THE TWO-PHASE MEMBRANE PROBLEM – AN INTERSECTION-COMPARISON APPROACH TO THE REGULARITY AT BRANCH POINTS

HENRIK SHAHGHOLIAN AND GEORG S. WEISS

ABSTRACT. For the two-phase membrane problem $\Delta u = \frac{\lambda_+}{2}\chi_{\{u>0\}} - \frac{\lambda_-}{2}\chi_{\{u<0\}}$, where $\lambda_+ > 0$ and $\lambda_- > 0$, we prove in two dimensions that the free boundary is in a neighborhood of each "branch point" the union of two C^1-graphs. We also obtain a stability result with respect to perturbations of the boundary data. Our analysis uses an intersection-comparison approach based on the Aleksandrov reflection.

In higher dimensions we show that the free boundary has finite $(n-1)$-dimensional Hausdorff measure.

1. Introduction

In this paper we study the regularity of the obstacle-problem-like equation

$$\Delta u = \frac{\lambda_+}{2}\chi_{\{u>0\}} - \frac{\lambda_-}{2}\chi_{\{u<0\}} \qquad \text{in } \Omega, \tag{1.1}$$

where $\lambda_+ > 0, \lambda_- > 0$ and $\Omega \subset \mathbf{R}^n$ is a given domain. Physically the equation arises for example as the "two-phase membrane problem": consider an elastic membrane touching the phase boundary between two liquid/gaseous phases with different viscosity, for example a water surface. If the membrane is pulled away from the phase boundary in both phases, then the equilibrium state can be described by equation (1.1).

Properties of the solution etc. have been derived by the authors in [17] and [14]. Moreover, in [13], the authors gave a complete characterization of global two-phase solutions satisfying a quadratic growth condition at the two-phase free boundary point 0 and at infinity. It turned out that each such solution coincides after rotation with the one-dimensional solution $u(x) = \frac{\lambda_+}{4}\max(x_n,0)^2 - \frac{\lambda_-}{4}\min(x_n,0)^2$. In particular this implies that each blow-up limit u_0 at so-called "branch points", $\Omega \cap \partial\{u > 0\} \cap \partial\{u < 0\} \cap \{\nabla u = 0\}$, is after rotation of the form $u_0(x) =$

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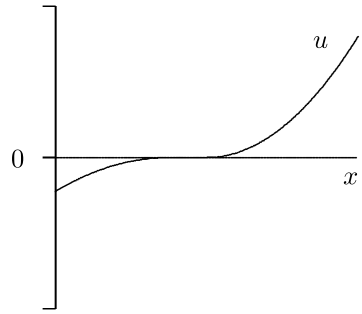

FIGURE 1. A solution in 1d

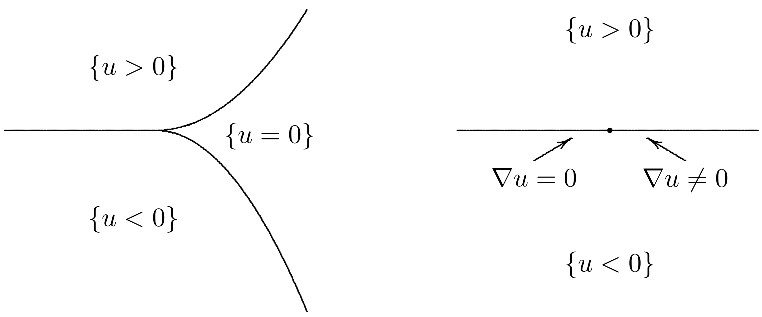

FIGURE 2. Examples of branch points

$\frac{\lambda_+}{4}\max(x_n,0)^2 - \frac{\lambda_-}{4}\min(x_n,0)^2$. Note that the nomenclature "branch point" is abusive in the sense that it does *not necessarily* imply a bifurcation of the free boundary at that point (see Figure 2). Also there *are* one-phase bifurcation points of the free boundary that are not included in our class of branch points. Nevertheless it makes sense to speak of branch points because *generically* a bifurcation occurs at those points.

In this paper we prove (cf. Theorem 4.1) that *in two dimensions* the free boundary is in a neighborhood of each branch point the union of (at most) two C^1-graphs. As application we obtain the following stability result: If the free boundary contains no singular one-phase point for certain boundary data (B_0), then for boundary data (B) close to (B_0) the free boundary consists of C^1-arcs converging to those of (B) (cf. Theorem 5.1).

In higher dimensions we derive an estimate for the $(n-1)$-dimensional Hausdorff measure of the free boundary.

Unfortunately the known techniques seem to be insufficient to do a conclusive analysis at branch points. One reason is that the density of the monotonicity formula by H.W. Alt-L.A. Caffarelli-A. Friedman takes the value 0 at branch points.

The situation is complicated by the fact that the limit manifold of all possible

blow-ups at branch points (including the case of varying centers) is not a one-dimensional or even smooth manifold, but has a more involved structure. Also the convergence to blow-up limits is close to the branch-point *not uniform!* Here we use an intersection-comparison approach based on the Aleksandrov reflection to show that – although the flow with respect to the limit manifold may not slow down when blowing up – the free boundaries are still *uniformly graphs* (see Proposition 4.2). The approach in Proposition 4.2 uses – apart from the reflection invariance – very little information about the underlying PDE and so yields a general approach to the regularity of free boundaries in two space dimensions provided that there is some information on the blow-up limits.
The Aleksandrov reflection has been recently used to prove regularity in geometric parabolic PDE ([9], [10], [11]). In contrast to those results, where structural conditions for the initial data are preserved under the flow, our results are completely local.
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2. Notation and Technical tools

Throughout this article $\mathbf{R}^n$ will be equipped with the Euclidean inner product $x \cdot y$ and the induced norm $|x|$. $B_r(x)$ will denote the open n-dimensional ball of center x, radius r and volume $r^n \omega_n$. When the center is not specified, it is assumed to be 0.

