WHEN DOES THE FREE BOUNDARY ENTER INTO CORNER POINTS OF THE FIXED BOUNDARY ?

TO NINA NIKOLAEVNA URALTSEVA ON THE OCCASION OF HER 70TH BIRTHDAY

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ABSTRACT. Our prime goal in this note is to lay the ground for studying free boundaries close to the corner points of a fixed, Lipschitz boundary. Our study is restricted to 2-space dimensions, and to the obstacle problem. Our main result states that the free boundary can not enter into a corner x^0 of the fixed boundary, if the (interior) angle is less than π, provided the boundary datum is zero close to the point x^0. For larger angles and other boundary datum the free boundary may enter into corners, as discussed in the text.

1. Introduction

This note concerns analysis of the free boundary for the obstacle problem (in 2-space dimensions) close to a fixed boundary, which has corners. Our hope is that methods and ideas in this paper can be carried over to higher dimensions, and more general situations.

To fix the idea, let f, and g be given functions. Consider a solution to the obstacle problem (see [F] ,[R])

$$\Delta u = f\chi_{\{u>0\}}, \qquad u \geq 0 \quad \text{in } B_1^*, \qquad u = g \quad \text{on } \partial B_1^*, \tag{1.1}$$

where χ is the characteristic function, and

$$B_r^* = \{x : \ |x| < r\} \cap \{x_2 > \psi(x_1)\},$$

with ψ a Lipschitz function defined on the interval $\{-2 < x_1 < +2\}$, satisfying $\psi(0) = 0$.

For simplicity, and clarity of the exposition, we assume that for some $r_0 > 0$

$$\psi \text{ is linear on both sides of } 0, \quad f \equiv 1, \quad g = 0, \quad \text{in } B_{r_0}, \tag{1.2}$$

and $g \geq 0$ everywhere. For (reasonably) general data f, g, and ψ one needs to work out some technical details, that might become quite involved.

We will also need to distinguish between free boundary points inside B_1^* and those on the boundary of B_1^*. Hence we define

$$\Gamma := \partial\{u > 0\} \cap B_1^*,$$

and

$$\Gamma^* := \{x \in \partial\{u > 0\} : \ \limsup_{y \to x} |\nabla u(x)| = 0, \quad \text{with } u(y) > 0\}.$$

Date: July 24, 2004.

2000 *Mathematics Subject Classification.* Primary 35R35.

Key words and phrases: free boundary, obstacle problem, corners.

Supported in part by the Swedish Research Council.

The author thanks Arshak Petrosyan for his fruitful comments.

The reason for taking limit superior above is the fact that, in general, at certain boundary points the gradient may not exist. Observe also that $\Gamma^* \setminus \Gamma \subset \partial B_1^*$.

In this paper we will show that the free boundary Γ can never enter into corners with angle θ_0 less than π. For angles strictly larger than π we show that the free boundary always enters into the corner, provided the latter is a free boundary point. For $\theta_0 = \pi$ both situations may occur.

The analysis for angles less than $\pi/2$ seem to be much more easier than that of $\theta_0 > \pi/2$.

Let us formulate our main result.

Theorem 1.1. *Let u solve the obstacle problem (1.1), with f, g, and B_1^* as in (1.2). Denote the interior angle of the domain B_1^* at the origin by θ_0. Then the following hold.*

(I) *If $\theta_0 \leq \pi/2$. Then*

$$0 \notin \Gamma^*. \tag{1.3}$$

(II) *If $\pi/2 < \theta_0 < \pi$, and $0 \in \Gamma^*$, then*

$$0 \quad \textit{is an isolated point of } \Gamma^*. \tag{1.4}$$

(III) *If $\theta_0 > \pi$, and $0 \in \Gamma^*$, then $0 \in \overline{\Gamma}$.*

