# Aleksandrov and Kelvin Reflection and the Regularity of Free Boundaries

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**Abstract.** The first part of this paper is an announcement of a result to appear. We apply the Aleksandrov reflection to obtain regularity and stability of the free boundaries in the *two-dimensional* problem

$$\Delta u = \frac{\lambda_+}{2}\chi_{\{u>0\}} - \frac{\lambda_-}{2}\chi_{\{u<0\}} ,$$

where $\lambda_+ > 0$ and $\lambda_- > 0$ .
In the second part we show that the Kelvin reflection can be used in a similar way to obtain regularity of the classical obstacle problem

$$\Delta u = \chi_{\{u>0\}}$$

*in higher dimensions.*

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**Keywords.** Free boundary, singular point, branch point, obstacle problem, regularity, Kelvin transform, global solution, blow-up, monotonicity formula.

## 1. Part I

From the physical point of view, the problem

$$\Delta u = \frac{\lambda_+}{2}\chi_{\{u>0\}} - \frac{\lambda_-}{2}\chi_{\{u<0\}} \qquad \text{in } \Omega, \tag{1.1}$$

where $\lambda_+ > 0, \lambda_- > 0$ and $\Omega \subset \mathbf{R}^n$ (cf. Fig. 1) arises for example as the "two-phase membrane problem": consider an elastic membrane touching the planar phase

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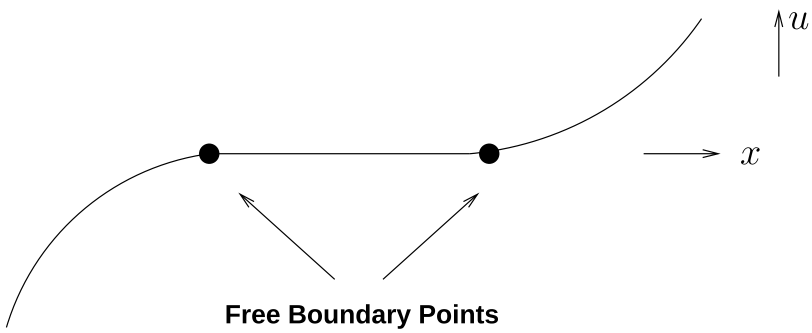


FIGURE 1. Example of a One-Dimensional Membrane

boundary between two liquid/gaseous phases with constant densities $\rho_1 > \rho_2$ in a gravity field, for example water and air. If the constant density $\rho_m$ of the membrane satisfies $\rho_1 > \rho_m > \rho_2$, then the membrane is being buoyed up in the phase with higher density and pulled down in the phase with lesser density, so the equilibrium state can be described by equation (1.1). Notice that (1.1) arises also as limiting case in the model of temperature control through the interior described in [7, 2.3.2] as $h_1, h_2 \to 0$.
Properties of the solution etc. have been derived by the G.S. Weiss in [19] and by N. Uraltseva in [16]. Moreover, in [15], H. Shahgholian-N. Uraltseva-G.S. Weiss gave a complete characterization of global two-phase solutions satisfying a quadratic growth condition at the two-phase free boundary point 0 and at infinity. It turned out that each such solution coincides after rotation with the one-dimensional solution $u(x) = \frac{\lambda_+}{4}\max(x_n,0)^2 - \frac{\lambda_-}{4}\min(x_n,0)^2$. In particular this implies that each blow-up limit $u_0$ at so-called "branch points" (see Fig. 2), $\Omega \cap \partial\{u > 0\} \cap \partial\{u < 0\} \cap \{\nabla u = 0\}$, is after rotation of the form $u_0(x) = \frac{\lambda_+}{4}\max(x_n,0)^2 - \frac{\lambda_-}{4}\min(x_n,0)^2$.

In this paper we prove (cf. Theorem 1.5) that *in two dimensions* the free boundary is in a neighborhood of each branch point the union of (at most) two $C^1$-graphs. As application we obtain the following stability result: If the free boundary contains no singular one-phase point for certain boundary data $(B_0)$, then for boundary data $(B)$ close to $(B_0)$ the free boundary consists of $C^1$-arcs converging to those of $(B)$ (cf. Theorem 1.7).