We will use $\partial_e u = \nabla u \cdot e$ for the directional derivative.

When considering a set A , χ_A shall stand for the characteristic function of A , while ν shall typically denote the outward normal to a given boundary.

Let $\lambda_+ > 0$ and $\lambda_- > 0$, $n \geq 2$, let Ω be a bounded open subset of $\mathbf{R}^n$ with Lipschitz boundary and assume that $u_D \in W^{1,2}(\Omega)$. From [17] we know then that there exists a "solution", i.e. a function $u \in W^{2,2}(\Omega)$ solving the strong equation $\Delta u = \frac{\lambda_+}{2}\,\chi_{\{u>0\}} - \frac{\lambda_-}{2}\,\chi_{\{u<0\}}$ a.e. in Ω, and attaining the boundary data u_D in L^2 . The boundary condition may be replaced by other, more general boundary conditions.

The tools at our disposition include two powerful monotonicity formulae. One is the monotonicity formula introduced in [16] by one of the authors for a class of semilinear free boundary problems (see also [15]). The second monotonicity formula has been introduced by H.W. Alt-L.A. Caffarelli-A. Friedman in [1]. What we are actually going to apply in section 3 is a stronger statement than the one in [1].

For the sake of completeness let us state both monotonicity formulae here.

Theorem 2.1 (Weiss's Monotonicity Formula). *Suppose that $B_\delta(x_0) \subset \Omega$. Then for all $0 < \rho < \sigma < \delta$ the function*

$$\Phi_{x_0}(r) := r^{-n-2} \int_{B_r(x_0)} \Big(|\nabla u|^2 \; + \; \lambda_+ \max(u,0) \; + \; \lambda_- \max(-u,0) \Big) \\ - 2\, r^{-n-3} \int_{\partial B_r(x_0)} u^2 \, d\mathcal{H}^{n-1} \; ,$$

defined in $(0,\delta)$, satisfies the monotonicity formula

$$\Phi_{x_0}(\sigma) \; - \; \Phi_{x_0}(\rho) \; = \; \int_\rho^\sigma r^{-n-2} \int_{\partial B_r(x_0)} 2 \left(\nabla u \cdot \nu - 2\, \frac{u}{r} \right)^2 \, d\mathcal{H}^{n-1} \, dr \; \geq 0 \; .$$

For a proof see [16].

In section 3 we are going to need the following stronger version of the Alt-Caffarelli-Friedman monotonicity formula.

Theorem 2.2 (Alt-Caffarelli-Friedman Monotonicity Formula). *Let h_1 and h_2 be continuous non-negative subharmonic $W^{1,2}$-functions in $B_R(z)$ satisfying $h_1 h_2 = 0$ in $B_R(z)$ as well as $h_1(z) = h_2(z) = 0$.*
Then for

$$\Psi_z(r, h_1, h_2) := r^{-4} \int_{B_r(z)} \frac{|\nabla h_1(x)|^2}{|x-z|^{n-2}} \, dx \int_{B_r(z)} \frac{|\nabla h_2(x)|^2}{|x-z|^{n-2}} \, dx \; ,$$

and for $0 < \rho < r < \sigma < R$, we have $\Psi_z(\rho) \leq \Psi_z(\sigma)$. Moreover, if equality holds for some $0 < \rho < r < \sigma < R$ then one of the following is true:
(A) $h_1 = 0$ in $B_\sigma(z)$ or $h_2 = 0$ in $B_\sigma(z)$,
(B) for $i = 1, 2$, and $\rho < r < \sigma$, supp $(h_i) \cap \partial B_r(z)$ is a half-sphere and $h_i \Delta h_i = 0$ in $B_\sigma(z) \setminus B_\rho(z)$ in the sense of measures.

For a proof of this version of monotonicity see [13]. We also refer to [1], for the original proof.

It is noteworthy that

$$\Psi_z(r, (\partial_e u)^+, (\partial_e u)^-) = \Psi_0(1, (\partial_e u_r)^+, (\partial_e u_r)^-) \text{ and } \Phi_z(r,u) = \Phi_0(1, u_r),$$

where

$$u_r(x) = \frac{u(rx+z)}{r^2}.$$

It is in fact possible to apply Theorem 2.2 to the positive and negative part of directional derivatives of u : due to N. Uraltseva, the functions $\max(\partial_e u, 0)$ and $-\min(\partial_e u, 0)$ are subharmonic in Ω (see Lemma 2 in [14]).

