tangentially)}$$

where $\tau_0 = (\tau_0^1, \tau_0^2)$.

Next by choosing different points x on the regular part of the free boundary, e.g. by going around the x_3-axis we may choose any vector τ_0 in the plain which comes from τ_j (this depends on the point). In particular choosing $\tau_0 = (a_1, a_2)$ we arrive at a contradiction with the above

$$0 = 2a_1 \tau_0^1 + 2a_2 \tau_0^2.$$

A basic tool in this paper will be the following monotonicity lemma.

Lemma 1.6. *(See [ACF; Lemma 5.1].) Let h_1, h_2 be two non-negative continuous sub-solutions of $\Delta u = 0$ in $B(x^0, R)$ $(R > 0)$. Assume further that $h_1 h_2 = 0$ and that $h_1(x^0) = h_2(x^0) = 0$. Then the following function is monotone in r $(0 < r < R)$*

$$\varphi(r) = \varphi(r, h_1, h_2, x^0) := \frac{1}{r^4} \left(\int_{B(x^0,r)} \frac{|\nabla h_1|^2}{|x - x^0|^{n-2}} \right) \left(\int_{B(x^0,r)} \frac{|\nabla h_2|^2}{|x - x^0|^{n-2}} \right).$$

The problem with Lemma 1.3 is that it does not apply when Δh_i is bounded from below. We intend to apply this lemma to the directional derivatives $D_e u$ of solutions to (1.1). Since $\Delta (D_e u)^{\pm} \geq -C$ we need a different version of Lemma 1.3.

The next lemma is a new type of monotonicity lemma. The advantage of it is that it relaxes the subharmonicity condition and allows the solutions to have bounded Laplacian only.

Lemma 1.7 (CJK; Theorem 1.3)**.** *Recall the assumptions in Lemma 1.3, and replace the subharmonicity assumption by the bounded-ness of the Laplacian of h_i, i.e., assume $\Delta h_i \geq -1$. Suppose moreover $|h_i(x)| \leq C|x|^\beta$ for some $\beta > 0$. Then*

$$\varphi(s_1) \leq (1 + s_2^\beta)\varphi(s_2) + C s_2^\beta, \tag{1.2}$$

where $0 < s_1 \leq s_2 \leq R$.

In Lemma 1.7 if we have $\Delta h_i \geq -C_i$ then we can replace h_i by h_i / C_i and change the constant C in (1.2). The reader may easily verify that any function verifying (1.2) must have a limit as $r \to 0^+$, i.e.,

$$\lim_{r \to 0^+} \varphi(r) = \text{exists} . \tag{1.3}$$

We refer to Lemma 1.6 as the monotonicity lemma and to Lemma 1.7 as the almost monotonicity lemma.

In the sequel, while applying the monotonicity formulas, we'll use the notation $\varphi(r, D_e u)$ with $u \in P_1(M)$ and h_1, h_2 replaced by $(D_e u)^{\pm}$. Here, e is a unit vector and

$$(D_e u)^+ = \max(D_e u, 0) \qquad (D_e u)^- = \max(-D_e u, 0).$$

Observe also that $(D_e u)^+$ are subsolutions.

Before continuing with our results we need to recall several facts about blow-up techniques. This will especially be helpful for non-specialists. We gather these in the below remark.

General Remarks. We will need several concepts as well as several facts that the non-specialist reader may be unfamiliar with. However, all these can be penetrated in literature and research papers; see e.g. [Ca1–3] and [CKS].

1) *Scaling:* For u a solution to (1.1) we set

$$u_r(x) = \frac{u(rx + x^0)}{r^2},$$

which is the so called "correct" scaling of u at $x^0 \in \partial\Omega$; since one expects u to behave quadratically near the free boundary.

2) *Global Solution:* A solution to $(\Delta u - 1)u = 0$ in $\mathbf{R}^n$, $u \in W^{2,p}_{loc}(\mathbf{R}^n)$, $(p > n)$with quadratic growth, is called a global solution.

3) *Blow-ups:* Let now Ω_r denote the set $\{x : rx \in \Omega\}$, and u_r be the scaling of u. If u is $C^{1,1}$ or even if $\sup_{B(0,r)} |u| \le Cr^2$ then we see that u_r is bounded and defined in $B(0,R)$ for any R, provided r is small enough. Hence by standard compactness methods in elliptic theory, since $\Delta u_r = \chi_{\Omega_r}$, we may let r tend to zero and obtain (for a subsequence) a global solution. This process is referred to as blowing up, and the global solution thus obtained is called a blow-up of u.