(IV) *If $\theta_0 = \pi$, then all possibilities may occur.*

When the boundary datum g is zero at the corner point x_0, but it is not identically zero close to the corner point, then naturally the free boundary has the possibility of entering into the corner. In this case however one may distinct between the behavior of g. Indeed one can show that if the behavior of g is better than quadratic $g(x) = o(|x-x_0|^2)$ then, in the case (I) above, the free boundary can enter into the corner only in a tangential fashion, and in a way that the set $\{u > 0\}$ close to the corner point is tangential to the fixed boundary. If $\sup g(x) \geq C|x|^2$ then the free boundary can enter into the corner non-tangentially. These analysis, however, are outside the scope of this paper. It is also noteworthy that statement (II) above can be proved to hold in a uniform fashion, i.e. for $x^0 \in \Gamma^*$ we have $|x^0| \geq d = d(n, \sup_{B_1^*} u)$. We hope to come back to this in a forthcoming paper.

At this point we want to discuss a simple case.

Example 1.2. Let us consider the problem in the domain $D = \{x_1 > 0,\ x_2 > 0\} \cap B_1$, with $f \equiv 1$, $g(x_1, 0) = 0$, $g(0, x_2) = 0$ (for $0 < x_1, x_2 < 1/2$). Consider a solution u to the above obstacle problem.

Now, if the free boundary enters into the origin, then locally close to the origin, we expect the solution to behave like a homogeneous function of degree two

$$u(x) = r^2\phi(\theta);$$

even though it is not clear at this moment why this is the case! This is what we will prove later on.

Once we have the above situation then we can compute the solution using polar coordinates and reducing the equation to

$$\phi_{\theta\theta} + 4\phi = 1, \qquad \phi(0) = \phi(\pi/2) = 0.$$

This, however, proves to have no solution. consequently, the free boundary could not enter into the origin.

2. The growth of the solution close to corners

To prove our main theorem, we need to establish two basic properties of the solutions: non-degeneracy, quadratic growth.

First we claim the following non-degeneracy

2.1. **Non-degeneracy.** For a solution u to the obstacle problem we have

$$\sup_{B(x^0,r)} u \geq u(x^0) + \frac{r^2}{4}, \qquad x^0 \in \overline{\{u > 0\}}. \tag{2.1}$$

The proof of this can be found in [Ca1], for the interior case. The same proof works in the presence of a fixed boundary and zero data (see [SU]).

2.2. **Quadratic growth.** In order to prove the main result we need a scaling and blow-up argument. However, our equation is invariant only under quadratic scaling $u(rx)/r^2$. The problem is that the solution is not necessarily of quadratic growth, when $\pi/2 < \theta_0 < \pi$. Indeed, if we let $u = x_2(x_2 + ax_1)/2 + \mathrm{Im}(z^{\pi/\theta_0})$, for appropriate a, then with correct boundary values u solves the obstacle problem and has a growth of order $\pi/\theta_0 < 2$. Hence the function $u(rx)/r^2$ will not be bounded. For angle $\theta_0 \geq \pi$ the quadratic growth holds always, as shown below.

We use the idea of "homogeneous" scaling. So let us set

$$M_k = \sup_{B_{2^{-k}}(0)} u(x),$$

then for u a solution to the obstacle problem, with $0 \in \Gamma^*$, it holds that:
(i) If $\theta_0 < \pi$, and there exists $x^0 \in \Gamma(u)$,
then, for some $C > 0$, we have

$$4^{k_0+1} M_{k_0+1} < \max\{4^{k_0} M_{k_0}, C \sup_{B_1^*} u\}, \tag{2.2}$$

where k_0 is such that $2^{-k_0-2} \leq |x^0| \leq 2^{-k_0-1}$.
(ii) If $\theta_0 \geq \pi$ then we have

$$\sup_{B(0,r)} u \leq Mr^2. \tag{2.3}$$

The constant M above depends on the supremum norm of u only.

When the angle of opening of the corner is small enough, say smaller than or equal to $\pi/2$, one can use a comparison with the harmonic function Cx_1x_2 (after appropriate rotation) for large C, to conclude $u(x) \leq |x|^2$. For even smaller angles $\theta_0 < \pi/2$ we may use harmonic barriers such as $\mathrm{Im} z^{\pi/\theta_0}$ to conclude $u(x) \leq |x|^{\pi/\theta_0}$.