A quadratic growth estimate near the set $\Omega \cap \{u=0\} \cap \{\nabla u = 0\}$ had already been proved in [17] for more general coefficients λ_+ and λ_- , but local $W^{2,\infty}$- or $C^{1,1}$-regularity of the solution has been shown for the first time in [14]. See also [12]. So we know that

$$u \in W^{2,\infty}_{\mathrm{loc}}(\Omega) \, . \tag{2.1}$$

Lemma 2.3. *Let u be a solution of (1.1) in B_1 and suppose that the origin is a free boundary point. Then the following statements are equivalent:*
1) Either $\nabla u(0) \neq 0$, or $\lim_{r\to 0} \Psi_0(r, (\partial_e u)^+, (\partial_e u)^-) = 0$ for each direction e.
2) Either $\nabla u(0) \neq 0$, or each blow-up limit

$$u_0(x) = \lim_{m\to\infty} \frac{u(r_m x)}{r_m^2}$$

is after rotation of the form

$$u_0(x) = a_1 \frac{\lambda_+}{4} \max(x_1, 0)^2 \; - a_2 \frac{\lambda_-}{4} \min(x_1, 0)^2$$

where $a_1, a_2 \in \{0, 1\}$ and $a_1 + a_2 \neq 0$.
3) Either $\nabla u(0) \neq 0$, or at least one blow-up limit

$$u_0(x) = \lim_{m\to\infty} \frac{u(r_m x)}{r_m^2}$$

is after rotation of the form

$$u_0(x) = a_1 \frac{\lambda_+}{4} \max(x_1, 0)^2 \; - a_2 \frac{\lambda_-}{4} \min(x_1, 0)^2$$

where $a_1, a_2 \in \{0, 1\}$.
4) The origin is not a one-phase singular free boundary point, i.e. no blow-up limit

$$u_0(x) = \lim_{m\to\infty} \frac{u(r_m x)}{r_m^2}$$

is allowed to be a non-negative/non-positive homogeneous polynomial of degree 2.

Proof. "1) $\Rightarrow$ 2) :" In the case $\nabla u(0) \neq 0$, we obtain – using for example the lower semicontinuity of the weighted L^2-norm $f \mapsto \int_{B_1} |x|^{2-n} f^2(x)\, dx$ with respect to weak convergence – that

$$0 = \lim_{m\to\infty} \Psi_0(1, (\partial_e u_{r_m})^+, (\partial_e u_{r_m})^-) \geq \int_{B_1} \frac{|\nabla(\partial_e u_0)^+(x)|^2}{|x|^{n-2}} dx \int_{B_1} \frac{|\nabla(\partial_e u_0)^-(x)|^2}{|x|^{n-2}} dx \; .$$

Thus Theorem 2.2 (A) applies, and we obtain that for each direction e, either $\partial_e u_0 \geq 0$ in $\mathbf{R}^n$ or $\partial_e u_0 \leq 0$ in $\mathbf{R}^n$. It follows that after rotation, u_0 is a function depending only on the x_1 variable, and we obtain 2).
"2) $\Rightarrow$ 3)" is trivial. "3) $\Rightarrow$ 1)" holds because the function in 2) is one-dimensional and because the limit $\lim_{r\to 0} \Psi_0(r, (\partial_e u)^+, (\partial_e u)^-) = 0$ exists.
"3) $\Leftrightarrow$ 4) :" From the monotonicity formula 2.1 (cf. [16, Theorem 4.1]) it follows that in the case $\nabla u(0) = 0$, u_0 is a 2-homogeneous solution of the same equation. These solutions have been characterized (cf. [13, Theorem 4.3], and the only possibilities are the solutions in 2) *and* certain non-negative/non-positive homogeneous polynomials of degree 2.

3. Classification of Global Solutions

In what follows, I shall be an index set in a metric space.

We define the class

$$
\begin{aligned}
&M^* := \{u : B_1(0) \to \mathbf{R} \ : \\
&u(x_1,\dots,x_n) = \beta_1 \left(\frac{\lambda_+}{4} \max(x_1,0)^2 \; - \frac{\lambda_-}{4} \min(x_1 - \tau, 0)^2 \right) + \beta_2 x_1, \\
&\text{where } \tau \in [-1,0], 0 \le \beta_1 \le a, 0 \le \beta_2 \le b, 0 < c \le \beta_1 + \beta_2, \\
&\text{and } \beta_2 \neq 0 \text{ implies } \tau = 0\}.
\end{aligned} \tag{3.1}
$$

The class M is then defined as all rotated elements of M^*, i.e.

$$
M := \{u : B_1(0) \to \mathbf{R} \ : u = v \circ U \text{ where } U \text{ is a rotation, } v \in M^*\}. \tag{3.2}
$$

Observe that singular one-phase solutions are excluded from M.

Theorem 3.1. *Let $(u_\alpha)_{\alpha \in I}$ be a family of solutions of (1.1) in B_1 that is bounded in $W^{2,\infty}(B_1)$, and suppose that $0 \in \Omega \cap (\partial\{u_{\alpha_0} > 0\} \cup \partial\{u_{\alpha_0} < 0\})$ for some $\alpha_0 \in I$, and either $\nabla u_{\alpha_0}(0) \neq 0$ or $\lim_{r\to 0} \Psi_0(r, (\partial_e u_{\alpha_0})^+, (\partial_e u_{\alpha_0})^-) = 0$ for each direction e; this means by Lemma 2.3 that 0 is not a singular one-phase free boundary point. Define further S_r by*

$$
r^{n-1} S_r^2(y, u_\alpha) = \int_{\partial B_r(y)} u_\alpha^2,
$$

Then, if $u_\alpha \to u_{\alpha_0}$ in $L^1(B_1)$ as $\alpha \to \alpha_0, \partial\{u_\alpha > 0\} \ni \ y \to 0$ and $r \to 0$, all possible limit functions of the family

$$
\frac{u_\alpha(y + r\cdot)}{S_r(y, u_\alpha)},
$$

belong to M for some a, b, c as above.

