CONVEXITY AND UNIQUENESS IN AN INVERSE PROBLEM OF POTENTIAL THEORY

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ABSTRACT. Let Ω_1 and Ω_2 be two bounded domains in $\mathbb{R}^n$ whose intersection is convex. Suppose moreover that their volume potential coincide in the complement of their union. Then $\Omega_1 = \Omega_2$.

In 1938, P.S. Novikov [4] proved that if two bounded, convex (or even starshaped) domains have the same volume potential in the complement of their union, then they are identical. In [5] L. Zalcman asks whether this is true if just one of the domains is assumed to be convex. Also, cf. ([3]; page 86). In our efforts to investigate this problem we where able to prove the following. If the intersection of two bounded domains, which have the same volume potential in the complement of their union, is convex, then they are identical.

Let us introduce some notations. For a bounded domain D we define U^D to be the volume potential of D, i.e.

$$U^D(x) = c_n \int_D \frac{dy}{|x-y|^{n-2}}, \qquad x \in \mathbb{R}^n,$$

where c_n is a normalization factor so that $\Delta U^D = 1$ in D. We also define the *modified Schwarz potential* (MSP) of a bounded domain D with respect to μ to be the solution of $\Delta u = \chi_D - \mu$ (in the sense of distribution) and $u = 0$ on $\mathbb{R}^n \setminus D$, where μ is a distribution supported in $\overline{D}$ and χ_D is the characteristic function of D.

THEOREM 1. *Let Ω_1 and Ω_2 be two bounded domains in $\mathbb{R}^n$, whose intersection $\Omega_1 \cap \Omega_2$ is convex. Suppose moreover $U^{\Omega_1} = U^{\Omega_2}$ in $\mathbb{R}^n \setminus (\Omega_1 \cup \Omega_2)$. Then $\Omega_1 = \Omega_2$.*

REMARK 1. *In theorem 1 we see, by putting $|x|^{n-2}U^{\Omega_1} = |x|^{n-2}U^{\Omega_2}$ and letting $|x| \to \infty$, that Ω_1 and Ω_2 have the same volume, therefore $\Omega_1 \setminus \Omega_2$ and $\Omega_2 \setminus \Omega_1$ are non-empty unless they are identical. It also follows that $\Omega_1 \cap \Omega_2$ is not empty, else U^{Ω_1} can be continued harmonicly into $\mathbb{R}^n$ and violate Liouville's theorem.*

The proof of theorem 1 is based on the following lemmas.

LEMMA 2. *(Gustafsson) In theorem 1, let $\Omega_1 \not\equiv \Omega_2$. Then each domain Ω_j $(j = 1, 2)$ admits a MSP u_j, which is $C(\mathbb{R}^n)$, zero outside Ω_j and satisfies $\Delta u_j = \chi_{\Omega_j} - \mu$, where μ is a distribution supported in $\overline{\Omega}_1 \cap \overline{\Omega}_2$.*

PROOF. Define

$$U = \begin{cases} U^{\Omega_1} & \text{in } \mathbb{R}^n \setminus \Omega_1, \\ U^{\Omega_2} & \text{in } \mathbb{R}^n \setminus \Omega_2. \end{cases}$$

As $U^{\Omega_1} = U^{\Omega_2}$ in $\mathbb{R}^n \setminus (\Omega_1 \cup \Omega_2)$ this definition is consistent. Now extend U to $\mathbb{R}^n$ as a continuous function and set $\mu = \Delta U$. Then μ is a distribution supported in $\overline{\Omega}_1 \cap \overline{\Omega}_2$. Set $u_j = U^{\Omega_j} - U$ for $j = 1, 2$. Then u_j satisfies the conditions stated in the lemma. □

REMARK 2. *In lemma 2 we can even assume the function u_1 (u_2) to be C^1 in a neighbourhood of any point $y \in \partial\Omega_2 \cap \Omega_1$ $(\partial\Omega_1 \cap \Omega_2)$. This will be needed for lemma 4, where we will use the fact that the derivative of u_1 (u_2) at these points exists.*

The following lemma is a modification of a result due to L.A. Caffarelli ([1]; lemma 1).

LEMMA 3. *For Ω_j $(j = 1, 2)$ in theorem 1, $\Omega_1 \not\equiv \Omega_2$ and u_j in lemma 2 the following hold:*

$$\sup_{\partial\Omega_2} u_1 \geq \frac{d_1^2}{2}, \qquad \sup_{\partial\Omega_1} u_2 \geq \frac{d_2^2}{2},$$

where $d_1 = \sup_{x \in \Omega_1} dist(x, \Omega_1 \cap \Omega_2)$ and $d_2 = \sup_{x \in \Omega_2} dist(x, \Omega_1 \cap \Omega_2)$.

PROOF. Let $y \in \partial(\Omega_1 \cap \Omega_2)$ and $z \in \partial\Omega_1$ be two points such that $d_1 = |z - y|$. We may, by rotation and translation, assume y is the origin and $z = (z_1, 0')$. Now set $\Omega_1' = \{x \in \Omega_1 : x_1 > 0\}$ and let $\{z^j\}$ be a sequence in Ω_1' converging to z and satisfying $u_1(z^j) \geq 0$ (as $\Delta u_1 \geq 0$ near z the existence of such points is obvious). Define now

$$w(x) = u_1(x) - u_1(z^j) - \frac{1}{2}(x_1 - z_1^j)^2 \qquad \text{in } \Omega_1',$$

and observe that $w(z^j) = 0$. Then w, being harmonic in Ω_1', attains its positive maximum on the boundary of Ω_1'. Since $w < 0$ on $\partial\Omega_1$ the maximum value is attained at $x_1 = 0$ and it is positive. Now letting $z^j \to z$ we obtain

$$\sup_{\partial\Omega_2} u_1 \geq \sup_{x_1 = 0} u_1 \geq \frac{1}{2} z_1^2 = \frac{d_1^2}{2},$$

where the first inequality is a consequence of the maximum principle, applied to u_1 in $\Omega_1 \setminus \Omega_2$. One, similarly, proves the second statement. Thus the lemma is proved. □