4) *Non-degeneracy:* The reader may have wondered what happens if the function u_r under the blow-up process converges (degenerates) identically to zero. Indeed, this can not happen due to the very simple fact that

$$\sup_{B(0,r)} u \ge \frac{r^2}{2n}.$$

The proof of this is standard and can be found in [Ca1]; observe that the assumption $u \ge 0$ in [Ca1] is superfluous (cf. [CKS; (4.1)]). Therefore

$$\sup_{B(0,R)} u_r(x) \ge \frac{R^2}{2n},$$

and thus the obvious non-degeneracy.

5) *Hausdorff measure of $\partial\Omega$:* It can be proven using techniques of [Ca3], that "locally" the free boundary $\partial\Omega$, has finite $(n-1)$-Hausdorff measure. See [Ca3; Corollary 4] and [CKS; General Remarks].

6) *Polynomial solution:* Next, consider a solution u to (1.1) which is also $C^{1,1}$ (by [CKS]). Then any blow-up sequence u_{r_j} of u that converges to a global solution has the obvious property that the blow-up limit u_0 has a quadratic growth near the infinity point. Now suppose the set $\{u = 0\}$ has empty interior. Then by the above $\mathbf{R}^n \setminus \Omega$ has zero Lebesgue measure. Hence $\Delta u_0 = 1$ almost everywhere. In particular, Liouville's theorem applies to conclude that u_0 is a homogeneous polynomial of degree two.

7) *$W^{2,p}$-convergence of blow-ups:* Suppose u is a solution and u_0 is a blow up of u through some sequence u_j. Then one may show that the convergence of u_j is not only in $C^{1,\alpha}$ but also in $W^{2,p}_{loc}(\mathbf{R}^n)$. This fact follows very easily by using certain properties of the solutions, such as non-degeneracy and that the set $\partial\Omega_0$ has zero Lebesgue measure One may even show that the free boundary $\partial\Omega_j$ converges to

$\partial\Omega_0$ in the usual Hausdorff metric. As a simple exercise from this it follows that the convergence of u_j to u_0 is in $W^{2,2}$; see [CKS; General Remarks].

This fact is used in the case of blowing up the solution in the monotonicity formula (Lemma 1.2), since here we need the convergence in $W^{2,2}$.

8) $\underline{\varphi(0^+, D_e u)}$: The limit value of the function $\varphi(r, D_e u)$, as r tends to zero, will play a crucial role in the analysis of singular points. In this part we will discuss some facts about $\varphi(0^+, D_e u)$. So suppose $0 \in S_u(a,k)$. Then there exists a blow-up of u at the origin giving rise to a polynomial solution

$$u_0(x) = \frac{1}{2}\sum_{i=1}^{n-k} \lambda_i x_i^2 = \frac{1}{2} x^t A x,$$

in a rotated system. Here A is the symmetric diagonal matrix with entries $a_{ii} = \lambda_i$. For convenience we'll also assume

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_{n-k}|,$$

with $|\lambda_{n-k}| \geq a$, and $\sum \lambda_i = 1$. From here it follows that

$$D_e u_0 = 2\sum_{i=1}^{n-k} \lambda_i e_i x_i, \qquad e = (e_1, \cdots, e_n).$$

Now if $e \in \mathrm{Ker}(A)$, then $D_e u_0 = 0$. When $e \notin \mathrm{Ker}(A)$ interesting things happen. Indeed, let e_λ be an eigenvector for A with eigenvalue λ. Define

$$K_\varepsilon(A, e_\lambda) := \{x : \ |x \cdot e_\lambda| > \varepsilon |x|\}, \tag{1.4}$$

then for $e = x/|x| \in K_\varepsilon(A, e_\lambda)$ we'll have

$$|\nabla D_e u_0|^2 = 4\sum_{i=1}^{n-k} (\lambda_i e_i)^2 \geq \frac{4}{n-k}\left(\sum_{i=1}^{n-k} |\lambda_i e_i|\right)^2 \geq \frac{4}{n-k}\lambda^2 \varepsilon^2.$$

In particular for φ as in the next lemma we obtain

$$\varphi(0^+, D_e u_0) = C\lambda^4 \varepsilon^4.$$

In a similar fashion, we may take a (lower dimensional) plane

$$\Pi_m = \{x_1 = \cdots x_m = 0\}, \qquad m \leq n-k,$$

to obtain for $e \notin \Pi_m$ the estimate

$$|\nabla D_e u_0|^2 = 4\sum_{i=1}^{n-k} (\lambda_i e_i)^2 \geq \frac{4}{n-k}\left(\sum_{i=1}^{n-k} |\lambda_i e_i|\right)^2 \geq \frac{4}{n-k}(\lambda_{(m-1)})^2 \varepsilon^2,$$

where ε is the angle between e and the projection of e on the (lower dimensional) plane Π_m.