Proof. As the statement holds by the implicit function theorem in the case $\nabla u_{\alpha_0}(0) \neq 0$, we may assume $\nabla u_{\alpha_0}(0) = 0$ and $\lim_{r\to 0} \Psi_0(r, (\partial_e u_{\alpha_0})^+, (\partial_e u_{\alpha_0})^-) = 0$ for each direction e. Consider sequences $u_j := u_{\alpha_j} \to u_{\alpha_0}$, $\partial\{u_j > 0\} \ni \ y_j \to 0, r_j \to 0$ and scaled functions

$$
v_j(x) = \frac{u_j(y_j + r_j x)}{S_{r_j}(y_j, u_j)}.
$$

A straightforward analysis of the limits of v_j will yield the statement of our theorem. First, setting

$$
T_j := \frac{r_j^2}{S_{r_j}(y_j, u_j)},
$$

we see that T_j is uniformly bounded from above, due to the non-degeneracy [13, Lemma 3.7]. Next, by the bounds on the second derivatives,

$$
|D^2 v_j(x)| = \frac{r_j^2}{S_{r_j}(y_j, u_j)} |D^2 u_j(y_j + r_j x)| \le C T_j \le C_0, \qquad x \in B_{1/(2r_j)},
$$

so that the $W^{2,\infty}$-norm of v_j is locally uniformly bounded. Now as the free boundary has zero Lebesgue measure [17, Theorem 5.1] one can infer as in [6], General Remarks, that v_j has a subsequence converging strongly in $W^{2,p}_{\mathrm{loc}}(\mathbf{R}^n)$. Let v be a limit function. The assumption $\lim_{r\to 0}\Psi_0(r,(\partial_e u_{\alpha_0})^+,(\partial_e u_{\alpha_0})^-)=0$ implies now by the monotonicity formula Theorem 2.2 that for each $R\in(0,\infty)$ and $\delta>0$,

$$\delta \ge \Psi_{y_j}(Rr,(\partial_e u_j)^+,(\partial_e u_j)^-) \ge \Psi_{y_j}(Rr_j,(\partial_e u_j)^+,(\partial_e u_j)^-)$$
$$= \Psi_0(R,(\partial_e v_j)^+,(\partial_e v_j)^-)\frac{S_{r_j}(y_j,u_j)}{r_j^2}$$

if we choose first r small and then j sufficiently large.
Consequently $\Psi_0(R,(\partial_e v)^+,(\partial_e v)^-)=0$ for every $R\in(0,\infty)$ and every direction e. But then Theorem 2.2 (A) applies, and for each direction e, either $\partial_e v\ge 0$ in $\mathbf{R}^n$ or $\partial_e v\le 0$ in $\mathbf{R}^n$. In particular v is one dimensional. As $\int_{\partial B_1(0)}|v|^2 = \lim_{j\to\infty}\int_{\partial B_1(0)}|v_j|^2=1$, we obtain that $v\in M$.

4. Uniform regularity of the free boundary close to branch points

This chapter contains the main result of this paper.

Theorem 4.1. *Let $n=2$, let $(u_\alpha)_{\alpha\in I}$ be a family of solutions of (1.1) in B_1 that is bounded in $W^{2,\infty}(B_1)$, and suppose that for some $\alpha_0\in I$, a blow-up limit*

$$\lim_{m\to\infty}\frac{u_{\alpha_0}(r_m\cdot)}{r_m^2}$$

is contained in M^*.
Then, if $u_\alpha\to u_{\alpha_0}$ in $L^1(B_1)$ as $\alpha\to\alpha_0$, $B_{r_0}\cap\partial\{u_\alpha>0\}$ and $B_{r_0}\cap\partial\{u_\alpha<0\}$ are C^1-graphs uniformly in $\alpha\in N_\kappa(\alpha_0)$ for some $r_0>0$ and $\kappa>0$; here the direction of every graph is the same, and $N_\kappa(\alpha_0)$ is a given open neighborhood of α_0.

The crucial tool in the proof of the theorem is the following proposition which uses an Aleksandrov reflection approach.

Proposition 4.2. *Let $n=2$, let $(u_\alpha)_{\alpha\in I}$ be a family of solutions of (1.1) in B_1 that is bounded in $W^{2,\infty}(B_1)$, and suppose that for some $\alpha_0\in I$, a blow-up limit*

$$\lim_{m\to\infty}\frac{u_{\alpha_0}(r_m\cdot)}{r_m^2}$$

is contained in M^*.
Then, given $\epsilon\in(0,1/8)$ there exist positive κ,δ and ρ such that for $\alpha\in N_\kappa(\alpha_0), y\in B_\delta\cap\partial\{u_\alpha>0\}$ and $r\in(0,\rho)$, the scaled function

$$u_r(x)=\frac{u_\alpha(rx+y)}{S_r(y,u_\alpha)} \tag{4.1}$$

satisfies

$$\operatorname{dist}(u_r,M^*)=\inf_{v\in M^*}\sup_{B_1(0)}|v(x)-u_r(x)|<\epsilon.$$

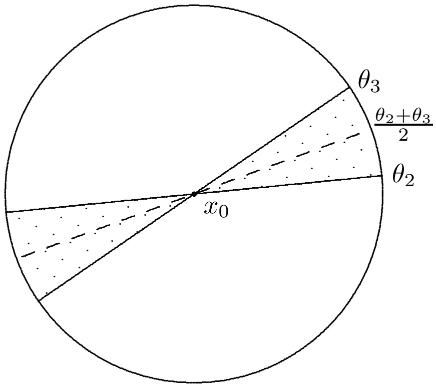

FIGURE 3. Turning free boundary

The result implies that we have uniform control of the rotation of the free boundaries. In particular, this implies uniform cone-flatness of the free boundaries.

