

# A PARABOLIC TWO-PHASE OBSTACLE-LIKE EQUATION

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ABSTRACT. For the parabolic obstacle-problem-like equation

$$\Delta u - \partial_t u = \lambda_+ \chi_{\{u > 0\}} - \lambda_- \chi_{\{u < 0\}},$$

where  $\lambda_+$  and  $\lambda_-$  are positive Lipschitz functions, we prove in arbitrary finite dimension that the free boundary  $\partial\{u > 0\} \cup \partial\{u < 0\}$  is in a neighborhood of each “branch point” the union of two Lipschitz graphs that are continuously differentiable with respect to the space variables. The result extends the elliptic paper [11] to the parabolic case. There are substantial difficulties in the parabolic case due to the fact that the time derivative of the solution is in general not a continuous function.

Our result is optimal in the sense that the graphs are in general not better than Lipschitz, as shown by a counter-example.

## CONTENTS

|  |    |
|--|----|
| 1. Introduction                                  | 2  |
| 1.1. Background and main result                  | 2  |
| 2. Notation                                      | 4  |
| 3. A supremum-mean-value estimate                | 4  |
| 4. Non-degeneracy and regularity of the solution | 5  |
| 5. Vanishing time derivative                     | 7  |
| 6. Directional monotonicity                      | 7  |
| 7. The set of non-vanishing gradient             | 9  |
| 8. Global solutions                              | 10 |
| 9. Uniform closeness to $h$                      | 13 |
| 10. Continuity of the time derivative            | 14 |
| 11. Directional Monotonicity II                  | 15 |

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|                               |    |
|-------------------------------|----|
| 12. Proof of the main theorem | 16 |
| References                    | 20 |

## 1. INTRODUCTION

1.1. **Background and main result.** In this paper we study the regularity of the parabolic obstacle-problem-like equation

$$(1.1) \quad \Delta u - \partial_t u = \lambda_+ \chi_{\{u > 0\}} - \lambda_- \chi_{\{u < 0\}} \quad \text{in } (0, T) \times \Omega,$$

where  $T < +\infty$ ,  $\lambda_+ > 0$ ,  $\lambda_- > 0$  are Lipschitz functions and  $\Omega \subset \mathbf{R}^n$  is a given domain. The problem arises as limiting case in the model of temperature control through the interior described in [4, 2.3.2] as  $h_1, h_2 \rightarrow 0$ .

We are interested in the regularity of the free boundary  $\partial\{u > 0\} \cup \partial\{u < 0\}$ . As the one-phase case (i.e. the case of a non-negative or non-positive solution) is covered by classical results, and regularity of the set  $\{u = 0\} \cap \{\nabla u \neq 0\}$  can be obtained via the implicit function theorem (see Section 7 for higher regularity), the research focuses on the study of  $\partial\{u > 0\} \cap \partial\{u < 0\} \cap \{\nabla u = 0\}$ .

In the stationary case — the two-phase membrane problem — the authors proved ([12] and [11]) that the free boundary  $\partial\{u > 0\} \cup \partial\{u < 0\}$  is in a neighborhood of each branch point, i.e. a point in the set  $\Omega \cap \partial\{u > 0\} \cap \partial\{u < 0\} \cap \{\nabla u = 0\}$ , the union of (at most) two  $C^1$ -graphs. Note that the definition of “branch point” does not necessarily imply a bifurcation as that in Figure 1.

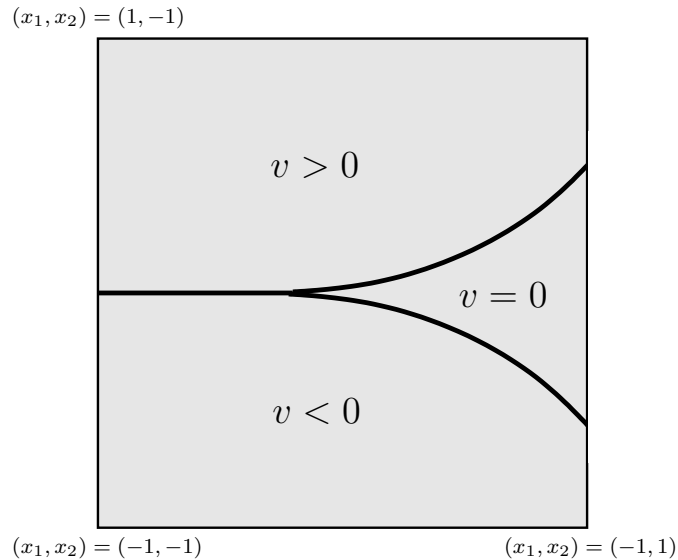


FIGURE 1. Example of a Stationary Branch Point

We formulate the main result in this paper.

**Theorem 1.1.** *Suppose that*

$$0 < \lambda_{\min} \leq \inf_{Q_1(0)} \min(\lambda_+, \lambda_-), \quad \sup_{Q_1(0)} \max(|\nabla \lambda_+|, |\nabla \lambda_-|, |\partial_t \lambda_+|, |\partial_t \lambda_-|) < +\infty$$

and that  $u$  is a weak solution of

$$\Delta u - \partial_t u = \lambda_+ \chi_{\{u > 0\}} - \lambda_- \chi_{\{u < 0\}} \text{ in } Q_1(0);$$

here  $Q_1(0)$  is the parabolic cylinder  $(-1, 1) \times B_1(0)$ .

Then there are constants  $\sigma > 0$  and  $r_0 > 0$  such that

$$(1.2) \quad u(0) = 0, |\nabla u(0)| \leq \sigma, \text{pardist}(0, \{u > 0\}) \leq \sigma \text{ and } \text{pardist}(0, \{u < 0\}) \leq \sigma$$

imply  $\partial\{u > 0\} \cap Q_{r_0}(0)$  and  $\partial\{u < 0\} \cap Q_{r_0}(0)$  being graphs of Lipschitz functions (in some space direction) that are continuously differentiable with respect to the space variables. The constants  $\sigma, r_0$ , the Lipschitz norms and the modulus of continuity of the spatial normal vectors to these surfaces depend only on  $\inf_{Q_1(0)} \min(\lambda_+, \lambda_-)$ , the Lipschitz norms of  $\lambda_{\pm}$ , the supremum norm of  $u$  and the space dimension  $n$ .

Moreover the regularity above is optimal in the sense that the graphs are in general not better than Lipschitz.

**Corollary 1.2.** *Suppose that*

$$0 < \lambda_{\min} \leq \inf_{Q_1(0)} \min(\lambda_+, \lambda_-), \quad \sup_{Q_1(0)} \max(|\nabla \lambda_+|, |\nabla \lambda_-|, |\partial_t \lambda_+|, |\partial_t \lambda_-|) < +\infty$$

and that  $u$  is a weak solution of

$$\Delta u - \partial_t u = \lambda_+ \chi_{\{u > 0\}} - \lambda_- \chi_{\{u < 0\}} \text{ in } Q_1(0).$$

Then there is a constant  $r_0 > 0$  such that if the origin is a branch point, then  $\partial\{u > 0\} \cap Q_{r_0}(0)$  and  $\partial\{u < 0\} \cap Q_{r_0}(0)$  are graphs of Lipschitz functions (in some space direction) that are continuously differentiable with respect to the space variables. The constant  $r_0$ , the Lipschitz norms and the modulus of continuity of the spatial normal vectors to these surfaces depend only on  $\inf_{Q_1(0)} \min(\lambda_+, \lambda_-)$ , the Lipschitz norms of  $\lambda_{\pm}$ , the supremum norm of  $u$  and the space dimension  $n$ .

As to the proof we extend the method of [11] to the parabolic case. There is however a substantial difficulty as the time derivative  $\partial_t u$  is in general not continuous, so that it is not possible to apply directly the comparison principle. We deal with that problem by a *two-stage* proof of directional monotonicity and by establishing alternative tools for the time derivative.