LEMMA 4. *Let Ω_1 and Ω_2 be as in theorem 1, $\Omega_1 \not\equiv \Omega_2$ and u_1, u_2 as in lemma 2. Then at least one of the following holds.*

(1) *There is a point $y^1 \in \Omega_1 \setminus \Omega_2$ such that $\dfrac{d_1^2}{2} < -u_1(y^1)$ and $|\nabla u_1|(y^1) = 0$.*

(2) *There is a point $y^2 \in \Omega_2 \setminus \Omega_1$ such that $\dfrac{d_2^2}{2} < -u_2(y^2)$ and $|\nabla u_2|(y^2) = 0$.*

PROOF. Let u_j $(j = 1, 2)$ be the MSP of Ω_j with respect to μ (see lemma 2). Set

$$W_1 = \{x \in \Omega_1 \setminus \Omega_2 : u_1(x) < 0\}, \qquad W_2 = \{x \in \Omega_2 \setminus \Omega_1 : u_2(x) < 0\},$$

and define $v = (u_1 - u_2)^2$. Then v is subharmonic in $\mathbb{R}^n \setminus (\overline{W_1 \cup W_2})$ and it tends to zero at infinity. Hence, by the maximum principle, v attains its maximum value at $y^1 \in \partial W_1 \cup \partial W_2$. If $W_1 \cup W_2$ is empty, then $u_1 = u_2$ which implies $\Omega_1 = \Omega_2$, and the proof will be completed. So assume it is not empty. Then, as $u_1 = u_2 = 0$ on $(\partial W_1 \cup \partial W_2) \setminus (\partial\Omega_1 \cup \partial\Omega_2)$, we conclude that $y^1 \in (\partial\Omega_1 \cup \partial\Omega_2) \cap (\partial W_1 \cup \partial W_2)$, i.e. $y^1 \in (\partial\Omega_1 \cap \partial W_2) \cup (\partial\Omega_2 \cap \partial W_1)$. Let, without loss of generality, $y^1 \in \partial\Omega_2 \cap \partial W_1$. Then

$$-u_1(y^1) = \sup_{\partial\Omega_2} \sqrt{v} > \sup_{\partial\Omega_2} u_1 \geq \frac{d_1^2}{2};$$

the last inequality follows from lemma 3. By remark 2 we may assume v is C^1 at y^1. Now either $|\nabla v|(y^1) = 0$, which (since u_2 has vanishing Cauchy-data) will complete the proof, or $\nu.\nabla v(y^1) > 0$ for some vector ν pointing outward Ω_2. Since u_2 has vanishing Cauchy-data at y^1 we obtain $\nu.\nabla u_1^2(y^1) > 0$. As $u_1(y^1) < 0$ we arrive at $\nu.\nabla u_1(y^1) < 0$. Thus u_1 decreases strictly in direction ν. Therefore there is another point in $\Omega_1 \setminus \overline{\Omega_2}$, which we (again) label y^1, at which u_1 attains its minimum value and consequently $|\nabla u_1|(y^1) = 0$. This completes the proof. □

PROOF OF THEOREM 1. Suppose the conclusion in the theorem doesn't hold. Then, by lemma 2 and 4, we may assume that the MSP u_1 of Ω_1 exists and satisfies:

(1) $\Delta u_1 = 1$ in $\Omega_1 \setminus \Omega_2$ (which, by remark 1, is not empty),

(2) there is a point $y^1 \in \Omega_1 \setminus \Omega_2$ such that $\dfrac{d_1^2}{2} < -u_1(y^1)$ and $|\nabla u_1|(y^1) = 0$.

We may obviously assume that $u_1(y^1) = \inf_{\Omega_1 \setminus \Omega_2} u_1$. Now by rotation and translation we may assume y^1 is the origin and $\Omega_1 \cap \Omega_2$ is in the half-space $\{x : x_1 < 0\}$. Set $v = \frac{1}{2}x_1^2 - u_1$ in $\Omega_1' = \{x \in \Omega_1 : x_1 > 0\}$. Then v, being harmonic in Ω_1', attains its maximum value on Ω_1' at the boundary $\partial\Omega_1'$. By (2) and that $u_1 = 0$ on $\partial\Omega_1$ we conclude that this maximum value is attained on $\partial\Omega_1' \setminus \partial\Omega_1$. Hence at the origin. Therefore, by Hopf's maximum principle [2],

$$0 > \frac{\partial v}{\partial x_1}(0) = 0,$$

since both $\frac{1}{2}x_1$ and u_1 has vanishing gradient at the origin. Thus a contradiction is obtained and the theorem is proved. □

References

1. L.A. Caffarelli, *Compactness methods in free boundary problems*, Partial Differential Equations **5** (1980), 427–448.
2. E. Hopf, *A remark on linear elliptic differential equations of second order*, Proc. AMS 3 (1952), 791–793.
3. V. Isakov, *Inverse Source Problems*, vol. 34, AMS, Providence, Rhode Island, 1990.
4. P.S. Novikov, *Sur le probème inverse du potentiel*, Dokl. Akad. Nauk SSSR **18** (1938), 165–168.
5. L. Zalcman, *Some inverse problems of potential theory*, Contemporary Math. **63** (1987), 337–350.