Proof. First, by Theorem 3.1, for any $\tilde\epsilon > 0$ there are positive $\tilde\kappa, \tilde\delta$ and $\tilde\rho$ such that

$$\mathrm{dist}(u_r, M) < \tilde\epsilon \text{ for } \alpha \in N_{\tilde\kappa}(\alpha_0), y \in \partial\{u_\alpha > 0\} \cap B_{\tilde\delta} \text{ and } r \in (0, \tilde\rho).$$

Now if the statement of the theorem does not hold, then there are positive r_0 and r_1 as well as two counterclockwise rotations U_{θ_0} and U_{θ_1} of non-negative angle θ_0 and θ_1, respectively, satisfying $|\theta_0 - \theta_1| \ge c_1\epsilon > 0$ and

$$\mathrm{dist}(u_{r_0} \circ U_{\theta_0}, M^*) \le \tilde\epsilon \qquad \text{as well as} \qquad \mathrm{dist}(u_{r_1} \circ U_{\theta_1}, M^*) \le \tilde\epsilon;$$

here c_1 is a constant depending on $(a, b, \lambda_+, \lambda_-)$.
Let now $M^{*,\theta} := \{v : B_1(0) \to \mathbf{R} : v \circ U_\theta \in M^*\}$ and observe that while we do not know at this stage whether $r \mapsto u_r$ is uniformly continuous on $(0, \tilde\rho)$, we do know that $t \mapsto u_{\exp(-t)}$ is uniformly continuous on $(t_0, +\infty)$. As each continuous connection of M^{*,θ_0} and M^{*,θ_1} in M must either contain for each $\theta \in [\theta_0, \theta_1]$ an element of $M^{*,\theta}$, or contain for each $\theta \in [-\pi, \pi) \setminus (\theta_0, \theta_1)$ an element of $M^{*,\theta}$, we obtain for small $\tilde\epsilon$ — depending on $(\epsilon, a, b, c, \lambda_+, \lambda_-)$ — also $c_2 \in [c_1/4, 3c_1/4]$ and $0 < r_2 < r_3 < 1$ as well as two rotations U_{θ_2} and U_{θ_3} satisfying $|\theta_3 - \theta_2| = c_2\epsilon$ such that

$$\mathrm{dist}(u_{r_2} \circ U_{\theta_2}, M^*) \le \tilde\epsilon \qquad \text{and} \qquad \mathrm{dist}(u_{r_3} \circ U_{\theta_3}, M^*) \le \tilde\epsilon.$$

We may assume that $\theta_3 - \theta_2 > 0$; if this is not the case, we apply the following part of the proof to $u_\alpha(x_1, 2y_2 - x_2)$ instead of $u_\alpha(x_1, x_2)$. Now set

$$\omega = \frac{c_2\epsilon}{2}, \qquad U = U_{\frac{\theta_2+\theta_3}{2}}.$$

Moreover, let

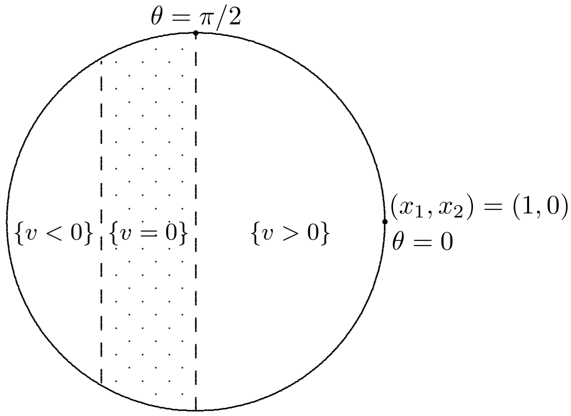

FIGURE 4. Example of v

$$\phi(r,\theta) := \frac{u_\alpha(y + rU(\cos\theta, \sin\theta))}{S_r(y, u_\alpha)} \ .$$

For each $0 < r < 1/2$, the function $\phi(r,\cdot)$ defines a function on the unit circle $[-\pi, \pi)$. The following part is inspired by applications of the Aleksandrov reflection (see for example [7], [8], [2]). There are however important differences: while the authors in [7], [8], [2] exclude *repetitive* behavior as $r \to 0$, for our application it is necessary to derive a contradiction from *just one turn of angle* $|\theta_0 - \theta_1|$. Moreover, our class M is not a one-dimensional or even a smooth manifold.

We consider

$$\xi(r,\theta) := \phi(r,\theta) - \phi(r,-\theta)$$

and observe that $\xi(r,0) = \xi(r,\pi) = 0$. In what follows we will prove that $\xi(r_3,\theta) \geq 0$ for $\theta \in [0,\pi]$ and $\frac{\partial\xi}{\partial\theta}(r_2, 0) \ < \ 0$ provided that $\tilde\epsilon$ has been chosen small enough (depending on $(\epsilon, a, b, c, \lambda_+, \lambda_-, \sup_{\alpha\in I}\sup_{B_1(0)}|u_\alpha|)$). By the comparison principle (applied to $S_r(y,u_\alpha)\phi(r,\theta)$ and $S_r(y,u_\alpha)\phi(r,-\theta)$ in the two-dimensional domain $[0,r_3) \times (0,\pi)$ with respect to the original coordinates x_1 and x_2) this yields a contradiction.