A crucial fact that can be deduced, at this moment, is the simple fact that

$$|\lambda_1| \geq \max_i \lambda_i \geq \frac{1}{n}, \tag{1.5}$$

which in conjunction with the above analysis shows that (in a rotated system)

$$|\nabla D_{x_1} u_0|^2 \geq C\varepsilon^2,$$

independently of the function u. In particular for all other singular points near the origin (which itself is assumed to be singular) we must have this estimate.

9) Semi-continuity of $\|A^x e\|$: From the monotonicity formulas it follows that

$$\lim_{x\to 0} \|A^x e\| \le \|A^0 e\|.$$

For a detailed proof one may apply [Ca3; Corollary 10] in an obvious manner. This in particular implies, at least heuristically at this moment, that

$$\mathrm{Ker}(A^0) \subset \mathrm{Ker}(\lim_{x\to 0} A^x).$$

Observe that at this moment we don't know whether the limit $\lim_{x\to 0} A^x$ exists.

2. Technical Lemmas

From General Remarks above, it follows that any blow-up of a solution to (1.1) at some singular point must be a polynomial solution. We show next that the matrices, representing two different polynomial blow-ups of the same function at a given singular point must have the same kernel.

Lemma 2.1. *Let $0 \in S_u$, with u and Ω as in (1.1), and Q_1, Q_2 be two different polynomial blow-ups of u, with corresponding matrices A and B. Then for any vector e*

$$\|Ae\| = \|Be\|, \qquad \text{and} \qquad A^2 = B^2.$$

In particular $KerA = KerB$.

Proof. Let $r_j \searrow 0$ be an arbitrary sequence, and set $u_{r_j} = u(r_j x)/r_j^2$. Suppose u_{r_j} converges, for a subsequence and in $C^{1,\alpha}_{loc}(\mathbf{R}^n)$, to a global solution Q_1. Since $0 \in S_u$ we'll have Q_1 is a polynomial in $\mathbf{R}^n$, i.e.,

$$Q_1 = \frac{1}{2} x^t A x = \frac{1}{2} \sum a_{ij} x_i x_j.$$

Here $A \in \mathcal{M}$ is a symmetric matrix with entries a_{ij}. Now let $t_j \searrow 0$ be another arbitrary sequence, and define accordingly u_{t_j}. Then a similar argument gives a limiting polynomial

$$Q_2 = \frac{1}{2} x^t B x = \frac{1}{2} \sum b_{ij} x_i x_j.$$

Here $B = (b_{ij}) \in \mathcal{M}$ is a symmetric matrix. We will show that $A^2 = B^2$. Let e be any arbitrary directional vector (unit length), and consider the monotonicity function for $D_e u$, by setting

$$\varphi(r, D_e u) = \frac{1}{r^4} \left(\int_{B(0,r)} \frac{|\nabla (D_e u)^+|^2}{|x|^{n-2}} \right) \left(\int_{B(0,r)} \frac{|\nabla (D_e u)^-|^2}{|x|^{n-2}} \right).$$

Then by Lemma 1.7, $\varphi(r, D_e u)$ is a almost monotone non-decreasing function of r. By scaling

$$\varphi(r, D_e u) = \varphi(1, D_e u_r). \tag{2.1}$$

Since φ is almost monotone the limit, as r tends to zero, exists and

$$\lim_{r\to 0} \varphi(r, D_e u) = C_e,$$

for some $C_e \ge 0$. Also the convergence of the functions u_{r_j} and u_{t_j} takes place in $W^{2,p}_{loc}(\mathbf{R}^n)$ (see General Remarks). Therefore we'll have (by (2.1))