## 2. NOTATION

Throughout this article  $\mathbf{R}^n$  will be equipped with the Euclidean inner product  $x \cdot y$  and the induced norm  $|x|$ ,  $B_r(x^0)$  will denote the open  $n$ -dimensional ball of center  $x^0$ , radius  $r$  and volume  $r^n \omega_n$ ,  $B'_r(0)$  the open  $n - 1$ -dimensional ball of center 0 and radius  $r$ , and  $e_i$  the  $i$ -th unit vector in  $\mathbf{R}^n$ . We define  $Q_r(t^0, x^0) := (t^0 - r^2, t^0 + r^2) \times B_r(x^0)$  to be the cylinder of radius  $r$  and height  $2r^2$ ,  $Q_r^-(t^0, x^0) := (t^0 - r^2, t^0) \times B_r(x^0)$  its “negative part” and  $Q_r^+(t^0, x^0) := (t^0, t^0 + r^2) \times B_r(x^0)$  its “positive part”. When omitted,  $x^0$  (or  $(t^0, x^0)$ , respectively) is assumed to be the origin. Moreover let  $\partial_{\text{par}} Q_r(t^0, x^0) := (t^0 - r^2, t^0 + r^2) \times \partial B_r(x^0) \cup \{t^0 - r^2\} \times B_r(x^0)$  denote the parabolic boundary of  $Q_r(t^0, x^0)$ . Let us also introduce the parabolic distance  $\text{pardist}((t, x), A) := \inf_{(s, y) \in A} \sqrt{|x - y|^2 + |t - s|}$ . Given a set  $A \subset \mathbf{R}^{n+1}$ , we denote its interior by  $A^\circ$  and its characteristic function by  $\chi_A$ . By  $\nabla u$  we mean the gradient with respect to the space variables. In the text we use the  $n$ -dimensional Lebesgue-measure  $\mathcal{L}^n$  and the  $m$ -dimensional Hausdorff measure  $\mathcal{H}^m$ . Finally,  $\mathbf{C}^{\beta, \mu} := \mathbf{H}^{\mu, \beta}$  denotes the parabolic Hölder-space as defined in [7].

## 3. A SUPREMUM-MEAN-VALUE ESTIMATE

In this section we show that at branch points the time derivative  $\partial_t u$ , in general a discontinuous function, satisfies a sup-mean-value estimate.

**Lemma 3.1.** *Let  $Q_{2r}^-(t^0, x^0) \subset (0, T) \times \Omega$  and let  $\lambda_+, \lambda_-$  be non-negative and Lipschitz continuous with respect to the time variable. Then each solution  $u$  of (1.1) satisfies*

$$\sup_{Q_1} |\partial_t u_{r_k}| = \sup_{Q_r^-(t^0, x^0)} |\partial_t u| \leq C \left( r^2 + \left( r^{-n-2} \int_{Q_{2r}^-(t^0, x^0)} |\partial_t u|^2 \right)^{\frac{1}{2}} \right).$$

*Proof.* Using the scaling invariance of the equation with respect to the scaling

$$u_r(t, x) = r^{-2} u(t^0 + r^2 t, x^0 + r x)$$

we may assume that  $r = 1/2, t^0 = 0$  and  $x^0 = 0$ .

Let  $H(t, x, z) = \lambda_+(t, x) \chi_{\{z > 0\}} - \lambda_-(t, x) \chi_{\{z < 0\}}$ . For

$$v(t, x) := \partial_t^\tau u(t, x) := \frac{u(t + \tau, x) - u(t)}{\tau}$$

and  $\eta \in L^2((-1, 1); W^{1,2}(B_1))$  such that  $\eta = 0$  on  $(-1, 0) \times \partial B_1$ , we calculate

$$(3.1) \quad \begin{aligned} & \int_{-1}^s \int_{B_1} (\eta \partial_t v + \nabla v \cdot \nabla \eta) \\ &= - \int_{-1}^s \int_{B_1} \eta \partial_t^\tau H(t, x, u(t, x)), \quad s \in (-1, 0). \end{aligned}$$

Here

$$\begin{aligned} \partial_t^\tau H(t, x, u(t, x)) &= \lambda_+(t^0 + r^2 t, x^0 + r x) \partial_t^\tau \chi_{\{u > 0\}} - \lambda_-(t^0 + r^2 t, x^0 + r x) \partial_t^\tau \chi_{\{u < 0\}} \\ &+ \chi_{\{u(t^0 + r^2(t+\tau), x^0 + r x) > 0\}} \partial_t^\tau \lambda_+ - \chi_{\{u(t^0 + r^2(t+\tau), x^0 + r x) < 0\}} \partial_t^\tau \lambda_- . \end{aligned}$$

Testing with  $\eta(t, x) := \zeta^2(x)\phi^2(t)\max(v(t, x) - k, 0)$  where  $k \geq 0$ ,  $\zeta \in C_0^{0,1}(B_1)$  and  $\phi \in C^{0,1}(-1, 1)$  such that  $\phi(t) \in [0, 1]$  and

$$\phi(t) := \begin{cases} 1, & t \geq -1/2 \\ 0, & t \leq -1, \end{cases}$$

and observing that

$$\max(v(t, x) - k, 0)\partial_t^\tau H(t, x, u(t, x)) \geq -C_1 r^2 \max(v(t, x) - k, 0)$$

we obtain

$$(3.2) \quad \sup_{-1 < s < 0} \int_{B_1} \phi^2(s)\zeta^2 \max(v(s, \cdot) - k, 0)^2 + \int_{-1}^s \int_{B_1} \phi^2 \zeta^2 |\nabla \max(v - k, 0)|^2 \\ \leq C_2 \int_{-1}^s \int_{B_1} [\max(v - k, 0)^2 (\phi^2 |\nabla \zeta|^2) + \phi |\partial_t \phi| \zeta^2 + r^2 \phi^2 \zeta^2 \max(v - k, 0)].$$

From the proof of [8, Theorem 4.7] we infer that

$$(3.3) \quad \sup_{Q_{1/2}^-} v \leq C_3 \left( r^2 + \left( \int_{Q_1^-} v^2 \right)^{\frac{1}{2}} \right).$$

Testing with  $\eta(t, x) := \zeta^2(x)\phi^2(t)\max(-v(t, x) - k, 0)$  where  $k \geq 0$ , we obtain in a similar way that

$$(3.4) \quad \sup_{Q_{1/2}^-} (-v) \leq C_3 \left( r^2 + \int_{Q_1^-} v^2 \right)^{\frac{1}{2}}.$$

Letting  $\tau \rightarrow 0$  and scaling back we obtain the statement.  $\square$

#### 4. NON-DEGENERACY AND REGULARITY OF THE SOLUTION

**Lemma 4.1** (Non-Degeneracy). *For every  $Q_{2r}(t^0, x^0) \subset (0, T) \times \Omega$  the following holds:*

$$1) \text{ If } (t^0, x^0) \in \partial\{u > 0\}, \text{ then } \sup_{Q_r^-(t^0, x^0)} u \geq \frac{1}{8n} \inf_{Q_r(t^0, x^0)} \lambda_+ r^2.$$

$$2) \text{ If } (t^0, x^0) \in \partial\{u < 0\}, \text{ then } \inf_{Q_r^-(t^0, x^0)} u \leq -\frac{1}{8n} \inf_{Q_r(t^0, x^0)} \lambda_- r^2.$$

*Proof.* We choose a sequence  $\{u > 0\} \ni (t^m, x^m) \rightarrow (t^0, x^0)$  as  $m \rightarrow \infty$ . Supposing that  $\sup_{Q_r^-(t^m, x^m)} u \leq \frac{1}{8n} \inf_{Q_r(t^0, x^0)} \lambda_+ r^2$ , the comparison principle yields that  $u(t, x) \leq v(t, x) := (\frac{t^m - t}{2} + \frac{1}{8n}|x - x_m|^2) \inf_{Q_r(t^0, x^0)} \lambda_+$  in  $Q_r^-(t^m, x^m)$ , a contradiction to the fact that  $u(t^m, x^m) > 0$ .

The estimate for  $\inf_{Q_r^-(t^0, x^0)} u$  is obtained the same way, replacing  $u$  by  $-u$  and  $\lambda_+$  by  $\lambda_-$ .  $\square$

**Lemma 4.2.** *Let  $\lambda_+, \lambda_- \in C_{\text{loc}}^{0,1}((0, T) \times \Omega)$ . Then each solution  $u$  of (1.1) satisfies the following:*

- 1)  $\partial_t u \in L_{\text{loc}}^\infty((0, T) \times \Omega)$ .
- 2)  $\partial_t \nabla u \in L_{\text{loc}}^2((0, T) \times \Omega)$ .

*Proof.* 1) follows from Lemma 3.1.

