A CHARACTERIZATION OF THE SPHERE IN TERMS OF SINGLE-LAYER POTENTIALS

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Abstract. Let Ω be a bounded smooth domain in R^n, and suppose the single-layer potential of $\partial\Omega$ coincides for $y \notin \overline{\Omega}$ with the function $c|y|^{2-n}$ $(c > 0)$. Then $\partial\Omega$ is a sphere centered at the origin.

Throughout this paper we assume $\Omega \subset \mathrm{R}^3$ (the case R^n is similar) is a bounded domain, and the boundary $\partial\Omega$ is smooth enough that for

$$u(y) = \int_{\partial\Omega} \frac{d\sigma_x}{|x-y|} \qquad \forall y \in \mathrm{R}^n, \tag{1}$$

one has

$$4\pi = \frac{\partial u}{\partial n^-} - \frac{\partial u}{\partial n^+} \qquad \text{on } \partial\Omega, \tag{2}$$

where $d\sigma_x$ denotes the surface measure, $+$ indicates the limit from the exterior and $-$ the limit from the interior (see [K], page 164), and n is the outward normal vector to $\partial\Omega$, which we assume to exist. If Ω has this property, then we say Ω is smooth.

THEOREM. *Let Ω be a bounded smooth domain in R^3 and suppose for some $c > 0$*

$$\int_{\partial\Omega} \frac{d\sigma_x}{|x-y|} = \frac{c}{|y|} \qquad \forall y \notin \bar{\Omega}. \tag{3}$$

Then, $\partial\Omega$ is a sphere centered at the origin.

PROOF: Since the single-layer potential is continuous in R^3 (see [K], page 160) (3) implies $0 \in \Omega$. Therefore we can take two balls B_1, B_2 both centered at the origin and such that B_1 is the largest ball in Ω and B_2 is the smallest ball containing Ω. Now the idea is to show that $\partial B_1 = \partial B_2 = \partial\Omega$.
Let now $y^1 \in \partial B_1 \cap \partial\Omega$ and $y^2 \in \partial B_2 \cap \partial\Omega$, then $|y^2| \geq |y^1|$. Thus it

suffices to prove $|y^2| \le |y^1|$.
Define u to be the function represented by (1). Then, by the maximum principle, (since u is harmonic in Ω) $\max u$ in Ω is attained on $\partial\Omega$, hence by (3) at y^1. Similarly, $\min u$ in Ω is attained at y^2, therefore

$$\frac{\partial u}{\partial n^-}(y^1) \ge 0 \qquad \text{and} \qquad \frac{\partial u}{\partial n^-}(y^2) \le 0.$$

Moreover at y^1 and y^2 we have

$$\frac{\partial}{\partial n^+} = \frac{\partial}{\partial |y|} \quad ,$$

which applied to (2) yields

$$4\pi = \frac{\partial u}{\partial n^-} - \frac{\partial u}{\partial n^+} = \frac{\partial u}{\partial n^-} - c\frac{\partial |y|^{-1}}{\partial |y|} \ge \frac{c}{|y|^2} \qquad \text{at} \quad y^1,$$

i.e. $|y^1| \ge \sqrt{\frac{c}{4\pi}}$. Similarly,

$$4\pi = \frac{\partial u}{\partial n^-} - \frac{\partial u}{\partial n^+} = \frac{\partial u}{\partial n^-} - c\frac{\partial |y|^{-1}}{\partial |y|} \le \frac{c}{|y|^2} \qquad \text{at} \quad y^2,$$

i.e. $|y^2| \le \sqrt{\frac{c}{4\pi}}$, and this completes the proof.
REMARK: As far as the proof of the theorem goes, we don't need equation (2) to hold on the entire boundary of Ω, but just at the points on $\partial\Omega$ where u attains its maximum and minimum.

The theorem can very easily be generalized to the case in which Ω is a union of disjoint domains or in which the density is assumed to be radially increasing. These generalizations, in the context of the *volume* potential, are dealt with in [ASZ].

We state these as corollaries, omitting the simple proofs.

COROLLARY 1. *Let*

$$\int_\Omega \frac{d\sigma_x}{|x-y|} = \sum_{j=1}^{m} \frac{c_j}{|x^j - y|} \qquad \forall y \notin \bar{\Omega},$$

where $\Omega = \cup_1^m \Omega_j$, $\{\Omega_j\}$ *are pairwise disjoint smooth domains and* $x^j \in \Omega_j$. *Then,* $\partial\Omega_j$ *is a sphere centered at* x^j, $\forall j$.

COROLLARY 2. *Let* $f = f(|x|)$ *be a continuous increasing function defined on* R^3 *and suppose*

$$\int_{\partial\Omega} \frac{f(|x|)d\sigma_x}{|x-y|} = \frac{c}{|y|} \qquad \forall y \notin \bar{\Omega}.$$

Then, $\partial\Omega$ *is a sphere centered at the origin.*

References

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