Let $\lambda_+ > 0$ and $\lambda_- > 0$, $n \geq 2$, let $\Omega$ be a bounded open subset of $\mathbf{R}^n$ with Lipschitz boundary and assume that $u_D \in W^{1,2}(\Omega)$. From [19] we know then that there exists a "solution", i.e. a function $u \in W^{2,2}(\Omega)$ solving the strong equation $\Delta u = \frac{\lambda_+}{2}\chi_{\{u>0\}} - \frac{\lambda_-}{2}\chi_{\{u<0\}}$ a.e. in $\Omega$, and attaining the boundary data $u_D$ in $L^2$. The boundary condition may be replaced by other, more general boundary conditions.

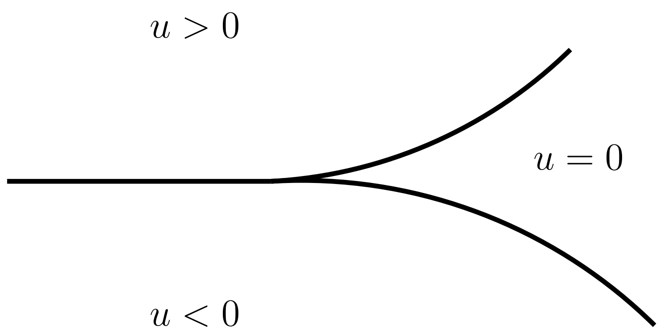


FIGURE 2. Example of a Branch Point

The tools at our disposition include two powerful monotonicity formulae. One is the monotonicity formula introduced in [18] by G.S. Weiss for a class of semilinear free boundary problems (see also [17]). The second monotonicity formula has been introduced by H.W. Alt-L.A. Caffarelli-A. Friedman in [1]. What we are actually going to apply in Theorem 1.4 is a stronger statement than the one in [1].

For the sake of completeness let us state both monotonicity formulae here.

**Theorem 1.1 (Weiss's Monotonicity Formula).** *Suppose that* $B_\delta(x_0) \subset \Omega$ . *Then for all* $0 < \rho < \sigma < \delta$ *the function*

$$\Phi_{x_0}(r) := r^{-n-2} \int_{B_r(x_0)} \Big( |\nabla u|^2 \;+\; \lambda_+ \max(u,0) \;+\; \lambda_- \max(-u,0) \Big) \\ - 2\, r^{-n-3} \int_{\partial B_r(x_0)} u^2 \, d\mathcal{H}^{n-1} ,$$

*defined in* $(0,\delta)$ , *satisfies the monotonicity formula*

$$\Phi_{x_0}(\sigma) \;-\; \Phi_{x_0}(\rho) \;=\; \int_\rho^\sigma r^{-n-2} \int_{\partial B_r(x_0)} 2 \left( \nabla u \cdot \nu - 2\, \frac{u}{r} \right)^2 \, d\mathcal{H}^{n-1} \, dr \;\geq 0 \; .$$

For a proof see [18].

In Theorem 1.4 we are going to need the following stronger version of the Alt-Caffarelli-Friedman monotonicity formula.

**Theorem 1.2 (Alt-Caffarelli-Friedman Monotonicity Formula).** *Let* $h_1$ *and* $h_2$ *be continuous non-negative subharmonic* $W^{1,2}$*-functions in* $B_R(z)$ *satisfying* $h_1 h_2 = 0$ *in* $B_R(z)$ *as well as* $h_1(z) = h_2(z) = 0$ .
*Then for*

$$\Psi_z(r, h_1, h_2) := r^{-4} \int_{B_r(z)} \frac{|\nabla h_1(x)|^2}{|x-z|^{n-2}} \, dx \int_{B_r(z)} \frac{|\nabla h_2(x)|^2}{|x-z|^{n-2}} \, dx ,$$

*and for* $0 < \rho < r < \sigma < R$, *we have* $\Psi_z(\rho) \le \Psi_z(\sigma)$. *Moreover, if equality holds for some* $0 < \rho < r < \sigma < R$ *then one of the following is true:*
*(A)* $h_1 = 0$ *in* $B_\sigma(z)$ *or* $h_2 = 0$ *in* $B_\sigma(z)$,
*(B) for* $i = 1, 2$, *and* $\rho < r < \sigma$, *supp* $(h_i) \cap \partial B_r(z)$ *is a half-sphere and* $h_i \Delta h_i = 0$ *in* $B_\sigma(z) \setminus B_\rho(z)$ *in the sense of measures.*

For a proof of this version of monotonicity see [15]. We also refer to [1], for the original proof.