2) follows from (3.2) with  $k = 0$  and from the analogous estimate for  $\max(-v, 0)$ .  $\square$

**Corollary 4.3.** *For every  $Q_{2r}(t^0, x^0) \subset (0, T) \times \Omega$ , there exists a constant  $c_0 > 0$  depending only on  $n$  and  $\|\partial_t u\|_{L^\infty(Q_r(t^0, x^0))}$  such that*

$$\begin{aligned} u \geq 0 \text{ in } Q_r^-(t^0, x^0) \text{ implies } u \geq 0 \text{ in } Q_{c_0 r}(t^0, x^0), \text{ and} \\ u \leq 0 \text{ in } Q_r^-(t^0, x^0) \text{ implies } u \leq 0 \text{ in } Q_{c_0 r}(t^0, x^0). \end{aligned}$$

*Proof.* Suppose towards a contradiction that  $u(t^1, x^1) < 0$  for some  $(t^1, x^1) \in Q_{c_0 r}^+(t^0, x^0)$ . Then there is a point  $(t^2, x^2) \in \partial\{u < 0\} \cap \overline{Q_{c_0 r}^+(t^0, x^0)}$ . Applying Lemma 4.1 at  $(t^2, x^2)$  with respect to the cylinder  $Q_{(1-c_0)r}(t^2, x^2)$  yields a contradiction to Lemma 4.2 1) provided that  $c_0$  has been chosen small enough.

The second estimate is proved in the same fashion.  $\square$

**Proposition 4.4.** *Let  $\lambda_+, \lambda_- \in C_{\text{loc}}^{0,1}((0, T) \times \Omega)$ . Then each solution  $u$  of (1.1) satisfies  $\nabla u \in C_{\text{loc}}^{1/2,1}((0, T) \times \Omega)$ , that is, the gradient is Lipschitz continuous with respect to the space variables and Hölder continuous with exponent  $1/2$  with respect to the time variable.*

*Proof.* Let us first show that for any  $e \in \partial B_1$ ,  $(\Delta - \partial_t)(\max(\partial_e u, 0)) \geq -C$  and  $(\Delta - \partial_t)(\max(-\partial_e u, 0)) \geq -C$  in  $\Omega$ . We give a formal proof that can be made rigorous translating everything into a weak formulation. In  $\{\partial_e u > 0\}$ ,

$$\begin{aligned} & (\Delta - \partial_t)(\partial_e u) \\ &= \frac{\partial_e u}{|\nabla u|} (\lambda_+ \mathcal{H}^{n-1}[\{\nabla u \neq 0\} \cap \partial\{u > 0\}] + \lambda_- \mathcal{H}^{n-1}[\{\nabla u \neq 0\} \cap \partial\{u < 0\}]) \\ & \quad + \partial_e \lambda_{+\chi_{\{u > 0\}}} - \partial_e \lambda_{-\chi_{\{u < 0\}}} \geq -C. \end{aligned}$$

As  $\partial_e u$  is continuous, we obtain  $(\Delta - \partial_t)(\max(\partial_e u, 0)) \geq -C$ .

Considering  $-e$  instead of  $e$  we obtain also  $(\Delta - \partial_t)(\max(-\partial_e u, 0)) \geq -C$ . But then the ‘‘almost monotonicity formula’’ Theorem I of [5] applies and we proceed as follows (cf. [9]): at each point  $(t^0, x^0) \in \{u \neq 0\} \cap \{\nabla u = 0\}$ , we obtain from the almost monotonicity formula that  $\nabla \partial_e u$  is bounded at  $(t^0, x^0)$  by a locally uniform constant.

At each point  $(t^0, x^0) \in \{u \neq 0\} \cap \{\nabla u \neq 0\}$ , we obtain in a similar way that for every  $e \perp \nabla u(t^0, x^0)$ ,  $|\nabla \partial_e u(t^0, x^0)|$  is bounded by a locally uniform constant. Let  $e_1 = \frac{\nabla u(t^0, x^0)}{|\nabla u(t^0, x^0)|}$ . Then  $-\partial_{11} u(t^0, x^0) = -\lambda_+ \chi_{\{u(t^0, x^0) > 0\}} + \lambda_- \chi_{\{u(t^0, x^0) < 0\}} - \partial_t u(t^0, x^0) + \sum_{j=2}^n \partial_{jj} u(t^0, x^0)$  is by Lemma 4.2 bounded by a locally uniform constant.  $\square$

**Corollary 4.5.**  $\mathcal{L}^{n+1}(\partial\{u > 0\} \cup \partial\{u < 0\}) = 0$

*Proof.* First, we obtain from Lemma 4.1, Lemma 4.2 and Proposition 4.4 that there exists a locally uniform constant  $c > 0$  such that for  $Q_{2r}(s, y) \subset (0, T) \times \Omega$ ,

$$\frac{\mathcal{L}^{n+1}(Q_r(s, y) \cap \{u > 0\})}{\mathcal{L}^{n+1}(Q_r)} \geq c > 0 \text{ if } (s, y) \in \partial\{u > 0\}$$

and  $\frac{\mathcal{L}^{n+1}(Q_r(s, y) \cap \{u < 0\})}{\mathcal{L}^{n+1}(Q_r)} \geq c > 0 \text{ if } (s, y) \in \partial\{u < 0\}.$

Since  $\chi_{\{u > 0\}} * \chi_{Q_r} / \mathcal{L}^{n+1}(Q_r) \rightarrow \chi_{\{u > 0\}}$  in  $L^1_{\text{loc}}((0, T) \times \Omega)$  as  $r \rightarrow 0$  and the analogous fact holds for  $\chi_{\{u < 0\}}$ , we obtain that  $\chi_{\{u > 0\}} \geq c > 0$   $\mathcal{L}^{n+1}$ -a.e. on  $\partial\{u > 0\}$  and  $\chi_{\{u < 0\}} \geq c > 0$   $\mathcal{L}^{n+1}$ -a.e. on  $\partial\{u < 0\}$ . Thus  $\mathcal{L}^{n+1}(\partial\{u > 0\} \cup \partial\{u < 0\}) = 0$ .  $\square$

## 5. VANISHING TIME DERIVATIVE

As a corollary of Lemma 3.1 we obtain now that at points at which the blow-up limit depends only on the space variables, the time derivative  $\partial_t u$  – in general a discontinuous function – attains the limit 0.

**Corollary 5.1.** *Let  $Q_{2r}(t^0, x^0) \subset (0, T) \times \Omega$  and suppose that for a sequence of solutions  $u_k$  in  $(0, T) \times \Omega$*

$$u_{r_k}(t, x) = r_k^{-2} u_k(t^k + r_k^2 t, x^k + r_k x) \rightarrow u_0(x) \text{ in } L^1_{\text{loc}}(\mathbf{R}^{n+1}) \text{ as } r_k \rightarrow 0.$$

*Then*

$$\sup_{Q_{r_k}(t^k, x^k)} |\partial_t u_k| \rightarrow 0$$

*as  $r_k \rightarrow 0$ .*

*Proof.* The statement follows from Lemma 3.1 and the fact that  $\partial_t u_{r_k}$  converges to 0 in  $L^2_{\text{loc}}(\mathbf{R}^{n+1})$  as  $r_k \rightarrow 0$ . The  $L^2$ -convergence in turn may be shown as follows: as  $\partial_t u_k$  is by Lemma 3.1 bounded in  $L^\infty(Q_r(t^0, x^0))$ , it is sufficient to prove a.e. convergence. For  $(s, y) \in \{u_0 = 0\}^0$  we obtain from Lemma 4.1 that  $u_{r_k} = 0$  in  $Q_\delta(s, y)$  for some  $\delta > 0$  and large  $k$ . For  $(s, y) \in \{u_0 > 0\} \cup \{u_0 < 0\}$ ,  $u_{r_k}$  converges in  $C^1(Q_\delta(s, y))$  for some  $\delta > 0$  as  $k \rightarrow \infty$ . Moreover we know from Corollary 4.5 that  $\mathcal{L}^{n+1}(\partial\{u_0 > 0\} \cup \partial\{u_0 < 0\}) = 0$ . It follows that  $\partial_t u_{r_k}$  converges  $\mathcal{L}^{n+1}$ -a.e. to  $\partial_t u_0$ .  $\square$

## 6. DIRECTIONAL MONOTONICITY

In a first stage, we show that if the solution is close to the one-dimensional solution

$$(6.1) \quad h(x) := \frac{\lambda_+(0)}{2} \max(x_1, 0)^2 - \frac{\lambda_-(0)}{2} \min(x_1, 0)^2.$$

then it is increasing in a cone of *spatial* directions. Later on we will extend the result to a cone of *tempo-spatial* directions.

**Proposition 6.1.** *Let  $0 < \lambda_{\min} \leq \inf_{Q_1(0)} \min(\lambda_+, \lambda_-)$ ,  $h$  as in (6.1), and let  $\varepsilon \in (0, 1)$ . Then each solution  $u$  of (1.1) in  $Q_1(0)$  such that*

$$\text{dist}_{L^\infty((-1,1); W^{1,\infty}(B_1))}(u, h) \leq \delta := \frac{\lambda_{\min}\varepsilon}{48n}$$

and

$$\sup_{Q_1(0)} \max(|\nabla\lambda_+|, |\nabla\lambda_-|) \leq \delta$$

satisfies  $\varepsilon^{-1}\partial_e u - |u| \geq 0$  in  $Q_{1/2}(0)$  for every  $e \in \partial B_1(0)$  such that  $e_1 \geq \varepsilon$ ; here  $e_1$  denotes the first component of the vector  $e$ .