$$C_e = \lim_{r\to 0} \varphi(r, D_e u) = \lim_{r\to 0} \varphi(1, D_e u_r).$$

Replacing r by r_j and then by t_j we obtain

$$C_e = \lim_{r_j \to 0} \varphi(1, D_e u_{r_j}) = \varphi(1, D_e Q_1),$$

and

$$C_e = \lim_{t_j \to 0} \varphi(1, D_e u_{t_j}) = \varphi(1, D_e Q_2).$$

Hence

$$\varphi(1, D_e Q_1) = \varphi(1, D_e Q_2). \tag{2.2}$$

Now inserting the polynomial representations of Q_1 and Q_2 in (2.2) we obtain for all directional vectors e

$$\|Ae\| = \|Be\|, \tag{2.3}$$

where $\|\cdot\|$ denotes the usual vector norm. From here we show that $A^2 = B^2$. Indeed, (2.3) and the symmetry of the matrices imply

$$(A^2 e, e) = (B^2 e, e) \qquad \forall\ e.$$

Using this we'll end up with

$$\begin{aligned}(A^2 x, y) &= \tfrac{1}{2}\big((A^2(x+y),(x+y)) - (A^2 x, x) - (A^2 y, y)\big)\\ &= \tfrac{1}{2}\big((B^2(x-y),(x-y)) - (B^2 x, x) - (B^2 y, y)\big) = (B^2 x, y),\end{aligned}$$

for all vectors x and y. Hence $A^2 = B^2$. □

From the above lemma it follows, using a contradictory argument, that if the origin is a singular point then the free boundary lies, locally, in a cusp like region, where the direction of the cusp is parallel to the kernel of the matrix A, in the representation of the blow-up of u. Obviously this implies that the free boundary is rectifiable (see [Si; Chapter 4]. This, however, is not enough for proving a C^1 regularity; see Lemma 2.3–2.4 below.

Remark. We want to point out a crucial fact about the matrices that appear in Lemma 2.1. A simple argument in matrix theory will reveal that the number of possible matrices in Lemma 2.1 is less than 2^{n-1}. This fact will be used in the proof of part (IV) of Theorem 1.5.

Now to see this fact let us take all possible matrices B that may appear in the proof of Lemma 2.1, i.e., all matrices $B \in \mathcal{M}$ such that $B^2 = A^2$ for a fixed matrix $A \in \mathcal{M}$. By rotation we may assume that A is diagonal. The problem is that A is allowed to have negative eigenvalues. Since by Lemma 2.1, A and B have the same kernel we may rearrange the coordinate system so that we only consider $m \times m$-matrices with nonzero eigenvalues. Also $m \leq n$. Let us also assume that A is diagonal with diagonal elements λ_j. Now $B^2 = A^2$ gives that B^2 has eigenvalues λ_j^2. Now let U_B be the orthogonal matrix which diagonalizes B. Then one can see that U_B also diagonalizes B^2. But $B^2 = A^2$ is fixed. Hence U_B is unique (up to 2^{n-1}-permutations of the column vectors) and independent of B. Therefore the maximal number of matrices B above must be smaller than or equal to 2^{n-1}.

For the next lemma recall the notation $\Lambda = \{u = |\nabla u|\}$.

Lemma 2.2. *Given $\delta > 0$ there exists R_δ such that if $u \in P_\infty(0, C_0)$ and $0, x^0 \in \Lambda(u)$ with $|x^0| = R \geq R_\delta$, then for $e_0 = x^0/|x^0|$ we have*

$$\|(D_{e_0} u)^+\|_{W^{1,2}_{B_1}} < \delta.$$

Proof. The lemma follows trivially if u is a polynomial. So suppose u is not a polynomial. Then, by [CKS] (Theorem II) we may only consider the subclass $\tilde{P}_\infty(0, C_0)$ of $P_\infty(0, C_0)$ that consists of convex solutions. Now suppose the statement of the lemma fails, then there exists a sequence $R_j \to \infty$, $u_j \in \tilde{P}_\infty(0, C_0)$, 0, $x^j \in \Lambda(u_j)$, with $|x^j| = R_j$ and such that

$$\|(D_{e_j} u_j)^+\|_{W^{1,2}_{B_1}} \geq \delta. \tag{2.4}$$

where $e_j = x^j/|x^j|$. Then, by convexity, the segment $l_j = [0, x^j] \subset \Lambda(u_j)$. Now (as usual) let us take a convergent subsequence of u_j, with the limit $u_0 \in \tilde{P}_\infty(0, C_0)$. Using the fact that (see [CKS; General Remarks])

$$\overline{\lim}\, \Lambda(u_j) \subset \Lambda(u_0),$$

we'll have that the limit function u_0 contains the ray $l_0 = \overline{\lim}\, l_j$ in $\Lambda(u_0)$. Now by the proof of Theorem II in [CKS; page 285] we have $D_{e_0} u_0 \leq 0$, where $e_0 = \lim_j x^j/|x^j|$ is the direction of the ray l_0. This contradicts (2.4). □