It is noteworthy that

$$\Psi_z(r, (\partial_e u)^+, (\partial_e u)^-) = \Psi_0(1, (\partial_e u_r)^+, (\partial_e u_r)^-) \text{ and } \Phi_z(r, u) = \Phi_0(1, u_r),$$

where

$$u_r(x) = \frac{u(rx + z)}{r^2}.$$

It is in fact possible to apply Theorem 1.2 to the positive and negative part of directional derivatives of $u$ : due to N. Uraltseva, the functions $\max(\partial_e u, 0)$ and $-\min(\partial_e u, 0)$ are subharmonic in $\Omega$ (see Lemma 2 in [16]).

A quadratic growth estimate near the set $\Omega \cap \{u = 0\} \cap \{\nabla u = 0\}$ had already been proved in [19] for more general coefficients $\lambda_+$ and $\lambda_-$ , but local $W^{2,\infty}$- or $C^{1,1}$-regularity of the solution has been shown for the first time in [16]. See also [14]. So we know that

$$u \in W^{2,\infty}_{\text{loc}}(\Omega)\,. \tag{1.2}$$

The following lemma relates the value of the density of the ACF-monotonicity formula to the structure of the singularity:

**Lemma 1.3.** *Let* $u$ *be a solution of (1.1) in* $B_1$ *and suppose that the origin is a free boundary point. Then the following statements are equivalent:*
*1) Either* $\nabla u(0) \neq 0$, *or* $\lim_{r\to 0} \Psi_0(r, (\partial_e u)^+, (\partial_e u)^-) = 0$ *for each direction* $e$.
*2) Either* $\nabla u(0) \neq 0$, *or each blow-up limit*

$$u_0(x) = \lim_{m\to\infty} \frac{u(r_m x)}{r_m^2}$$

*is after rotation of the form*

$$u_0(x) = a_1 \frac{\lambda_+}{4} \max(x_1, 0)^2 \; - a_2 \frac{\lambda_-}{4} \min(x_1, 0)^2$$

*where* $a_1, a_2 \in \{0, 1\}$ *and* $a_1 + a_2 \neq 0$.
*3) Either* $\nabla u(0) \neq 0$, *or at least one blow-up limit*

$$u_0(x) = \lim_{m\to\infty} \frac{u(r_m x)}{r_m^2}$$

*is after rotation of the form*

$$u_0(x) = a_1 \frac{\lambda_+}{4} \max(x_1, 0)^2 \; - a_2 \frac{\lambda_-}{4} \min(x_1, 0)^2$$

*where* $a_1, a_2 \in \{0,1\}$.

*4) The origin is not a one-phase singular free boundary point, i.e. no blow-up limit*

$$u_0(x) = \lim_{m\to\infty} \frac{u(r_m x)}{r_m^2}$$

*is allowed to be a non-negative/non-positive homogeneous polynomial of degree* 2.

Let us now define the class

$$\begin{aligned} M^* := \{u : B_1(0) \to \mathbf{R} \ :& \\ u(x_1,\dots,x_n) = \beta_1 &\left(\frac{\lambda_+}{4}\max(x_1,0)^2 \ - \frac{\lambda_-}{4}\min(x_1-\tau,0)^2\right) + \beta_2 x_1, \\ \text{where } \tau \in [-1,0], 0 \le \beta_1 \le a,& 0 \le \beta_2 \le b, 0 < c \le \beta_1+\beta_2, \\ \text{and } \beta_2 \neq 0 \text{ implies } \tau = 0\}.& \end{aligned} \tag{1.3}$$

The class $M$ is then defined as all rotated elements of $M^*$, i.e.

$$M := \{u : B_1(0) \to \mathbf{R} \ : u = v \circ U \text{ where } U \text{ is a rotation, } v \in M^*\}. \tag{1.4}$$

Observe that singular one-phase solutions are excluded from $M$.