For the general case we could not found any easy way out. However, we can prove (2.2)-(2.3) using elaborated techniques introduced in [KS], and later developed in [SU], and [ASU].

Proof of case (i): In order not to be repetitive we just sketch some details.

If (2.2) fails, then for every positive integer j, there exist $x^j \in \Gamma(u)$, such that

$$4^{k_j+1} M_{k_j+1} \geq \max\{4^{k_j} M_{k_j}, j\}. \tag{2.4}$$

Here $2^{-k_j-2} \leq |x^j| \leq 2^{-k_j-1}$, and $k_j \to \infty$.

Now defining

$$u_j(x) = \frac{u(2^{-k_j}x)}{M_{k_j+1}} \qquad \text{in} \qquad D_j := \{x : \ 2^{-k_j}x \in B_1^*\} = 2^{k_j} B_1^*,$$

and following the lines of the proof in [ASU], Lemma 3.1 (see also the proof of Theorem 1.3 in [BS]) we will end up with a limit function u_0 as $j \to \infty$ (for a subsequence). Moreover it follows that u_0 is harmonic in the set $D_0 := \lim D_j$ and more importantly

$$\sup_{B_{1/2}} u_0 = 1, \qquad u_0 \geq 0, \qquad u_0 \text{ on } \partial D_0. \tag{2.5}$$

It is not hard to realize that we can assume, after rotation, that

$$D_0 = \{(r, \theta) : 0 < r,\ 0 < \theta < \theta_0\},$$

where $\theta_0 > \pi/2$ by assumption.

Now $\tilde{x}^j = 2^{k_j} x^j \in \Gamma(u_j)$, and $1/4 \leq |\tilde{x}^j| \leq 1/2$. Hence the limit point (after passing to a subsequence) $\tilde{x} := \lim \tilde{x}^j$ is a free boundary point for u_0 and in particular $\nabla u_0(\tilde{x}) = 0$; the convergence is in $C^{1,\alpha}_{loc}$ in $\overline{D}_0 \setminus \{0\}$. To this end we can apply the strong maximum principle or the boundary Hopf lemma (depending on whether $\tilde{x} \in D_0$ or $\tilde{x} \in \partial D_0 \setminus \{0\}$) to reach a contradiction.

Proof of case (ii): For the case $\theta_0 \geq \pi$ we consider a conformal map

$$z \;\to\; z^{\theta_0/\pi},$$

and set $V(z) = u(z^{\theta_0/\pi})$ in $B^+_{r_0} = \{|x| < r_0,\ x_2 > 0\}$. The function V satisfies $|\Delta V(x)| \leq C|x|^{2(\theta_0/\pi - 1)}$ in $\{x_2 > 0\}$, and $V = 0$ on $\{x_2 = 0\} \cap B_{r_0}$. Therefore to prove a quadratic growth for u we need to prove a growth of order $|x|^{2\theta_0/\pi}$ for V. Hence we need to show

$$4^{(k+1)\theta_0/\pi} M_{k+1} < \max\{4^{k\theta_0/\pi} M_k, C \sup_{B^+_{r_0}} V\} \quad \text{for all} \quad k = 0, 1, 2, \ldots$$

Now going back to the start of the proof above by redoing everything we end up with a blow up function V_0 which is harmonic in the upper half plane and by local uniform $C^{1,\alpha}$-convergence up to $\{x_2 = 0\}$ we conclude that $|\nabla V_0(0)| = 0$, since $0 \in \Gamma^*(u)$. Again Hopf's lemma applies to reach a contradiction.

Having these two basic properties in our disposal, we can now prove the main result.

3. Proof of the main result

3.1. **Homogeneous solutions in Cones of given angle.** In this section we will classify homogeneous solutions in cones with given interior angle. More exactly we consider homogeneous solutions in D_0 (see above) with quadratic growth, and we give explicit formulas for the solutions.