Let us recall the definition of $K_\varepsilon(A)$ in (1.4). Now according to Lemma 2.1, all blow-ups of u at the origin have the same kernel. Using this fact in the definition of $K_\varepsilon(A)$ we see that $K_\varepsilon(A) = K_\varepsilon(B)$ if A and B are matrices that come from different blow-ups of u. This suggests to define the cones, using the function u itself, i.e., we define $K_\varepsilon(u) = K_\varepsilon(A)$, where A is any of the matrices arising in the blow-up of u.

Lemma 2.3. *Let $0 \in S_u$ with the corresponding blow-up matrix A. Fix $a > 0$ and suppose e_λ is an eigenvector corresponding to the eigenvalue λ with $|\lambda| \geq a$. Then given $\varepsilon > 0$, there exists $r_\varepsilon = r_\varepsilon(|\lambda|, C_0) > 0$ (independent of u) such that*

$$K_\varepsilon(u, e_\lambda) \cap B(0, r_\varepsilon) \subset \Omega.$$

Proof. If the conclusion of the lemma fails, then there exists $u_j \in P_1(0, C_0)$, with blow-up matrix A_j and its eigenvalue λ_j, $x^j \in \Lambda(u_j) \cap K_\varepsilon(u_j, e_{\lambda_j})$, $r_j = |x^j| \searrow 0$, and $|\lambda_j| > a$. Consider a scaling of u_j at the origin, in the following way. For $s > 0$ (s is large and will be chosen later) set

$$\tilde{u}_j(x) = u_j(sr_j x)/(sr_j)^2, \qquad sr_j < 1.$$

By usual compactness argument a subsequence (again labeled $\tilde{u}_j$) will converge to a global solution u_0 (since $r_j \to 0$), and $\tilde{x}^j = x^j/(sr_j) \in \Lambda(\tilde{u}_j) \cap K_\varepsilon(\tilde{u}_j)$ will converge to $x^0 \in \Lambda(\tilde{u}_0) \cap K_\varepsilon(\tilde{u}_0, e_{\lambda_0})$. Finally, using that $a < |\lambda_j| \leq C_0$ we'll have that λ_j shuld converge (up to a subsequence) to some limit value λ_0 with $|\lambda_0| \geq a$. Also $|\tilde{x}^j| = 1/s$ implies $|x^0| = 1/s$.

Next fix j. Then $0 \in S_{u_j}$. This in conjunction with fact 8) in General Remarks implies

$$C(a\varepsilon)^4 \leq C(\lambda_j \varepsilon)^4 = \lim_{r \to 0} \varphi(r, D_{e_j} u_j),$$

where $e_j = x^j/|x^j|$. Hence by Lemma 1.7

$$C(a\varepsilon)^4 - O(sr_j)^\beta \leq \varphi(sr_j, D_{e_j} u_j) = \varphi(1, D_{e_j} \tilde{u}_j).$$

As j tends to infinity we obtain

$$C(a\varepsilon)^4 \leq \varphi(1, D_{e_0} \tilde{u}_0),$$

where $e_0 = x^0/|x^0|$. Since $x^0 \in \partial\Omega(u_0)$ and since $|x^0| = 1/s$, we can apply Lemma 2.2 in the following way. For small δ we can choose large s so as to arrive at

$$\|(D_{e_0}u_0)^+\|_{W^{1,2}_{B_1}} < \delta.$$

Hence we end up with

$$C(a\varepsilon)^4 \le \varphi(1, D_{e_0}u_0) \le C\delta^2.$$

Choosing $\delta^2 = C(a\varepsilon)^4\varepsilon_0$ with ε_0 small enough we'll have a contradiction. □