*Proof.* First note that  $\varepsilon^{-1}\partial_e h - |h| \geq 0$  in  $Q_2(0)$ . It follows that

$$(6.2) \quad \varepsilon^{-1}\partial_e u - |u| \geq -3\delta\varepsilon^{-1} \text{ in } Q_1(0)$$

provided that  $\text{dist}_{L^\infty((-1,1); W^{1,\infty}(B_1))}(u, h) \leq \delta$ . Suppose now towards a contradiction that the statement is not true. Then there exist  $\lambda_+, \lambda_- \in (\lambda_{\min}, +\infty)$ ,  $(t^*, x^*) \in Q_{1/2}(0)$ ,  $e^*$ , and a solution  $u$  of (1.1) in  $Q_1(0)$  such that  $\text{dist}_{L^\infty((-1,1); W^{1,\infty}(B_1))}(u, h) \leq \delta$ ,

$$\sup_{Q_1(0)} \max(|\nabla\lambda_+|, |\nabla\lambda_-|) \leq \delta,$$

$e_1^* \geq \varepsilon$  and  $\varepsilon^{-1}\partial_{e^*} u(t^*, x^*) - |u(t^*, x^*)| < 0$ . For the positive constant  $c$  to be defined later the functions  $v := \varepsilon^{-1}\partial_{e^*} u - |u|$  and  $w := \varepsilon^{-1}\partial_{e^*} u - |u| + c|x - x^*|^2 - c(t - t^*)$  satisfy then the following: in the set  $D := Q_1(0) \cap \{v < 0\} \cap \{t < t^*\}$ ,

$$\begin{aligned} \Delta w - \partial_t w &\leq 2nc + c - \lambda_+ \chi_{\{u>0\}} - \lambda_- \chi_{\{u<0\}} \\ &+ \varepsilon^{-1}(\lambda_+ + \lambda_-) \nu_x \cdot e^* \mathcal{H}^{n-1}[(\{u=0\} \cap \{\nabla u \neq 0\}) \\ &+ \varepsilon^{-1}(\chi_{\{u>0\}} \partial_{e^*} \lambda_+ - \chi_{\{u<0\}} \partial_{e^*} \lambda_-) \end{aligned}$$

where  $\nu_x = \frac{\nabla u}{|\nabla u|}$ . As

$$\nu_x \cdot e^* < 0 \text{ on } \{u=0\} \cap \{v < 0\} = \{u=0\} \cap \{\varepsilon^{-1}\partial_{e^*} u < 0\},$$

we obtain by the definition of  $\delta$  that  $w$  is supercaloric in  $D$  provided that  $c$  has been chosen accordingly, say  $c := \lambda_{\min}/(4n)$ . It follows that the negative infimum of  $w$  is attained on

$$\partial_{\text{par}} D \subset (\partial_{\text{par}} Q_1(0) \cap \{t \leq t^*\}) \cup (Q_1(0) \cap \partial\{v < 0\}).$$

Consequently it is attained on  $\{t \leq t^*\} \cap \partial_{\text{par}} Q_1(0)$ , say at the point  $(\bar{t}, \bar{x}) \in \{t \leq t^*\} \cap \partial_{\text{par}} Q_1(0)$ . Since  $\text{pardist}((\bar{t}, \bar{x}), (t^*, x^*)) \geq 1/2$ , we obtain that

$$\begin{aligned} \varepsilon^{-1}\partial_{e^*} u(\bar{t}, \bar{x}) - |u(\bar{t}, \bar{x})| &= v(\bar{t}, \bar{x}) = w(\bar{t}, \bar{x}) - c|x^* - \bar{x}|^2 + c(\bar{t} - t^*) \\ &< -c/4 = -\lambda_{\min}/(16n). \end{aligned}$$

But this contradicts (6.2) in view of  $\delta = \frac{\lambda_{\min}\varepsilon}{48n}$ .  $\square$



## 7. THE SET OF NON-VANISHING GRADIENT

In the sequel we are going to need higher regularity of the level set  $\{u = 0\} \cap \{\nabla u \neq 0\}$ . Higher regularity can be obtained in a standard way using the von Mises transform:

**Lemma 7.1.** *The set  $\{u = 0\} \cap \{\nabla u \neq 0\}$  is locally in  $(0, T) \times \Omega$  a  $C^1$ -surface and  $\partial_t u$  is continuous on that surface.*

*Proof.* Let  $(t^0, x^0) \in \{u = 0\} \cap \{\nabla u \neq 0\}$ . We may assume that  $\nabla u(t^0, x^0) = \partial_1 u(t^0, x^0)$  and that in  $Q_\delta(t^0, x^0)$ ,  $u$  is strictly increasing in the  $x_1$ -direction and  $\{u = 0\}$  is the graph of a function, say  $x_1 = g(t, x')$  for  $(t, x) \in Q_\delta(t^0, x^0)$ , where  $g \in C^0((t^0 - \delta^2, t^0 + \delta^2); C^1(B'_\delta(x^0)))$ . It is sufficient to prove that  $g \in C^1(Q_{\delta/2}(t^0, x^0))$ . To do so, we use von Mises variables, i.e.

$$y = u(t, x_1, x') \text{ and } x_1 = v(t, y, x').$$

The calculation

$$\partial_1 v = 1/\partial_1 u, \partial_t v = -\partial_t u/\partial_1 u, \partial_i v = -\partial_i u/\partial_1 u \text{ for } 2 \leq i \leq n,$$

$$\partial_{ij} v + \partial_{i1} v \partial_j u + \partial_{1j} v \partial_i u + \partial_1 v \partial_{ij} u + \partial_{11} v \partial_i u \partial_j u = 0 \text{ for } 2 \leq i, j \leq n,$$

$$\partial_{i1} v \partial_1 u + \partial_{11} v \partial_1 u \partial_i u + \partial_1 v \partial_{i1} u = 0 \text{ for } 2 \leq i \leq n, \partial_{11} v = -\partial_{11} u/(\partial_1 u)^3$$

assures that  $\partial_t v$  and all spatial second derivatives of  $v$  are bounded. Moreover

$$\begin{aligned} & \left( \frac{-1 - |\nabla' v|^2}{(\partial_y v)^3} \partial_{yy} v \right) - \frac{\Delta' v}{\partial_y v} + 2 \frac{\nabla' v \cdot \nabla' \partial_y v}{(\partial_y v)^2} + \frac{\partial_t v}{\partial_y v} \\ &= \begin{cases} \lambda_+(t, v(t, y, x'), x'), & y > 0 \\ -\lambda_-(t, v(t, y, x'), x'), & y < 0 \end{cases}. \end{aligned}$$

Thus

$$\partial_t v - a_{ij}(\nabla v) \partial_{ij} v = f(t, y, x') \partial_y v := \begin{cases} -\lambda_+(t, v(t, y, x'), x') \partial_y v, & y > 0 \\ \lambda_-(t, v(t, y, x'), x') \partial_y v, & y < 0. \end{cases}$$

Provided that  $\delta$  has been chosen small enough,  $|\nabla' v| \leq 1/2$ ,  $0 < \partial_y v \leq C$  and the above equation is uniformly parabolic. Furthermore

$$\begin{aligned} & \partial_t \partial_t^h v - a_{ij}(\nabla v) \partial_{ij} \partial_t^h v - \frac{\partial a_{ij}(z_h)}{\partial p_k} \partial_{ij} v(t+h, y, x') \partial_k \partial_t^h v \\ &= f(t, y, x') \partial_y \partial_t^h v + \partial_y v(t+h, y, x') \partial_t^h f(t, y, x') \end{aligned}$$

where  $z_h = \theta(t, y, x') \nabla v(t+h, y, x') + (1 - \theta(t, y, x')) \nabla v(t, y, x')$  and  $\theta(t, y, x') \in [0, 1]$ . Since  $f(t, y, x')$ ,  $\partial_t^h f(t, y, x')$  and  $\frac{\partial a_{ij}(z_h)}{\partial p_k} \partial_{ij} v(t+h, y, x')$  are bounded uniformly in  $h$ , we obtain from [6] that  $\partial_t^h v$  is uniformly Hölder continuous with respect to  $h$  and that  $\partial_t u$  is Hölder continuous in  $Q_{\delta/2}(t^0, x^0)$ .  $\square$