**Theorem 1.4.** *Let* $(u_\alpha)_{\alpha\in I}$ *be a family of solutions of (*1.1*) in* $B_1$ *that is bounded in* $W^{2,\infty}(B_1)$, *and suppose that* $0 \in \Omega \cap (\partial\{u_{\alpha_0} > 0\} \cup \partial\{u_{\alpha_0} < 0\})$ *for some* $\alpha_0 \in I$, *and either* $\nabla u_{\alpha_0}(0) \neq 0$ *or* $\lim_{r\to 0}\Psi_0(r,(\partial_e u_{\alpha_0})^+,(\partial_e u_{\alpha_0})^-) = 0$ *for each direction* $e$; *this means by Lemma* 1.3 *that* 0 *is not a singular one-phase free boundary point. Define further* $S_r$ *by*

$$r^{n-1}S_r^2(y,u_\alpha) = \int_{\partial B_r(y)} u_\alpha^2,$$

*Then, if* $u_\alpha \to u_{\alpha_0}$ *in* $L^1(B_1)$ *as* $\alpha \to \alpha_0, \partial\{u_\alpha > 0\} \ni \ y \to 0$ *and* $r \to 0$, *all possible limit functions of the family*

$$\frac{u_\alpha(y + r\cdot)}{S_r(y,u_\alpha)},$$

*belong to* $M$ *for some* $a, b, c$ *as above.*

The following theorem contains our main result, i.e. regularity at branch points. Unfortunately the known techniques seem to be insufficient to do a conclusive analysis at branch points. One reason is that the density of the monotonicity formula by H.W. Alt-L.A. Caffarelli-A. Friedman takes the value 0 at branch points.

The situation is complicated by the fact that the limit manifold of all possible blow-ups at branch points (including the case of varying centers) is not a one-dimensional or even smooth manifold, but has a more involved structure. Also the convergence to blow-up limits is close to the branch-point *not uniform!* Here we use an intersection-comparison approach based on the Aleksandrov reflection to show that – although the flow with respect to the limit manifold may not slow down

when blowing up – the free boundaries are still *uniformly graphs* (see Proposition 1.6). The approach in Proposition 1.6 uses – apart from the reflection invariance – very little information about the underlying PDE and so yields a general approach to the regularity of free boundaries in two space dimensions provided that there is some information on the blow-up limits.
The Aleksandrov reflection has been recently used to prove regularity in geometric parabolic PDE ([10], [11], [12]). In contrast to those results, where structural conditions for the initial data are preserved under the flow, our results are completely local.

**Theorem 1.5.** *Let $n = 2$, let $(u_\alpha)_{\alpha \in I}$ be a family of solutions of (1.1) in $B_1$ that is bounded in $W^{2,\infty}(B_1)$, and suppose that for some $\alpha_0 \in I$, a blow-up limit*

$$\lim_{m\to\infty} \frac{u_{\alpha_0}(r_m \cdot)}{r_m^2}$$

*is contained in $M^*$.*
*Then, if $u_\alpha \to u_{\alpha_0}$ in $L^1(B_1)$ as $\alpha \to \alpha_0$, $B_{r_0} \cap \partial\{u_\alpha > 0\}$ and $B_{r_0} \cap \partial\{u_\alpha < 0\}$ are $C^1$-graphs uniformly in $\alpha \in N_\kappa(\alpha_0)$ for some $r_0 > 0$ and $\kappa > 0$; here the direction of every graph is the same, and $N_\kappa(\alpha_0)$ is a given open neighborhood of $\alpha_0$.*

The crucial tool in the proof of the theorem is the following proposition which uses an Aleksandrov reflection approach.