Let A^{x^0} be the matrix corresponding to the blow-up of u at x^0. In the next lemma using similar ideas as that of the proof in Lemma 2.3 we can prove that the kernel of A^{x^0} is continuous in x^0 for $x^0 \in S_u(a,k)$. Unfortunately the continuity depends strongly on the constant a. For this purpose we need a definition of distance of the matrices. For two $n \times n$-matrices A_1 and A_2 we define

$$\mathrm{dist}(A_1, A_2) := \mathcal{H} - \mathrm{dist}(\mathrm{Ker}(A_1) \cap B_1, \mathrm{Ker}(A_2) \cap B_1),$$

where $\mathcal{H} - \mathrm{dist}$ denotes the Hausdorff distance between sets. Here we have considered the linear space $\mathrm{Ker}(A_i)$ as set of points. Observe that by this definition $\mathrm{dist}(A_1, A_2) = 0$ if and only if $\mathrm{Ker}(A_1) = \mathrm{Ker}(A_2)$. In particular we may have two different matrices having zero distance.

Lemma 2.4. *Given $\varepsilon > 0$, there exists $r_\varepsilon = r_\varepsilon(a, k, C_0) > 0$ such that if $x^0, x^1 \in S_u(a,k)$ and $|x^0 - x^1| < r_\varepsilon$, then dist$(A^{x^0}, A^{x^1}) < \varepsilon$.*

Proof. The proof follows from Lemma 2.3. □

3. Proof of Theorem 1.5

Proof of (I). The first statement in Theorem 1.5 follows the same steps as that of [CKS], with minor changes. Indeed everywhere in Theorems I, and III in [CKS] when the monotonicity formula is used one needs to add a correction term r^β, which corresponds to the almost monotonicity lemma. It is not hard to check that at all other points of the proofs given in [CKS] for $f = 1$ works with small modifications for Lipschitz f.

The proof of Theorem II in [CKS] is unchanged since one only classifies global solutions with $f \equiv 1$. This depends on the fact that when we scale the functions in the proof of Theorem III in [CKS], the limit functions, are global solutions with $f \equiv 1$.

Proof of (II)-(III). These parts are easy (but probably not obvious) consequences of Lemmas 2.3–2.4 and Withney-type extension theorem (see [St; chapter 6]). We only treat case (III), since by (1.5) $S_u \subset S_u(1/n, n-1)$ part (II) will follow by part (III). Let $x^0 \in S_u(a,k)$. If x^0 is an isolated point of $S_u(a/2,k)$ then we are done. Let us assume x^0 is non-isolated in $S_u(a/2,k)$. Assume also $x^0 = 0$ (the origin). Denote by M^z the kernel of the matrix in the representation of the corresponding blow-up solution at the point $z \in S_u(a/2,k)$. Let also e_0 be any unit vector orthogonal to the kernel of M^0 (the matrix representation at the origin), and define

$$\Pi = \{x : \; x \cdot e_0 = 0\}.$$

By rotation we assume $\Pi = \{x_1 = 0\}$ and e_0 is directed in the positive x_1-axis.

Define the closed truncated cone

$$K(z,r) := \{x : \ 2|x_1 - z_1| \geq |x - z|\} \cap B(z,r),$$

with vertex at the point $z \in S_u(a/2,k)$, and for small r. By Lemma 2.3 for r small enough the cone $K(0,r)$ intersects the free boundary only at the origin.

Now choose $z_0 \in S_u(a/2,k) \cap B(0,r/2)$. Then, by taking r even smaller if necessary, we can apply Lemma 2.3–2.4 to conclude that $K(z_0,r) \cap \partial\Omega = \{z_0\}$. Here the continuity of the kernel of M^{z_0} in $z_0 \in S_u(a/2,k)$ plays an essential role. It also follows that the projection

$$P : \ S_u(a/2,k) \cap B(0,r/2) \quad \rightarrow \quad \Pi,$$

is one-to-one. Let $S_u^*(a/2,k)$ denote the image of $S_u(a/2,k)$ under P. Then the inverse mapping

$$P^* : S_u^*(a/2,k) \quad \rightarrow \quad \mathbf{R},$$

is well defined and it is C^1-function over the set $S_u^*(a/2,k)$; since the tangent space on $S_u(a/2,k)$ exists and varies continuously (Lemma 2.4). Moreover the C^1-norm is uniform for the class, as Lemma 2.2 suggests.