Theorem 3.1. *Let u be a homogeneous solution of degree two for the obstacle problem in the set*

$$D_0 = \{(r, \theta) : 0 < r,\ 0 < \theta < \theta_0\}.$$

In other words u satisfies

$$u(x) = r^2 \phi(\theta), \qquad \Delta u = \chi_{\{u>0\}}, \quad u \geq 0, \qquad \text{in } D_0$$

and $\phi(0) = \phi(\theta_0) = 0$. Then the following hold.

(I) *For $\pi/2 < \theta_0 \leq \pi$ we have*

$$u(x) = x_2(ax_1 + x_2/2) \qquad \text{in } D_0.$$

(II) *For* $\pi < \theta_0 \le 2\pi$ *we have, after rotation of the support of* u,

$$u(x) = (\max(x_2, 0))^2/2) \qquad \text{in } D_0.$$

Observe that for the case $0 < \theta_0 \le \pi/2$ we can impossibly have a solution, as simple calculations show (see Example 1.2 above).

Proof. We assume that the support of u is connected. Otherwise we restrict the solution to a connected part of its support.

Let also, by rotation invariance,

$$\tilde{D}_0 = \{u > 0\} = \{(r, \theta) : 0 < r,\ 0 < \theta < \tilde{\theta}_0 < \theta_0\}.$$

Now by use of the polar coordinates, for the Laplacian, one can solve the ode $\phi_{\theta\theta} + 4\phi = 1$, to find

$$\phi = 1/4 + A\cos 2\theta + B\sin 2\theta \qquad \text{in } \tilde{D}_0.$$

Using boundary data it follows that for Case (I) we have a solution of the given type above. For Case (II), one sees again that the problem has no solution with $\tilde{D}_0 = D_0$. Indeed, in this case again the only solution is given by type (I) solutions. But on the other hand we have $u \equiv 0$ in the set $D_0 \setminus \tilde{D}_0$. And therefore the extra (free) boundary condition $\phi_\theta(\tilde{\theta}_0) = 0$ plays a role. The only solution that is possible (and easily verified) is the one give in Case (II). □

3.2. **Homogeneity of blow up solutions.** If the solution function u, for the obstacle problem behaves well, in some good sense, then one expects that the blow up limit $u_0 = \lim_j u(r_j)/r_j^2$ (when it exists) should only reflect the properties of the second derivatives of the function u at the origin. Think of a case when u is at least C^2, then the higher order polynomials in its Taylor expansion around the origin should vanish (in the limit) upon a quadratic scaling as above. Hence we expect the blow up limit u_0 to be a degree two homogeneous function.

A (modified) monotonicity formula of G.S. Weiss [W1]-[W2] states that the function

$$W(r, u, x^0) = \frac{1}{r^4}\int_{B_r^*} \left(|\nabla u|^2 + 2u\right) - \frac{1}{r^5}\int_{\partial B_r^*} 2u^2, \tag{3.1}$$

is monotone increasing in r, for $0 < r < r_0$. Actually it is strictly increasing for $r < r_0$, unless it is homogeneous in B_r^*. Observe that in the original monotonicity formula of Weiss one has integration over the complete ball B_r. However, for zero or degree-two homogeneous boundary data (on the straight part of the boundary) the result still holds.

Suppose now we have a scaled function $u_{r_j}(x) = u(r_j x)/r_j^2$ and that there is a limit function $u_0 = \lim_j u_{r_j}$. Using the monotonicity formula of Weiss (over the domain B_r^*) we have, for $s < 1$,

$$Const. = W(0^+, u) = \lim_j W(r_j s, u) = \lim_j W(s, u_{r_j}) = W(s, u_0).$$

Since $W(s, u_0)$ is constant we must have

$$u_0 \qquad \text{is homogeneous.} \tag{3.2}$$

3.3. **A symmetric case and its consequences.** In the proof of the main theorem case (II), we will need a barrier from below that prevents the free boundary point coming to close to the origin.