## 8. GLOBAL SOLUTIONS

In this section we extend our characterization of elliptic global solutions [10, Theorem 4.3] to the parabolic case. We are going to need the following version of the Caffarelli-Kenig monotonicity formula of [3]:

**Theorem 8.1.** *Let*

$$\Phi(r, w) := \frac{1}{r^4} I(r, \max(w, 0)) I(r, \max(-w, 0))$$

where

$$I(r, v) := \int_{-r^2}^0 \int_{\mathbf{R}^n} |\nabla v|^2 G(t, x)$$

and  $G$  is the backwards heat kernel

$$G(t, x) = (4\pi(-t))^{n/2} \exp\left(\frac{|x|^2}{4t}\right).$$

If  $\max(w, 0)$  and  $\max(-w, 0)$  are continuous subcaloric functions, then  $r \mapsto \Phi(r, w)$  is non-decreasing, and  $\Phi(\sigma, w) = \Phi(\rho, w)$  for some  $0 < \rho < \sigma$  implies that either

(A)  $\nabla \max(w, 0) = 0$  in  $-\sigma^2 < t < 0$  or  $\nabla \max(-w, 0) = 0$  in  $-\sigma^2 < t < 0$ .

or

(B)  $\max(w, 0)(\partial_t - \Delta) \max(w, 0) = 0$  and  $\max(-w, 0)(\partial_t - \Delta) \max(-w, 0) = 0$  in  $-\sigma^2 < t < 0$  in the sense of measures.

*Proof.* For  $v := \max(w, 0)$  (or  $v := \max(-w, 0)$ , respectively) we calculate

$$I(r, v) = -\frac{1}{2} \int_{-r^2}^0 \int_{\mathbf{R}^n} G(t, x) (\partial_t - \Delta) v^2 + \int_{-r^2}^0 \int_{\mathbf{R}^n} G(t, x) v (\partial_t - \Delta) v,$$

$$I'(r, v) \geq 2r \int_{\mathbf{R}^n} |\nabla v|^2 G(-r^2, x).$$

In what follows we assume that  $I(r, v) \neq 0$ . It follows that

$$\frac{I'(r, v)}{I(r, v)} \geq 4r \frac{\int_{\mathbf{R}^n} |\nabla v(-r^2, x)|^2 G(-r^2, x)}{\int_{\mathbf{R}^n} v^2(-r^2, x) G(-r^2, x)}.$$

In the case  $I'(r, v) \neq 0$  the inequality is strict unless  $\int_{-r^2}^0 \int_{\mathbf{R}^n} v (\partial_t - \Delta) v = 0$ . Consequently  $\Phi(r, w) = 0$ , or else

$$\begin{aligned} \frac{\Phi'(r, w)}{\Phi(r, w)} &\geq \frac{4}{r} \left[ -1 + r^2 \frac{\int_{\mathbf{R}^n} |\nabla \max(w, 0)|^2 G(-r^2, x)}{\int_{\mathbf{R}^n} \max(w, 0)^2 G(-r^2, x)} \right. \\ &\quad \left. + r^2 \frac{\int_{\mathbf{R}^n} |\nabla \max(-w, 0)|^2 G(-r^2, x)}{\int_{\mathbf{R}^n} \max(-w, 0)^2 G(-r^2, x)} \right], \end{aligned}$$

where the inequality is strict unless both  $\int_{-r^2}^0 \int_{\mathbf{R}^n} \max(w, 0) (\partial_t - \Delta) \max(w, 0) = 0$  and  $\int_{-r^2}^0 \int_{\mathbf{R}^n} \max(-w, 0) (\partial_t - \Delta) \max(-w, 0) = 0$ . Moreover, by [3, Corollary 2.4.6], the right-hand side is non-negative.  $\square$

**Lemma 8.2.** *Let  $v^1, v^2$  be solutions of (1.1) in  $\mathbf{R}^{n+1}$  with such that  $v^1 = v^2$  in  $\{t < 0\}$  and  $v^1, v^2$  have polynomial growth with respect to the space variables. Then  $v^1 = v^2$  in  $\mathbf{R}^{n+1}$ .*

*Proof.* Multiplying the difference of the two equations by  $(v^1 - v^2)W$  where  $W(t, x) = G(t - T, x)$  and integrating, we obtain for each  $0 < T < +\infty$ ,  $0 < S < T$  and  $H$  defined in Lemma 3.1 that  $0 =$

$$\begin{aligned} & \int_0^S \int_{\mathbf{R}^n} W[|\nabla(v^1 - v^2)|^2 + (H(v^1) - H(v^2))(v^1 - v^2)] - \frac{1}{2} \int_0^S \int_{\mathbf{R}^n} (v^1 - v^2)^2 \partial_t W \\ & + \frac{1}{2} \int_{\mathbf{R}^n} W(S)(v^1(S) - v^2(S))^2 + \int_0^S \int_{\mathbf{R}^n} (v^1 - v^2) \nabla W \cdot \nabla(v^1 - v^2) \\ & \geq \frac{1}{2} \int_{\mathbf{R}^n} W(S)(v^1(S) - v^2(S))^2 + \frac{1}{2} \int_0^S \int_{\mathbf{R}^n} (v^1 - v^2)^2 [-\partial_t W - \Delta W] \\ & = \frac{1}{2} \int_{\mathbf{R}^n} W(S)(v^1(S) - v^2(S))^2. \end{aligned}$$

□

**Lemma 8.3.** *Assume that  $w$  is a backward self-similar solution with constant coefficients  $\lambda_+, \lambda_-$ , i.e.*

$$w(\theta^2 t, \theta x) = \theta^2 w(t, x) \text{ for all } \theta \geq 0, t < 0 \text{ and } x \in \mathbf{R}^n.$$

Then  $\nabla w = 0$  on  $\{w = 0\} \cap \{t < 0\}$ .

*Proof.* First, the self-similarity implies that

$$(8.1) \quad \partial_e w(\lambda^2 t, \lambda x) = \lambda \partial_e w(t, x) \text{ for all } e \in \partial B_1, \lambda \geq 0, t < 0 \text{ and } x \in \mathbf{R}^n.$$

Consequently the function  $r \mapsto \Phi(r, \partial_e w)$  of the monotonicity formula Theorem 8.1 is constant in  $(0, +\infty)$ , implying by Theorem 8.1 that either

(A)  $\nabla \max(\partial_e w, 0) = 0$  in  $\{t < 0\}$  or  $\nabla \max(-\partial_e w, 0) = 0$  in  $\{t < 0\}$ .

or

(B)  $\max(\partial_e w, 0)(\partial_t - \Delta) \max(\partial_e w, 0) = 0$  in  $\{t < 0\}$  and  $\max(-\partial_e w, 0)(\partial_t - \Delta) \max(-\partial_e w, 0) = 0$  in  $\{t < 0\}$  in the sense of measures.

Suppose now towards a contradiction that there is a point  $(t^1, x^1) \in \{t < 0\} \cap \{w = 0\} \cap \{\nabla w \neq 0\}$  and denote  $\nu = \frac{\nabla w}{|\nabla w|}$ ,  $\nu^0 = \frac{\nabla w(t^1, x^1)}{|\nabla w(t^1, x^1)|}$  and let  $Q_\kappa(t^1, x^1)$  such that  $\partial_{\nu^0} w > 0$  in  $Q_\kappa(t^1, x^1)$  and  $\{w = 0\} \cap Q_\kappa(t^1, x^1)$  is a  $C^1$ -surface. In the case  $\nu^0 \cdot e \neq 0$ ,

$$|(\partial_t - \Delta) \partial_e w|(Q_\kappa(t^1, x^1)) = |\lambda_+ + \lambda_-| \int_{t^1 - \kappa^2}^{t^1 + \kappa^2} \int_{B_\kappa(x^1) \cap \{w(t)=0\}} |e \cdot \nu| d\mathcal{H}^{n-1} dt \neq 0.$$

Thus (A) holds. From (8.1) we infer that  $\partial_e w \geq 0$  in  $\{t < 0\}$  if  $e \cdot \nu^0 > 0$  and  $\partial_e w \leq 0$  in  $\{t < 0\}$  if  $e \cdot \nu^0 < 0$ . Hence  $\partial_e w = 0$  in  $\{t < 0\}$  for all  $e \perp \nu^0$ . As in [2, p. 844] we may write

$$w(t, x) = -tf\left(\frac{x_n}{\sqrt{-t}}\right)$$

and calculate the 2-parameter family of solutions of the ODE which  $f(\xi) = w(-1, \xi)$  satisfies in  $(0, +\infty)$ ,

$$f(\xi) = \lambda_+ + C_1(\xi^2 - 2) + C_2 \left( -2\xi e^{\xi^2/4} + (\xi^2 - 2) \int_0^\xi e^{s^2/4} ds \right) \text{ in } \{f > 0\}$$

and

$$f(\xi) = -\lambda_- + C_3(\xi^2 - 2) + C_4 \left( -2\xi e^{\xi^2/4} + (\xi^2 - 2) \int_0^\xi e^{s^2/4} ds \right) \text{ in } \{f < 0\}.$$