Now by Withney's extension theorem we can extend P^* as a C^1-function (keeping the same uniform C^1-norm) into the entire Π. Also the graph of the extended function, denoted by Γ_{e_0}, is (uniformly) C^1 and it contains the set $S_u(a/2,k)$, locally near the origin, i.e.

$$S_u(a/2,k) \cap B(0,r) \subset \Gamma_{e_0},$$

for r small enough. Since for every direction e, orthogonal to $\mathrm{Ker}(M^0)$, we can repeat this argument to find Γ_e with the above properties, and since there are $(n-k)$ such independent directions e_j $(j = 1, \cdots, n-k)$, we will have

$$S_u(a/2,k) \cap B(0,r) \subset \Gamma := \bigcap_{j=1}^{n-k} \Gamma_{e_j}.$$

and that

$$\dim(\Gamma) = k.$$

Proof of (IV). Let us suppose that there are two different blow-ups u^1 and u^2 of the same solution u with singularity at the origin. Let also $\{r_j\}$ and $\{t_j\}$ be the corresponding blow-up sequences, so that

$$u_{r_j} \rightarrow u^1, \qquad u_{t_j} \rightarrow u^2 \qquad \text{in } W^{2,p}(\mathbf{R}^n). \tag{3.1}$$

Assume moreover

$$r_{j+1} < t_{j+1} < r_j < t_j,$$

and that

$$c_1 := u^1(x^0) < u^2(x^0) =: c_2 \tag{3.2}$$

for some $x^0 \in \partial B_1$. We will prove that for all values $c \in (c_1,c_2)$ there are blow-ups u^c such that $u^c(x^0) = c$. In particular we will have an infinite number of different blow-ups with the corresponding matrix A_c. Hence we will have an infinite number of matrices A_c satisfying the conditions of the remark preceding Lemma 2.2. But then according to the same remark, we must have a finite number of such matrices. Hence we should reach a contradiction. Now to complete the proof we will show that we have a blow-up u^c for each $c \in (c_1,c_2)$. So let us take $\varepsilon > 0$ small such that $c \in (c_1 + 2\varepsilon, c_2 - 2\varepsilon)$. Then we may choose t_j and r_j small enough such that $u_{r_j}(x^0) < c_1 + \varepsilon$ and $u_{t_j}(x^0) > c_2 - \varepsilon$. Next we observe that the function

$$t \ \rightarrow \ \frac{u(tx^0)}{t^2},$$

is continuous for $t > 0$. Hence for each interval (r_j, t_j) this function takes all intermediate values between $c_1 + \varepsilon$, and $c_2 - \varepsilon$, provided j is large enough. In particular there exists $\tau_j \in (r_j, t_j)$ such that

$$c = u_{\tau_j}(x^0) = \frac{u(\tau_j x^0)}{\tau_j^2}.$$

Therefore the limit function $u^c(x)$ (after subtracting a convergence subsequence) will satisfy $c = u^c(x^0)$. This completes the proof of part (IV).

Proof of (V). The last assertion can be proven easily by scaling. The continuity of $D_{ij}u$ up to the regular boundary points are classical; see e.g. [F; page 175].

Next, let $x^j \to 0 \in S_u$ non-tangentially, i.e., $\mathrm{dist}(x^j, \partial\Omega(u)) \geq C|x^j|$ for some $C > 0$. Define

$$u_j(x) = \frac{u(|x^j|x)}{|x^j|^2}; \qquad \tilde{x}^j = \frac{x^j}{|x^j|} \in \partial B_1.$$

Obviously

$$\mathrm{dist}(\tilde{x}^j, \partial\Omega(u_j)) \geq C.$$

Consequently $B(\tilde{x}^j, C) \subset \Omega(\tilde{u}_j)$. Hence for some limit function and in $C^2(B(\tilde{x}^j, C/2))$, $u_j \to u_0$. In particular

$$D^2 u_j(\tilde{x}^j) = D^2 u(x^j) \to D^2 u_0(x^0), \tag{3.3}$$

where $x^0 \in \partial B_1$ is the limit of $\tilde{x}^j$. Now by part (IV) of this theorem, any blow-ups at the origin converge to the same limit function. Hence $u_0 = (x^t A x)/2$ for some symmetric matrix A. Moreover A is independent of the blow-up, i.e., independent of the choice of x^j. This together with (3.3) gives that

$$D^2 u(x^j) \to D^2((x^t A x)/2) = \text{fixed}.$$

The theorem is proved. □

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University of Texas at Austin, Department of Mathematics, RLM 8.100 Austin, TX 78712-1082

E-mail address: caffarel@math.utexas.edu

Department of Mathematics, Royal Institute of Technology, 100 44 Stockholm, Sweden

E-mail address: henriks@math.kth.se