**Proposition 1.6.** *Let $n = 2$, let $(u_\alpha)_{\alpha \in I}$ be a family of solutions of (1.1) in $B_1$ that is bounded in $W^{2,\infty}(B_1)$, and suppose that for some $\alpha_0 \in I$, a blow-up limit*

$$\lim_{m\to\infty} \frac{u_{\alpha_0}(r_m \cdot)}{r_m^2}$$

*is contained in $M^*$.*
*Then, if $u_\alpha \to u_{\alpha_0}$ in $L^1(B_1)$ as $\alpha \to \alpha_0$, there exist for given $\epsilon \in (0, 1/8)$ positive $\kappa, \delta$ and $\rho$ such that for $\alpha \in N_\kappa(\alpha_0), y \in B_\delta \cap \partial\{u_\alpha > 0\}$ and $r \in (0, \rho)$, the scaled function*

$$u_r(x) = \frac{u_\alpha(rx + y)}{S_r(y, u_\alpha)} \tag{1.5}$$

*satisfies*

$$\mathrm{dist}(u_r, M^*) = \inf_{v \in M^*} \sup_{B_1(0)} |v(x) - u_r(x)| < \epsilon.$$

The idea of the proof is to reflect the solution at a plane as in Fig. 3 and to compare the reflected solution to the original solution. As a consequence we obtain the following stability result:

**Theorem 1.7.** *Let $\Omega \subset \mathbf{R}^2$ be a bounded Lipschitz domain and assume that for given Dirichlet data $u_D \in W^{1,2}(\Omega)$ the free boundary does not contain any one-phase singular free boundary point (cf. Lemma 1.3).*

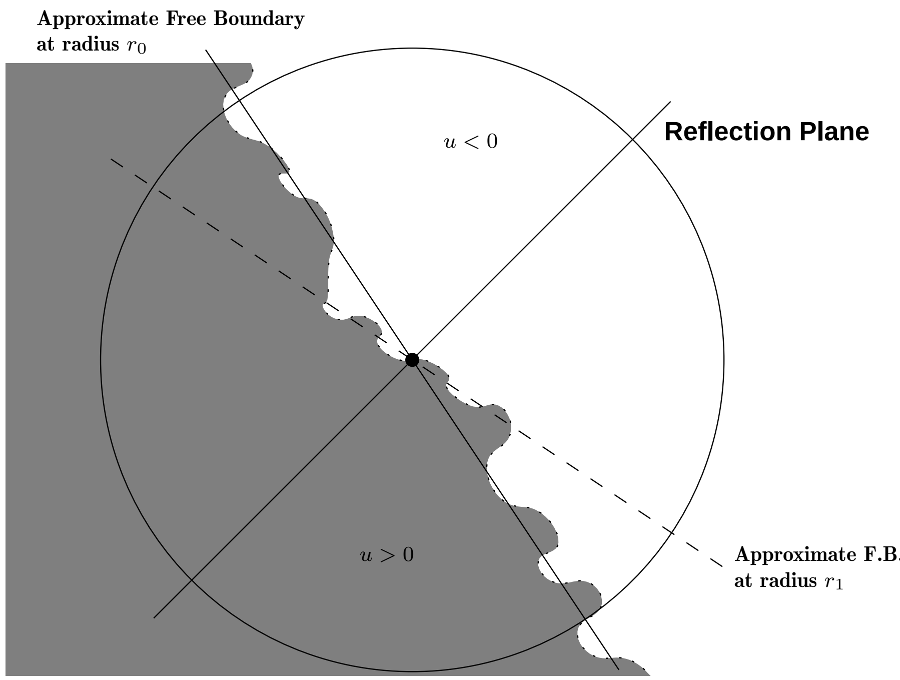


FIGURE 3. Turning Free Boundary

*Then for* $K \subset\subset \Omega$ *and* $\tilde{u}_D \in W^{1,2}(\Omega)$ *satisfying* $\sup_{\partial\Omega} |u_D - \tilde{u}_D| < \delta_K$, *there is* $\omega > 0$ *such that the free boundary is for every* $y \in K$ *in* $B_\omega(y)$ *the union of (at most) two* $C^1$*-graphs which approach those of the solution with respect to boundary data* $u_D$ *as* $\sup_{\partial\Omega} |u_D - \tilde{u}_D| \to 0$.