As  $w$  has polynomial growth towards infinity we conclude that  $0 = C_2 = C_4$  and that

$$f(\xi) = \lambda_+ + C_1(\xi^2 - 2) \text{ in } \{f > 0\}$$

and

$$f(\xi) = -\lambda_- + C_3(\xi^2 - 2) \text{ in } \{f < 0\}.$$

If  $f(a) = 0$  and  $f'(a) \neq 0$  for some  $a \in \mathbf{R}$  then  $C_1 = C_3 = -\lambda_+/(a^2 - 2) = \lambda_-/(a^2 - 2)$ , a contradiction. Therefore  $f(a) = 0$  implies  $f'(a) = 0$ . It follows that  $\nabla w = 0$  on  $\{w = 0\}$ .  $\square$

**Theorem 8.4.** *Let  $w$  be a global solution with constant coefficients  $\lambda_+, \lambda_-$  such that  $\partial_t w$  and  $D^2 w$  are bounded, and suppose that the origin (in time-space) is a branch point of  $w$ . Then after rotation*

$$w(t, x) = w^*(t, x) := \lambda_+ \max(x_n, 0)^2/2 - \lambda_- \max(-x_n, 0)^2/2 \text{ for } (t, x) \in \mathbf{R}^{n+1}.$$

*Proof.*

**Step 1:** Let us first assume that  $w$  is a backward self-similar solution. By Lemma 8.3  $\nabla w = 0$  on  $\{w = 0\} \cap \{t < 0\}$ . But then  $z_1 := \max(w, 0)$  and  $z_2 := \max(-w, 0)$  are in  $\{t \leq 0\}$  non-negative backward self-similar solutions. Concerning those, it has been shown in [2, Lemma 6.3] and [2, Theorem 8.1] that either  $z_j$  is a half-plane solution of the form  $z_j(t, x) = \lambda_\pm/2 \max(x \cdot e, 0)^2$  for some  $e \in \partial B_1$ , or  $z_j(t, x) = -a_0 t + \sum_{i=1}^n a_i x_i^2$  with non-negative constants  $a_i, 0 \leq i \leq n$ . In the latter case the symmetry of  $z_j$  implies that  $z_k = 0$  in  $\{t < 0\}$  for  $k \neq j$ , and by Corollary 4.3 the origin cannot be a branch point.

It follows that after rotation

$$w(t, x) = w^*(t, x) \text{ for } t < 0.$$

**Step 2:** In the case of a general solution  $w$  as in the statement of our theorem, we consider the blow-up  $w_0$  of  $w$  at the origin and the blow-down  $w_\infty$ . By the non-degeneracy Lemma 4.1 and [13, Theorem 4.1], both  $w_0$  and  $w_\infty$  satisfy the assumptions of Step 1. Thus both  $w_0$  and  $w_\infty$  are after rotation of the form  $\lambda_+ \max(x_n, 0)^2/2 - \lambda_- \max(-x_n, 0)^2/2$  for  $t < 0$ , and the monotonicity formula [13]

implies that  $w$  is backward self-similar. But then it follows from Step 1 that after rotation

$$w(t, x) = w^*(t, x) \text{ for } t < 0.$$

Last, we apply Lemma 8.2 to obtain the same for  $t \geq 0$ .  $\square$

## 9. UNIFORM CLOSENESS TO $h$

We are now ready to prove uniform closeness of the scaled solution to the global solution  $h$  of (6.1), assuming that we are in the setting of Theorem 1.1.

**Lemma 9.1.** *Let  $u$  be a solution of (1.1) in  $Q_1(0)$ . Then, given  $\delta > 0$ , there are constants  $r_\delta > 0, \sigma_\delta > 0$  (depending only on  $\inf_{Q_1(0)} \min(\lambda_+, \lambda_-)$ , the Lipschitz norms of  $\lambda_\pm$ , the supremum norm of  $u$  and the space dimension  $n$ ) such that the following holds:*

*If  $r \in (0, r_\delta]$ ,  $u(s, y) = 0$ ,  $|\nabla u(s, y)| \leq \sigma_\delta r$ ,  $\text{pardist}((s, y), \{u > 0\}) \leq \sigma_\delta r$  and  $\text{pardist}((s, y), \{u < 0\}) \leq \sigma_\delta r$  for some  $(s, y) \in Q_{1/2}(0)$  then in  $Q_r(s, y)$ , the solution  $u(s + \cdot, y + \cdot)$  is  $\delta r^2$ -close to a rotated version  $\tilde{h}$  of the one-dimensional solution  $h$  defined in (6.1), more precisely*

$$\begin{aligned} r^{-2} \sup_{Q_r(0)} |u(s + \cdot, y + \cdot) - \tilde{h}| + r^{-1} \sup_{Q_r(0)} |\nabla u(s + \cdot, y + \cdot) - \nabla \tilde{h}| + \sup_{Q_r(0)} |\partial_t u(s + \cdot, y + \cdot)| \\ \leq \delta. \end{aligned}$$

*Proof.* Suppose towards a contradiction that the statement of the lemma fails. Then for some  $\delta > 0$  there exist  $\sigma_j \rightarrow 0, r_j \rightarrow 0, (s^j, y^j) \rightarrow (s^0, y^0) \in \overline{Q_{1/2}}$ , a sequence  $u_j$  of solutions such that  $(s^j, y^j) \in Q_{1/2}(0)$ ,  $u_j(s^j, y^j) = 0$ ,  $|\nabla u_j(s^j, y^j)| \leq \sigma_j r_j$ ,  $\text{pardist}((s^j, y^j), \{u_j > 0\}) \leq \sigma_j r_j$ ,  $\text{pardist}((s^j, y^j), \{u_j < 0\}) \leq \sigma_j r_j$  and

$$\begin{aligned} r_j^{-2} \sup_{Q_1(0)} |u_j(s^j + r_j^2 \cdot, y^j + r_j \cdot) - \tilde{h}(r_j \cdot)| + r_j^{-1} \sup_{Q_1(0)} |\nabla u_j(s^j + r_j^2 \cdot, y^j + r_j \cdot) - \nabla \tilde{h}(r_j \cdot)| \\ + \sup_{Q_1(0)} |\partial_t u_j(s^j + r_j^2 \cdot, y^j + r_j \cdot)| > \delta \end{aligned}$$

for all possible rotations  $\tilde{h}$  of  $h$ .

We may define

$$U_j(x) := \frac{u_j(r_j^2 t + s^j, r_j x + y^j)}{r_j^2}$$

and arrive at

$$(9.1) \quad \|U_j - \tilde{h}\|_{W^{1,\infty}(Q_1)} > \delta,$$

for all possible rotations  $\tilde{h}$  of  $h$ .

Observe that  $U_j$  is a solution of (1.1) in  $Q_1$  with respect to the scaled coefficients  $\lambda_+(r_j^2 t + s^j, r_j x + y^j)$  and  $\lambda_-(r_j^2 t + s^j, r_j x + y^j)$ . Since  $U_j(0) = 0$ ,  $|\nabla U_j(0)| \leq \sigma_j$ ,  $\text{pardist}(0, \{U_j > 0\}) \leq \sigma_j$ ,  $\text{pardist}(0, \{U_j < 0\}) \leq \sigma_j$  and the derivatives  $D^2 U_j, \partial_t U_j$  are uniformly bounded, we obtain by standard compactness arguments a global limit solution  $U_0$  of (1.1) in  $\mathbf{R}^n$  with respect to  $\lambda_+(s^0, y^0)$  and  $\lambda_-(s^0, y^0)$

which satisfies  $0 \in \partial\{U_0 > 0\} \cap \partial\{U_0 < 0\} \cap \{\nabla U_0 = 0\}$ . By Theorem 8.4,  $U_0 = \tilde{h}$  where  $\tilde{h}$  is a rotated version of  $h$ . Thus  $U_j$  and  $\nabla U_j$  converge in  $Q_1$  uniformly to  $\tilde{h}$  and  $\nabla \tilde{h}$ , respectively, and by Corollary 5.1  $\partial_t U_j \rightarrow 0$  in  $L^\infty(Q_1)$  as  $j \rightarrow \infty$ . We obtain a contradiction to (9.1).  $\square$

## 10. CONTINUITY OF THE TIME DERIVATIVE

Assuming once more that we are in the setting of Theorem 1.1, we show in the present section that the time derivative of the solution is *continuous* in a suitable neighborhood of the origin.