Let P be a parallelogram generated by the vectors $\mathbf{v}_1 = (0,1)$, and $\mathbf{v}_2 = (\cos\theta_0, \sin\theta_0)$, where $\pi/2 < \theta_0 < \pi$. Obviously, for some $t, T > 0$,

$$D_t := \{(r,\theta) :\ r < t,\ 0 < \theta < \theta_0\} \subset P \subset D_T := \{(r,\theta) :\ r < T,\ 0 < \theta < \theta_0\}$$

(polar coordinates). Now, for appropriate constant a, we have $h := (ax_1 + x_2)x_2/2$ solves the obstacle problem in $D_1 := \{(r,\theta) :\ r < 1,\ 0 < \theta < \theta_0\}$ with the origin as the only free boundary point. For ε small let

$$h' = \frac{((a+\varepsilon)x_1 + x_2)(-\varepsilon x_1 + x_2)}{2(1 - \varepsilon(a+\varepsilon))}$$

and

$$D' = \{(r,\theta) :\ r < 1,\ 0 < \theta < \pi\ :\ h' > 0\} \subset D_1.$$

Obviously h' solves the obstacle problem in D', with boundary values h'. Extend h' to $\mathbf{R}^2$ by defining it to be zero outside D'. Now let u be the solution to the obstacle problem in P with boundary values which is the restriction of h' to ∂P. By comparison principle, see [F],

$$h' \le u \le C\ \mathrm{Im}(z^{\pi/\theta_0}), \qquad \text{in } P, \tag{3.3}$$

and in particular

$$0 \in \Gamma^*(u). \tag{3.4}$$

The idea is to conclude that for some $r_1 > 0$, $P \cap B_{r_1}$ has no free boundary points, and the only possible free boundary points, other than the origin, are on the boundary of $P \cap B_{r_1}$. We use the symmetry of the domain! Let now $l_1 = \{\mathbf{v}_2 + s\mathbf{v}_1,\ 0 < s < 1\}$, and $l_2 = \{\mathbf{v}_1 + s\mathbf{v}_2,\ 0 < s < 1\}$, be the two (upper) boundary segments of P. Then, since $\mathbf{v}_i \cdot \nabla h' \ge 0$ on l_i, $(i = 1, 2)$ we can use standard moving plane technique (see e.g. [GS]) to conclude that $\mathbf{v}_i \cdot \nabla u \ge 0$ in P' $(i = 1, 2)$, the parallelogram generated by the half-vectors $\mathbf{v}_1/2, \mathbf{v}_2/2$.

Now if we had a free boundary point $z \in P'$, then due to monotonicity of u in $\mathbf{v}_i$-directions, $i = 1, 2$, the parallelogram K bounded by the lines

$$\{z + s\mathbf{v}_1,\ s \in \mathbf{R}\}, \quad \{z + s\mathbf{v}_2,\ s \in \mathbf{R}\}, \quad \{s\mathbf{v}_1,\ s \in \mathbf{R}\}, \quad \{s\mathbf{v}_2,\ s \in \mathbf{R}\}$$

must belong to $\{u = 0\}$. Hence $u = 0$ in K and we have a contradiction to (3.4).

As a conclusion we have the following lemma.

Lemma 3.2. *Let u be a solution to the obstacle problem in P, with boundary values greater than or equal to h' on ∂P. Then there is a constant $r_1 > 0$ for which (interior of) $P \cap B_{r_1}$ has no free boundary points, and the only possible free boundary points must occur on the boundary of this set.*

Using this lemma in conjunction with Harnack's inequality we can obtain the following result.

Lemma 3.3. *Let u be a solution to the obstacle problem in B_T^*, with T as above, so that $P \subset D_T$. Suppose also that for some $y \in \partial P$ we have $u(y)$ is large enough and that there are no free boundary points in a δ-neighborhood of $\partial P \setminus B_{r_2}$, for some $r_2 > 0$. Then, for some $r_3 > 0$,*

$$\Gamma^*(u) \cap B_{r_3} = \{0\}.$$

Proof. By Lemma 3.2 it suffices to show that $u \geq h'$ on ∂P. To do so we first observe that if $u > 0$ in $B_{2r}(z)$ then the function $v(x) = u(x) + (2r)^2/4 - |x|^2/4$ is non-negative and harmonic in $B_{2r}(z)$. In particular by Harnack's inequality

$$\inf_{B_r(z)} v \geq 2c \sup_{B_r(z)} v$$

for some $c > 0$. Hence for any point z' in $B_r(z)$ we have $u(z') \geq cu(z)$ if $cu(z) \geq r^2$.