## 2. Part II

We are going to give a sketch of how a similar approach can be applied to the classical obstacle problem in higher dimensions.

The solution of the classical obstacle problem $u$ is non-negative and satisfies

$$\Delta u = \frac{1}{2}\chi_{\{u>0\}} \, . \tag{2.1}$$

Classical results include local $C^{1,1}$-regularity (see [8]) and non-degeneracy of the solution. Regularity of the free boundary in higher dimensions has first been proved by L.A. Caffarelli in [4]. Here we give an alternative proof of the fact that the free boundary is close to regular points the graph of a differentiable function. To this end we define the class

$$M^* := \{u : B_1(0) \to \mathbf{R} \;:\; u(x_1, \dots, x_n) = \frac{1}{4}\max(x_1, 0)^2 \;\}. \tag{2.2}$$

The class $M$ is then defined as all rotated elements of $M^*$, i.e.

$$M := \{u : B_1(0) \to \mathbf{R} \;: u = v \circ U \text{ where } U \text{ is a rotation, } v \in M^*\}. \tag{2.3}$$

Moreover for any $\gamma \in (0, \pi)$ the class $M^\gamma$ is defined as

$$\begin{aligned} M^\gamma := \{&u : B_1(0) \to \mathbf{R} \;: u = v \circ U \text{ where } U \text{ is a rotation,} \\ &v \in M^*, \text{ and } \sup_{e \in \partial B_1} |\arccos((Ue) \cdot e)| \le \gamma\}. \end{aligned} \tag{2.4}$$

**Lemma 2.1.** *Let $u$ be a $W^{2,\infty}(B_1)$-solution of (2.1). If one blow-up limit $u_0$ of the blow-up sequence $u(x_0 + r_j\cdot)/r_j^2$ as $r_j \to 0$ is contained in $M$ then all blow-up limits of $u(x_k + r_k\cdot)/r_k^2$ as $x_k \to x_0$ and $r_k \to 0$ are contained in $M$.*

*Proof.* This follows from the upper semicontinuity of the density $x \mapsto \Phi_x(0+)$, from the consequent homogeneity of blow-up limits of $u(x_k + r_k\cdot)/r_k^2$ and from the known fact that each non-trivial homogeneous solution of degree 2 is either contained in $M$ or a quadratic homogeneous polynomial (cf. [4]). $\square$

**Theorem 2.2.** *Let $u$ be a $W^{2,\infty}(B_1)$-solution and suppose that a blow-up limit*

$$\lim_{m\to\infty} \frac{u(r_m\cdot)}{r_m^2}$$

*is contained in $M^*$.*
*Then the free boundary $\partial\{u > 0\}$ is in some open neighborhood of $x_0$ the graph of a differentiable, Lipschitz continuous function.*

Theorem 2.2 follows from the combination of Lipschitz continuity (see the following Proposition) and flatness (see Lemma 2.1). The following Proposition based on the Kelvin transform is crucial.

**Proposition 2.3.** *Let $u$ be a $W^{2,\infty}(B_1)$-solution and suppose that a blow-up limit*

$$\lim_{m\to\infty} \frac{u(r_m\cdot)}{r_m^2}$$

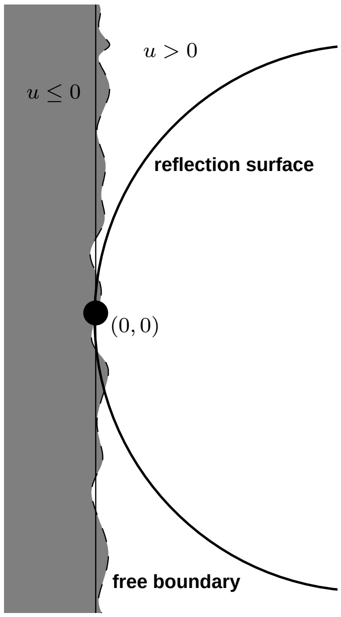


FIGURE 4. Reflection at a sphere cap

*is contained in* $M^*$.
*Then there exist for* $\epsilon \in (0,1)$ *positive* $\delta$ *and* $\rho$ *such that for* $y \in B_\delta \cap \partial\{u > 0\}$ *and* $r \in (0,\rho)$, *the scaled function*

$$u_r(x) = \frac{u(y+rx)}{r^2} \tag{2.5}$$

*satisfies*

$$\mathrm{dist}(u_r, M^{\gamma_0}) = \inf_{v \in M^{\gamma_0}} \sup_{B_1(0)} |v(x) - u_r(x)| < \epsilon,$$

*where* $\gamma_0 = \pi/2 - 1/10$.