Let us prove $\frac{\partial\xi}{\partial\theta}(r_3, 0) \ > \ 0$ as well as $\xi(r_3,\theta) \geq 0$ for $\theta \in [0,\pi]$. The partial derivative estimate at r_2 is obtained in the same way. Take $v \in M^*$ such that

$$\sup_{B_1(0)} |v - u_{r_3} \circ U_{\theta_3}| = \mathrm{dist}(u_{r_3} \circ U_{\theta_3}, M^*) \leq \tilde\epsilon$$

(note that we do not need the axiom of choice to do so) and define

$$\phi_0(\theta) := v(\cos\theta, \sin\theta),$$

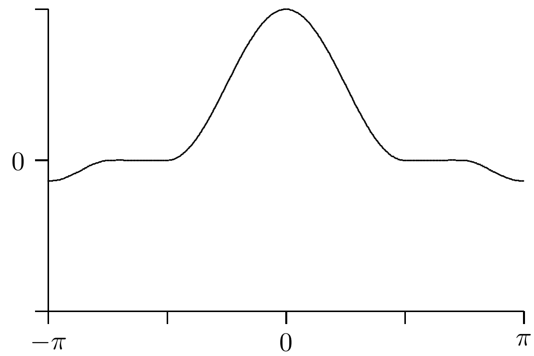

FIGURE 5. Example of ϕ_0

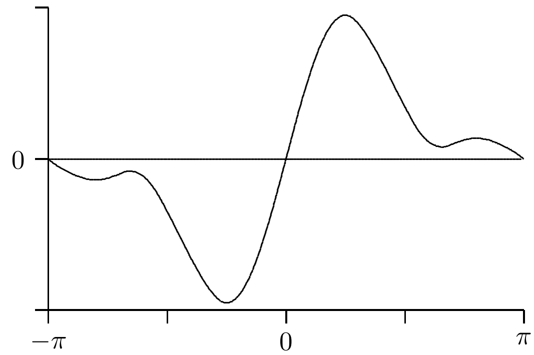

FIGURE 6. Example of ξ_0^γ

$$\sigma := \sup\{\theta \in (0,\pi) : \phi_0(\theta) = 0\}$$

$$\text{and } \xi_0^\gamma(\theta) := \phi_0(\gamma+\theta) - \phi_0(\gamma-\theta).$$

Observe that we may assume $\sigma = \pi$ or $\sigma \le 3\pi/4$. If this is not the case we change r_3 to $r_3/2$ where we still have flatness in the same direction.

Since ϕ_0 is an even 2π-periodic function which is decreasing on $(0,\pi)$, $\xi_0^\gamma(\theta) \ge 0$ for $-\pi \le \gamma \le 0, 0 \le \theta \le \pi$. Note that in the case of ϕ_0 being strictly decreasing in $(0,\pi)$, we also obtain $\xi_0^\gamma(\theta) > 0$ for $-\pi < \gamma < 0, 0 < \theta < \pi$. As $u_{r_3} \circ U_{\theta_3}$ is close to v (and thus $\xi(r_3,\cdot)$ close to $\xi_0^{-\omega}$) we expect the same to hold for $\xi(r_3,\cdot)$. In order to prove this rigorously we proceed as follows:

1) $\frac{\partial}{\partial\theta}\xi_0^\gamma(0) = 2\phi_0'(\gamma) \ge c_3 = c_3(\gamma, c, \lambda_+, \lambda_-) > 0$ for $\gamma \in (-\pi/2, 0)$, and for $\sigma \ne \pi$, $\frac{\partial}{\partial\theta}\xi_0^\gamma(\pi) = 2\phi_0'(\gamma+\pi) \le -c_3 < 0$ for $\gamma \in (-\pi+\sigma, 0)$. Consequently, for small $\tilde\epsilon$ (depending on $(\epsilon, a, b, c, \lambda_+, \lambda_-, \sup_{\alpha\in I}\sup_{B_1(0)}|u_\alpha|)$), $\frac{\partial}{\partial\theta}\xi(r_3, 0) \ge c_3/2 > 0$ and, in the case $\sigma \ne \pi$, $\frac{\partial}{\partial\theta}\xi(r_3,\pi) \le -c_3/2 < 0$. It follows that there is $c_4 = c_4(\epsilon, a, b, c, \lambda_+, \lambda_-, \sup_{\alpha\in I}\sup_{B_1(0)}|u_\alpha|)$ such that $\xi(r_3,\cdot) > 0$ in $(0, c_4)$ and, in the case $\sigma \ne \pi$, $\xi(r_3,\cdot) > 0$ in $(\pi - c_4, \pi)$.

2) Next, since $\phi_0(\gamma+\theta) \ge 0$ for $\theta \in [-\sigma-\gamma, \sigma-\gamma]$ and $\phi_0(\gamma-\theta) \le 0$ for

$\theta \in [\pi/2+\gamma, \pi+\gamma]$, making use of the non-degeneracy [13, Lemma 3.7], we see that for small $\tilde\epsilon$ (depending on $(\epsilon, a, b, c, \lambda_+, \lambda_-, \sup_{\alpha\in I}\sup_{B_1(0)}|u_\alpha|)$),

$$\phi(r_3,\theta) \geq 0 \text{ for } 0 \leq \theta \leq \sigma+\omega-\omega/2$$

and

$$\phi(r_3,-\theta) \leq 0 \text{ for } \pi - 3\omega/2 \geq \theta \geq \pi/2-\omega+\omega/2.$$