Using a chain of Harnack inequality, and assuming $u(y)$ is large enough, and $r = r_\delta$ is small enough, we can conclude that $u \geq h'$ on ∂P. Here we have used the fact that $h' = 0$ on $\partial P \setminus P'$, so that the Harnack's chain can be terminated on the set $\partial P \cap \overline{P'}$ before reaching end points. Now Lemma 3.2 applies to deduce the result.

The final conclusion from this part is the following lemma.

Lemma 3.4. *Let u be a solution to the obstacle problem in B_1^*, and t, T be as above. Let now, for k large, $B^*_{T2^{-k-1}} \setminus B_{t2^{-k-2}}$ be empty of free boundary points. Then either*

$$M_{k+1} \leq C4^{-k-1},$$

or

$$B^*_{r_3 2^{-k-1}} \cap \Gamma^*(u) = \emptyset.$$

Proof. Define

$$\tilde{u}(x) = \frac{u(2^{-k-1}x)}{4^{-k-1}},$$

and apply the previous lemma.

3.4. **Proof of the main Theorem.** To prove the main result we will use a contradictory argument. So suppose we are given a a function u solving the obstacle problem with $0 \in \Gamma^*$. Now define

$$u_j(x) := \frac{u(r_j x)}{r_j^2}, \tag{3.5}$$

and assume that u_j is locally, uniformly bounded from above.

Then $\{u_j\}$ solves a new obstacle problem in $D_{r_j} := \{x : r_j x \in B^*_{1/r_j}\}$. Now using compactness arguments we can deduce that a subsequence converges to a homogeneous solution u_0 in the limit-cone $D_0 = \lim_j D_{r_j}$ (for more details of such arguments see [SU], proof of Theorem D). Moreover, by (3.2) the limit function u_0 is a homogeneous function, and $0 \in \Gamma^*(u_0)$. Observe that by non-degeneracy (2.1), we have $u_0 \not\equiv 0$.

Let us now look at all possible cases of the main theorem.

Case (I): If $\theta_0 = \pi/2$, then obviously by quadratic growth (u is bounded by Cx_1x_2, for large C), we have that u_j in (3.5) is bounded, and hence we can argue as above to end up with a homogeneous solution u_0, which by classification of homogeneous solutions does not exists. See the lines following Theorem 3.1, and/or Example 1.2.

If $\theta_0 < \pi/2$, then barriers $C\mathrm{Im}(z^{\pi/\theta_0})$ above and non-degeneracy (see (2.1)) shows that the origin can impossibly be a free boundary point.

Case (II): Let $0 \in \Gamma^*$, and suppose it is not an isolated free boundary point. Then we have a sequence $x^j \in \Gamma$ such that $x^j \to 0$. Then, by (2.2) and Lemma 3.4, we have either

$$4^{k+1}M_{k+1} \le \max(4^k M_k, C \sup_{B_1^*} u),$$

or

$$M_{k+1} \le C4^{-k-1},$$

for all $k = 1, 2, \cdots$ An iteration argument shows that $M_k \le C4^{-k}$. Hence u_j is also bounded in this case. Let us now take $r_j = 2|x^j|$. It is apparent that the point $\tilde{x}^j := x^j/r_j \in D_{r_j} \cap \partial B_{1/2}$, and $\tilde{x}^j \in \Gamma(u_j)$. In the limit we will then have $x^0 = \lim_j \tilde{x}^j \in \Gamma^*(u_0)$. A point of caution is that x^0 may be on $\partial D_0 \cap \partial B_{1/2}$, but we still have $\lim_j \nabla u_j(\tilde{x}^j) = 0$. In other words the $C^{1,\alpha}_{loc}$ convergence is valid up to $\overline{D_0 \cap B_1} \setminus \{0\}$.

Hence the homogeneous solution u_0 will have at least two free boundary points, the origin and x^0. This will contradict the classification of homogeneous solutions, in the second case.