**Proposition 10.1.** *Let  $u$  be a solution of (1.1) in  $Q_1$ . Then there are positive constants  $\tilde{r}$  and  $\tilde{\sigma}$  (depending on  $\inf_{Q_1} \min(\lambda_+, \lambda_-)$ , the Lipschitz norms of  $\lambda_\pm$ , the supremum norm of  $u$  and the space dimension  $n$ ) such that the following holds. If  $u(0) = 0$ ,  $|\nabla u(0)| \leq \tilde{\sigma}\tilde{r}$ ,  $\text{pardist}(0, \{u > 0\}) \leq \tilde{\sigma}\tilde{r}$  and  $\text{pardist}(0, \{u < 0\}) \leq \tilde{\sigma}\tilde{r}$  then each blow-up limit at a point  $(t^1, x^1) \in Q_{\tilde{r}} \cap \{u = 0\} \cap \{\nabla u = 0\}$  is time-independent.*

*Proof.* Let us consider  $(t^1, x^1) \in \{u = 0\} \cap \{\nabla u = 0\}$ . As the statement of the Proposition is by Theorem 8.4 true when  $(t^1, x^1)$  is a branch point, we may assume that  $u \geq 0$  in some neighborhood of  $(t^1, x^1)$ . From Lemma 9.1 (with  $\delta := \inf_{Q_1} \min(\lambda_+, \lambda_-)/(96n)$ ) and Proposition 6.1 we know that  $u$  is non-decreasing, say in the direction  $e$  for every  $e$  close to  $x_n$  in  $Q_{\tilde{r}}$  and that  $|\partial_t u| \leq \inf_{Q_{\tilde{r}}} \min(\lambda_+, \lambda_-)/4$  in  $Q_{\tilde{r}}$ .

From [13, Theorem 4.1] we infer now that each blow-up limit  $z$  at  $(t^1, x^1)$  is a non-negative backward self-similar solution. Concerning those, it has been shown in [2, Lemma 6.3] and [2, Theorem 8.1] that either  $z$  is a half-plane solution of the form  $z(t, x) = \lambda_+(t^1, x^1)/2 \max(x \cdot e, 0)^2$  for some  $e \in \partial B_1$ , or  $z(t, x) = -a_0 t + \sum_{i=1}^n a_i x_i^2$  with non-negative constants  $a_i$ ,  $0 \leq i \leq n$  satisfying  $a_0 + 2 \sum_{i=1}^n a_i = \lambda_+(t^1, x^1)$ . As  $a_0 \leq \lambda_+(t^1, x^1)/2$ , it follows in this case that at least one  $a_i$ ,  $1 \leq i \leq n$  is strictly positive which contradicts the fact that  $z$  is non-decreasing in every direction  $e$  as above. Consequently  $z(t, x) = \lambda_+(t^1, x^1)/2 \max(x \cdot e, 0)^2$  in  $\{t < 0\}$ , and Lemma 8.2 implies that  $\partial_t z = 0$  in  $\mathbf{R}^{n+1}$ .  $\square$

**Corollary 10.2.** *Let  $u$  be a solution of (1.1) in  $Q_1$ . Then there are positive constants  $\tilde{r}$  and  $\tilde{\sigma}$  (depending on  $\inf_{Q_1} \min(\lambda_+, \lambda_-)$ , the Lipschitz norms of  $\lambda_\pm$ , the supremum norm of  $u$  and the space dimension  $n$ ) such that the following holds. If  $u(0) = 0$ ,  $|\nabla u(0)| \leq \tilde{\sigma}\tilde{r}$ ,  $\text{pardist}(0, \{u > 0\}) \leq \tilde{\sigma}\tilde{r}$  and  $\text{pardist}(0, \{u < 0\}) \leq \tilde{\sigma}\tilde{r}$  then  $\partial_t u$  is continuous in  $Q_{\tilde{r}}$ .*

*Proof.* The corollary follows immediately from Lemma 7.1, Proposition 10.1 and Corollary 5.1.  $\square$

## 11. DIRECTIONAL MONOTONICITY II

It is now possible to extend the directional monotonicity result of Section 6 to a directional monotonicity result with respect to time-space variables.

**Proposition 11.1.** *Let  $0 < \lambda_{\min} \leq \inf_{Q_1(0)} \min(1, \lambda_+, \lambda_-)$ ,  $h$  as in (6.1), let  $\varepsilon \in (0, 1)$  and let  $\tilde{r}$  and  $\tilde{\sigma}$  be the constants of Corollary 10.2. Then each solution  $u$  of (1.1) in  $Q_1(0)$  such that*

$$\text{dist}_{W^{1,\infty}(Q_1(0))}(u, h) \leq \delta := \frac{\lambda_{\min} \varepsilon}{48n} \tilde{r}^2 \tilde{\sigma}^2$$

and

$$\sup_{Q_1(0)} \max(|\nabla \lambda_+|, |\partial_t \lambda_+|, |\nabla \lambda_-|, |\partial_t \lambda_-|) \leq \delta$$

satisfies  $\varepsilon^{-1} \alpha \partial_t u + \varepsilon^{-1} \partial_e u - |u| \geq 0$  in  $Q_{1/2}(0)$  for every  $\alpha \in [-1, 1]$  and every  $e \in \partial B_1(0)$  such that  $e_1 \geq \varepsilon$ ; here  $e_1$  denotes the first component of the vector  $e$ .

*Proof.* First note that  $Q_1 \cap \{u = 0\}$  is by the assumptions contained in the strip  $|x_1| < \tilde{\sigma} \tilde{r}/2$ , implying by Corollary 10.2 and Lemma 7.1 that  $\partial_t u$  is continuous in  $Q_1$ . We know that  $\varepsilon^{-1} \alpha \partial_t h + \varepsilon^{-1} \partial_e h - |h| \geq 0$  in  $Q_1$ . It follows that

$$(11.1) \quad \varepsilon^{-1} \alpha \partial_t u + \varepsilon^{-1} \partial_e u - |u| \geq -3\delta \varepsilon^{-1} \text{ in } Q_1$$

provided that  $\text{dist}_{W^{1,\infty}(Q_1(0))}(u, h) \leq \delta$ . Suppose now towards a contradiction that the statement is not true. Then there exist  $\lambda_+, \lambda_- \in (\lambda_{\min}, +\infty)$ ,  $(t^*, x^*) \in Q_{1/2}(0)$ ,  $\alpha^*, e^*$ , and a solution  $u$  of (1.1) in  $Q_1(0)$  such that  $\text{dist}_{W^{1,\infty}(Q_1(0))}(u, h) \leq \delta$ ,

$$\sup_{Q_1(0)} \max(|\nabla \lambda_+|, |\partial_t \lambda_+|, |\nabla \lambda_-|, |\partial_t \lambda_-|) \leq \delta,$$

$|\alpha^*| \leq 1$ ,  $e_1^* \geq \varepsilon$  and  $\varepsilon^{-1} \alpha^* \partial_t u(t^*, x^*) + \varepsilon^{-1} \partial_{e^*} u(t^*, x^*) - |u(t^*, x^*)| < 0$ . For the positive constant  $c$  to be defined later the functions  $v := \varepsilon^{-1} \alpha^* \partial_t u + \varepsilon^{-1} \partial_{e^*} u - |u|$  and  $w := \varepsilon^{-1} \alpha^* \partial_t u + \varepsilon^{-1} \partial_{e^*} u - |u| + c|x - x^*|^2 - c(t - t^*)$  satisfy then by the definition of  $\delta$  the following: in the set  $D := Q_1(0) \cap \{v < 0\} \cap \{t < t^*\}$ ,

$$\begin{aligned} \Delta w - \partial_t w &\leq 2nc + c - \lambda_+ \chi_{\{u > 0\}} - \lambda_- \chi_{\{u < 0\}} \\ &+ \varepsilon^{-1} (\lambda_+ + \lambda_-) \nu_x \cdot e^* \mathcal{H}^n[\{u = 0\} \cap \{\nabla u \neq 0\}] \\ &+ \varepsilon^{-1} (\lambda_+ + \lambda_-) \nu_t \alpha^* \mathcal{H}^n[\{u = 0\} \cap \{\nabla u \neq 0\}] \\ &+ \varepsilon^{-1} (\chi_{\{u > 0\}} (\alpha^* \partial_t + \partial_{e^*}) \lambda_+ - \chi_{\{u < 0\}} (\alpha^* \partial_t + \partial_{e^*}) \lambda_-) \end{aligned}$$

where  $\nu = \frac{(\partial_t u, \nabla u)}{|\partial_t u, \nabla u|}$ . As

$$\nu \cdot (\alpha^*, e^*) \leq 0 \text{ on } \{u = 0\} \cap \{v < 0\} = \{u = 0\} \cap \{\varepsilon^{-1} \alpha^* \partial_t u + \varepsilon^{-1} \partial_{e^*} u < 0\},$$

we obtain by the definition of  $\delta$  that  $w$  is supercaloric in  $D$  provided that  $c$  has been chosen accordingly, say  $c := \lambda_{\min}/(4n)$ . It follows that the negative infimum of  $w$  is attained on

$$\partial_{\text{par}} D \subset (\partial_{\text{par}} Q_1(0) \cap \{t < t^*\}) \cup (Q_1(0) \cap \partial\{v < 0\}).$$