Consequently

$$\xi(r_3,\theta) \geq 0 \text{ for } \pi/2-\omega/2 \leq \theta \leq \sigma+\omega/2$$

and $\tilde\epsilon$ as above. Observe that $\sigma = \pi$ and $\pi/2 \leq \sigma \leq 3\pi/4$ are both allowed here.
3) Last, in $[c_4, \pi/2-\omega/4] \cup [\sigma+\omega/4, \pi-c_4]$, we obtain by the assumed range for σ that

$$\xi_0^{-\omega}(\theta) = \phi_0(\theta-\omega) - \phi_0(\theta+\omega) \geq c_5 = c_5(\epsilon, c, \lambda_+, \lambda_-) > 0,$$

so that $\xi(r_3,\cdot) \geq c_5/2 > 0$ in $[c_4, \pi/2-\omega/4] \cup [\sigma+\omega/4, \pi-c_4]$ for small $\tilde\epsilon$.

Combining 1)-3) we obtain the desired estimate, i.e. $\frac{\partial\xi}{\partial\theta}(r_3, 0) > 0$ as well as $\xi(r_3,\theta) \geq 0$ for $\theta \in [0,\pi]$.

Proof of Theorem 4.1: By Proposition 4.2 we know that g_α^+, g_α^- defined by

$$g_\alpha^+(x_2) = \sup\{x_1 \,:\, (x_1,x_2) \in B_\delta \cap \{u_\alpha = 0\}\}$$

$$\text{and } g_\alpha^-(x_2) = \inf\{x_1 \,:\, (x_1,x_2) \in B_\delta \cap \{u_\alpha = 0\}\}$$

are bounded in $C^1([-\delta/2, \delta/2])$.
We maintain that $u_\alpha = 0$ in $B_{\tilde\delta} \cap \{g_\alpha^- < x_n < g_\alpha^+\}$ for $\alpha \in N_{\tilde\kappa}(\alpha_0)$. Suppose this is not true: then, replacing if necessary u by $-u$ and exchanging λ_+ and λ_-, there are $y, z \in B_{\tilde\delta} \cap \partial\{u_\alpha > 0\}$ such that $y_2 = z_2$ and $u_\alpha > 0$ on the straight line segment between y and z. But then, setting $r = 2|y-z|$, we obtain that

$$u_r(x) = \frac{u_\alpha(rx+y)}{S_r(y, u_\alpha)}$$

does not satisfy $\mathrm{dist}(u_r, M^*) \leq \tilde\epsilon$, a contradiction to Proposition 4.2 provided that $\tilde\kappa$ and $\tilde\delta$ have been chosen small enough.

5. Stability of the free boundary

Theorem 5.1. *Let $\Omega \subset \mathbf{R}^2$ be a bounded Lipschitz domain and assume that for given Dirichlet data $u_D \in W^{1,2}(\Omega)$ the free boundary does not contain any one-phase singular free boundary point (cf. Lemma 2.3).*
Then for $K \subset\subset \Omega$ and $\tilde u_D \in W^{1,2}(\Omega)$ satisfying $\sup_{\partial\Omega}|u_D - \tilde u_D| < \delta_K$, there is $\omega > 0$ such that the free boundary is for every $y \in K$ in $B_\omega(y)$ the union of (at most) two C^1-graphs which approach those of the solution with respect to boundary data u_D as $\sup_{\partial\Omega}|u_D - \tilde u_D| \to 0$.

Proof. Let u and $\tilde{u}$ be the solutions with respect to u_D and $\tilde{u}_D$, respectively. By the comparison principle, $\sup_\Omega |u - \tilde{u}| \to 0$ as $\sup_{\partial\Omega} |u_D - \tilde{u}_D| \to 0$. Consequently, $\tilde{u} \to u$ in $C^{1,\beta}_{\rm loc}(\Omega)$ as $\sup_{\partial\Omega} |u_D - \tilde{u}_D| \to 0$. But then Theorem 4.1 applies, and the free boundary of $\tilde{u}$ is in $B_\omega(y)$ the union of two C^1-graphs which are bounded in C^1. More precisely, fixing $z \in \Omega \cap (\partial\{u > 0\} \cup \partial\{u < 0\})$ and translating and rotating once, we obtain $r_0 > 0$ such that $\partial\{\tilde{u} > 0\} \cup \partial\{\tilde{u} < 0\}$ is for $\sup_{\partial\Omega} |u_D - \tilde{u}_D| < \delta_K$ in B_{r_0} the union of the graphs of the C^1-functions $\tilde{g}^+$ and $\tilde{g}^-$ in the direction of e_2; moreover, the C^1-norms of $\tilde{g}^+$ and $\tilde{g}^-$ are bounded as $\tilde{u}_D \to u_D$. Suppose now that

$$\sup_{[-r_0/2, r_0/2]} |\tilde{g}^+ - \tilde{g}^-| \geq c_1 > 0$$

for some sequence $\tilde{u}_D \to u_D$. Then the fact that u and $\tilde{u}$ are near free boundary points after rescaling close to M (Theorem 3.1), implies that

$$\sup_{B_{1/2}} |\tilde{u} - u| \geq c_2 > 0$$

for the same sequence, and we obtain a contradiction.

6. Finite $n-1$-dimensional Hausdorff measure of the free boundary

In this section we assume that $n \geq 2$.