Case (III): Now if the angle of opening of D_0 is larger than π, and the origin is a free boundary point, $0 \in \Gamma^*$, then, according to the classification theorem of homogeneous solutions, the limit function u_0 must be of the form (II) in Theorem 3.1. But this just implies that $0 \in \overline{\Gamma}$. Indeed we have a slightly better result. Namely, $\mathrm{vol}(\{u = 0\} \cap B_r^*) \ge c_0|B_r^*|$, for some c_0 depending on the ingredients, and the angle θ_0. We leave the verification of the latter to the reader.

Finally in the case $\theta_0 = \pi$ one may show several possibilities for the behavior of the free boundary. E.g. if we let the boundary data be small on $\partial B_1 \setminus \{x_2 = 0\}$, and identically zero on $\{x_2 = 0\}$ then $0 \notin \Gamma^*$. This follows easily by the non-degeneracy (2.1).

The example $x_2^2/2 + ax_2$ also shows the possibility of the origin being or not being a free boundary point, depending on $a = 0$ or not.

For a tangential touch, we may just take any explicit example for which the non-coincident set is convex and take a supporting line locally at the touching point, and call it the fixed boundary. Here is how. Let u be the difference of a multiple of the fundamental solution for the Laplacian and a touching parabola, $u = AF(x) - P(x) = -A\log|x| + |x|^2/4 + B$. One can fix the values A, B so that in the ball $B_3(0)$ we have $\Delta u = 1 - c\delta_0$, for appropriate constant c. And moreover $u = \nabla u = 0$ on $\partial B_3(0)$. Now translate u upwards so that the support of u touches x_1-axis tangentially. Now in B_1^+ we have (for the translated function $\tilde{u}$) $\Delta\tilde{u} = \chi_{\{\tilde{u}>0\}}$ and it solves the obstacle problem. The free boundary also touches the origin tangentially.

4. Appendix

Our complicated argument for proving Case II in the main theorem can be replaced by a different, still similar, approach that was kindly suggested to us by Arshak Petrosyan.

Going back to (2.2) we may replace the condition $x^0 \in \Gamma(u)$ by a weaker one, namely

$$u(x^0) < h(x^0). \tag{4.1}$$

Now we can argue in the same way, and since

$$0 \le \frac{h(2^{-k}x)}{M_{k+1}} \le C\frac{4^{-k}}{M_{k+1}} \ \to\ 0,$$

we obtain, as in the proof of (2.2), a limit function with a free boundary point $\tilde{x} \in \partial D_0$

If $\tilde{x}$ becomes a boundary point then we need to show $\nabla u_0(\tilde{x}) = 0$, in order to obtain the final contradiction argument. Let x_j be the points for which $u(x_j) < h(x_j)$. Upon re-scaling assume $|\tilde{x}_j| = 1$. Also $\tilde{x}_j$ are interior points, since the inequality is strict. Let y_j be the closest point on the boundary to $\tilde{x}_j$. Let also $\mathbf{e}$ be the interior unit vector orthogonal to the edge of the angle to which we approach. Then, by the mean value theorem, there exists a point z_j^0 on the segment $[y_j, \tilde{x}_j]$ such that

$$D_e u(z_j) \le D_e h(z_j),$$

since $u(\tilde{x}_j) < h(\tilde{x}_j)$ and $u(y_j) = h(y_j)$. Thus, in the limit, one will have

$$D_e u_0(\tilde{x}) \le 0.$$

This implies $D_e u_0(\tilde{x}) = 0$, since $\mathbf{e}$ is the interior normal. Hence $\nabla u_0(\tilde{x}) = 0$.

So we have either of the following:

a) For some small $r > 0$, $u(x) \ge h(x)$ for all x with $|x| = r$ and, by (strong) comparison principle (up to the boundary), we have a barrier from below, and thus there are no free boundary points in B_r, except the origin.

b) For every $r > 0$ there is a point x with $|x| = r$ such that $u(x) < h(x)$ and we can iterate (2.2) to prove that $u(x) \le C|x|^2$.

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