*Proof.* First, by continuity and by Lemma 2.1, for any $\tilde{\epsilon} > 0$ there are positive $\tilde{\kappa}, \tilde{\delta}$ and $\tilde{\rho}$ such that

$$\mathrm{dist}(u_{\tilde{\rho}}, M^*) < \tilde{\epsilon} \text{ for } \alpha \in N_{\tilde{\kappa}}(\alpha_0) \text{ and } y \in \partial\{u_\alpha > 0\} \cap B_{\tilde{\delta}}$$

$$\text{and } \operatorname{dist}(u_r, M) < \tilde\epsilon \text{ for } \alpha \in N_{\tilde\kappa}(\alpha_0), y \in \partial\{u_\alpha > 0\} \cap B_{\tilde\delta} \text{ and } r \in (0, \tilde\rho).$$

Now if the statement of the theorem does not hold, then there are positive $r_0$ and a rotation $U_{\theta_0}$ satisfying $\arccos((U_{\theta_0} e) \cdot e) \ge \pi/2 - \gamma_0 - c_1\epsilon > 0$ as well as

$$\operatorname{dist}(u_{r_0} \circ U_{\theta_0}, M^*) \le \tilde\epsilon;$$

here $c_1$ is a constant depending on $(a, b, \lambda_+, \lambda_-)$. It is important for what follows that $\alpha$ and $y$ are the same for $u_{r_0}$ and $u_{\tilde\rho}$. In the remainder of the proof $\alpha$ and $y$ are fixed. Let us now take an arbitrary rotation $V$ such that $\sup_{e\in\partial B_1} |\arccos((Ve) \cdot e)| \le 2(\pi/2 - \gamma_0)$, let $w := u_{\tilde\rho} \circ V$ and define $v$ define by the Kelvin transform (cf. [13, Theorem 4.13], i.e.

$$v(x) := |x - 2e_1|^{2-n} w(\frac{x - 2e_1}{|x - 2e_1|^2})$$

(see Fig. 4). The function $v$ satisfies in $B_1$ the equation

$$\Delta v(x) = |x - 2e_1|^{-n-2} \chi_{\{v>0\}} \ .$$

By the $C^1$-closeness of $u_{\tilde\rho}$ to $M^*$ we know that $w \ge v$ on $\partial(B_1(0) \cap B_1(2e_1))$. Thus

$$\int_{B_1(0)\cap B_1(2e_1)} |\nabla \max(v - w, 0)|^2 \ = \ \int_{B_1(0)\cap B_1(2e_1)} \Delta(w - v) \max(v - w, 0)$$

$$= \int_{B_1(0)\cap B_1(2e_1)} \max(v - w, 0) \left(\chi_{\{w>0\}} - |x - 2e_1|^{-n-2} \chi_{\{v>0\}}\right) \ \le \ 0,$$

implying that $w \ge v$ in $B_1(0) \cap B_1(2e_1)$. Consequently

$$u_{r_0} \ge |\frac{r_0}{\tilde\rho} x - 2e_1|^{2-n} u_{r_0}(V(\frac{x - (\tilde\rho/r_0)2e_1}{|(r_0/\tilde\rho)x - 2e_1|^2})) \text{ in } B_1(0) \cap B_1(2e_1) \ ,$$

a contradiction to

$$\operatorname{dist}(u_{r_0} \circ U_{\theta_0}, M^*) \le \tilde\epsilon$$

in view of $\arccos((U_{\theta_0} e) \cdot e) \ge \pi/2 - \gamma_0 - c_1\epsilon > 0$ and the arbitrary choice of $V$. $\square$

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