We first show that the free boundary has finite perimeter, which can be done as in [3]: Set

$$\beta(u) := \lambda_+ \chi_{\{u>0\}} - \lambda_- \chi_{\{u<0\}}$$

and define

$$\psi_\epsilon(t) = \begin{cases} 1 & \text{for } t > \epsilon, \\ -1 & \text{for } t < -\epsilon, \text{ and} \\ t/\epsilon & \text{when } -\epsilon \leq t \leq \epsilon. \end{cases}$$

Now, if η is a cut-off function, we obtain, differentiating the equation $\Delta u = \beta(u)$, multiplying by $\psi_\epsilon(\partial_i u)\eta$ and integrating over Ω, that

$$\int_\Omega \beta'(u)\partial_i u \psi_\epsilon(\partial_i u)\eta = \int_\Omega \partial_i \Delta u \psi_\epsilon(\partial_i u)\eta = -\int_\Omega \psi'_\epsilon |\nabla \partial_i u|^2 \eta - \int_\Omega \psi_\epsilon(\partial_i u)\partial_i \nabla u \cdot \nabla \eta.$$

The first integral on the right-hand side of the equality being non-positive and the second one bounded implies, letting ϵ tend to zero, that

$$\int_\Omega |\nabla \beta(u)|\eta \leq C_1 .$$

Here we used the fact that ψ_ϵ converges to the sign function as $\epsilon \to 0$, and that $\beta' \geq 0$. The above calculation can be made rigorous regularizing the equation by $\Delta u_\delta = \beta_\delta(u_\delta)$ where β_δ is a smooth increasing function tending to β as $\delta \to 0$; we let first ϵ and then δ go to 0.

Using in the above regularization the assumption $\min(\lambda_+, \lambda_-) > 0$ as well as the lower semicontinuity of the BV-norm, we obtain that the sets $\{u > 0\}$ and $\{u < 0\}$

are locally in Ω sets of finite perimeter. Since the set $\{u = 0\} \cap \{\nabla u \neq 0\}$ is locally in Ω a $C^{1,1}$-surface, the finite perimeter estimate tells us that

$$\mathcal{H}^{n-1}\left(\{u = 0\} \cap \{\nabla u \neq 0\} \cap K\right) < +\infty \text{ for each } K \subset\subset \Omega\ .$$

Note that the above estimate implies also that

$$\int_\Omega |\nabla \Delta u| \eta \leq C_2 \int_\Omega |\nabla \eta|\ . \tag{6.1}$$

This estimate in turn can be used to prove as in [4] that $(\partial\{u > 0\} \cup \partial\{u < 0\}) \cap \{\nabla u = 0\}$ has locally in Ω finite $n-1$-dimensional Hausdorff measure: for ψ_ϵ and η as above,

$$\int_\Omega \eta \left(\nabla \psi_\epsilon(\partial_i u) \cdot \nabla \partial_i u + \psi_\epsilon(\partial_i u) \Delta \partial_i u\right) = -\int_\Omega \psi_\epsilon(\partial_i u) \nabla \eta \cdot \nabla \partial_i u\ . \tag{6.2}$$

Using estimate (6.1), we deduce that

$$\int_{\{0<|\partial_i u|<\epsilon\}\cap\Omega} \eta |\nabla \partial_i u|^2 \leq C_3 \epsilon \left(\int_\Omega \eta |\Delta \partial_i u| + \int_\Omega |D^2 u|\,|\nabla \eta|\right) \leq C_4 \epsilon \int_\Omega |\nabla \eta|.$$

Take now – using Vitali's covering theorem – for each $\epsilon > 0$ a covering $\bigcup_{j=1}^m B_\epsilon(x_j)$ of $(\partial\{u > 0\} \cup \partial\{u < 0\}) \cap \{\nabla u = 0\} \cap \{\eta > 1\}$ such that $x_j \in (\partial\{u > 0\} \cup \partial\{u < 0\}) \cap \{\nabla u = 0\}$ and $\sum_{j=1}^m \chi_{B_\epsilon(x_j)}(y) \leq C_5$ for all $y \in \Omega$; here C_5 depends only on the dimension n. From the local $C^{1,1}$-regularity (2.1) and the non-degeneracy [13, Lemma 3.7] we conclude as in [4] that

$$\mathcal{L}^n(\{0 < |\nabla u| < \epsilon\} \cap B_\epsilon(x_j)) \geq c_6 \epsilon^n$$

where c_6 does not depend on ϵ or j. It follows that

$$\sum_{j=1}^m \epsilon^{n-1} \leq \frac{1}{c_6^n} \frac{1}{\epsilon} \sum_{j=1}^m |\{0 < |\nabla u| < \epsilon\} \cap B_\epsilon(x_j)| \leq C_7 \frac{1}{\epsilon} \sum_{j=1}^m \int_{\{0<|\nabla u|<\epsilon\}\cap B_\epsilon(x_j)} |\Delta u|^2$$

$$\leq C_8 \frac{1}{\epsilon} \sum_{j=1}^m \sum_{i=1}^n \int_{\{0<|\partial_i u|<\epsilon\}\cap B_\epsilon(x_j)} |\partial_{ii} u|^2 \leq C_9 \frac{1}{\epsilon} \sum_{i=1}^n \int_{\{0<|\partial_i u|<\epsilon\}} \eta |\partial_{ii} u|^2 \leq C_{10}$$

where C_{10} does not depend on ϵ.
We obtain:

Theorem 6.1. *Let u be a solution of (1.1) in Ω. Then $\partial\{u > 0\} \cup \partial\{u < 0\}$ is locally in Ω a set of finite $n-1$-dimensional Hausdorff measure.*

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