Consequently it is attained on  $\{t < t^*\} \cap \partial_{\text{par}}Q_1(0)$ , say at the point  $(\bar{t}, \bar{x}) \in \{t < t^*\} \cap \partial_{\text{par}}Q_1(0)$ . Since  $\text{pardist}((\bar{t}, \bar{x}), (t^*, x^*)) \geq 1/2$ , we obtain that

$$\begin{aligned} & \varepsilon^{-1}\alpha^*\partial_t u(\bar{t}, \bar{x}) + \varepsilon^{-1}\partial_{e^*} u(\bar{t}, \bar{x}) - |u(\bar{t}, \bar{x})| \\ &= v(\bar{t}, \bar{x}) = w(\bar{t}, \bar{x}) - c|x^* - \bar{x}|^2 + c(\bar{t} - t^*) < -c/4 = -\lambda_{\min}/(16n). \end{aligned}$$

But this contradicts (11.1) in view of  $\delta = \frac{\lambda_{\min}\varepsilon}{48n}\tilde{r}^2\tilde{\sigma}^2$ .  $\square$

## 12. PROOF OF THE MAIN THEOREM

The theorem is proven in several simple steps, using mainly Proposition 11.1, and Lemma 9.1. Note that the proof can be simplified substantially in the case that we are dealing not with a whole class of solutions but a single solution.

**Part I:** In this first part we prove uniform Lipschitz regularity and continuous differentiability with respect to the space variables.

**Step 1 (Directional monotonicity):** Given  $\varepsilon > 0$ , there are  $\sigma_\varepsilon > 0$  and  $r_\varepsilon > 0$  (depending only on the parameters of the statement) such that  $2\alpha\varepsilon^{-1}r_\varepsilon^2\partial_t u + 2\varepsilon^{-1}r_\varepsilon\partial_e u - |u| \geq 0$  in  $Q_{r_\varepsilon/2}(y)$  for every  $\alpha \in [-1, 1]$ . The inequality holds for every  $(s, y) \in Q_{1/2}(0)$  satisfying  $u(s, y) = 0$ ,  $|\nabla u(s, y)| \leq \sigma_\varepsilon r_\varepsilon$ ,  $\text{pardist}((s, y), \{u > 0\}) \leq \sigma_\varepsilon r_\varepsilon$  and  $\text{pardist}((s, y), \{u < 0\}) \leq \sigma_\varepsilon r_\varepsilon$ , for some unit vector  $\nu_\varepsilon(s, y)$  and for every  $e \in \partial B_1$  satisfying  $e \cdot \nu_\varepsilon(s, y) \geq \frac{\varepsilon}{2}$ . In particular, for  $\varepsilon = 1$ , the solution  $u$  is by condition (1.2) with  $\sigma = \sigma_1 r_1$  non-decreasing in  $Q_{r_1/2}(0)$  in direction  $(r_1, e)$  for every  $e \in \partial B_1(0)$  such that  $e \cdot \nu_\varepsilon(0) \geq \frac{1}{2}$ .

*Proof:* By Lemma 9.1 there are  $\sigma_\varepsilon > 0$  and  $r_\varepsilon > 0$  as above such that the scaled function  $u_{r_\varepsilon}(t, x) = u(s + r_\varepsilon^2 t, y + r_\varepsilon x)/r_\varepsilon^2$  is  $\delta := \varepsilon \frac{\lambda_{\min}}{64n} \tilde{r}^2 \tilde{\sigma}^2$ -close in  $C^1(Q_1(0))$  to a rotated version  $\tilde{h}$  of  $h$  in  $Q_1$ . Let  $\nu_\varepsilon(s, y)$  be the accordingly rotated version of the unit vector  $e_1$ . Since  $u_{r_\varepsilon}$  solves (1.1) with respect to  $\lambda_+(r_\varepsilon^2 \cdot + s, r_\varepsilon \cdot + y)$  and  $\lambda_-(r_\varepsilon^2 \cdot + s, r_\varepsilon \cdot + y)$ , and since  $\max(|\nabla(\lambda_+(r_\varepsilon^2 \cdot + s, r_\varepsilon \cdot + y))|, |\nabla(\lambda_-(r_\varepsilon^2 \cdot + s, r_\varepsilon \cdot + y))|, |\partial_t(\lambda_+(r_\varepsilon^2 \cdot + s, r_\varepsilon \cdot + y))|, |\partial_t(\lambda_-(r_\varepsilon^2 \cdot + s, r_\varepsilon \cdot + y))|) \leq C_1 r_\varepsilon$ , we may choose  $r_\varepsilon < \delta/C_1$  in order to apply Proposition 11.1 to  $u_{r_\varepsilon}$  in  $Q_1$  and to conclude that  $2\alpha\varepsilon^{-1}\partial_t u_{r_\varepsilon} + 2\varepsilon^{-1}\partial_e u_{r_\varepsilon} - |u_{r_\varepsilon}| \geq 0$  in  $Q_{1/2}(0)$  for every  $\alpha \in [-1, 1]$  and every  $e \in \partial B_1(0)$  such that  $e \cdot \nu_\varepsilon(s, y) \geq \varepsilon/2$ . Scaling back we obtain the statement of Step 1.

**Step 2 (Lipschitz continuity):**  $\partial\{u > 0\} \cap Q_{r_1/2}(0)$  and  $\partial\{u < 0\} \cap Q_{r_1/2}(0)$  are Lipschitz graphs in the direction of  $(0, \nu_\varepsilon(0))$  with spatial Lipschitz norms less than 1 and temporal Lipschitz norms less than  $r_1^{-1}$ . Moreover, for each  $\varepsilon \in (0, 1)$  and  $(s, y) \in \{u = 0\} \cap Q_{1/2}$  satisfying  $|\nabla u(s, y)| \leq \sigma_\varepsilon r_\varepsilon$ ,  $\text{pardist}((s, y), \{u > 0\}) \leq \sigma_\varepsilon r_\varepsilon$  and  $\text{pardist}((s, y), \{u < 0\}) \leq \sigma_\varepsilon r_\varepsilon$ , the free boundaries  $\partial\{u > 0\} \cap Q_{r_\varepsilon/2}(s, y)$  and  $\partial\{u < 0\} \cap Q_{r_\varepsilon/2}(s, y)$  are Lipschitz graphs (in the direction of  $\nu_\varepsilon(s, y)$ ) with spatial Lipschitz norms not greater than  $\varepsilon$ .

*Proof:* This follows from the monotonicity obtained in Step 1.

**Step 3 (Existence of a spatial tangent plane at points  $(s, y) \in \partial\{u >$**



$0\} \cap \partial\{u < 0\} \cap Q_{r_1/2}(0)$  **satisfying**  $|\nabla u(s, y)| = 0$ ): The Lipschitz graphs of Step 2 are both differentiable with respect to the space variables at the point  $(s, y)$ , and the two spatial tangent planes at  $(s, y)$  coincide.

*Proof:* This follows from Step 2 by letting  $\varepsilon$  tend to zero.

**Step 4 (One-phase points are regular):** If  $(s, y) \in Q_{r_1/2}(0)$  is a free boundary point and the solution  $u$  is non-negative or non-positive in  $Q_\delta(s, y)$ , then the free boundary is the graph of a  $C^{1,\alpha}$ -function in  $Q_{c_1\delta}(s, y)$ , where  $c_1$  and the  $C^{1,\alpha}$ -norm depend only on the parameters in the statement. Consequently, in  $Q_{r_1/2}(0)$ , there exist no singular one-phase free boundary points.

*Proof:* By Step 2, the sets  $\{u > 0\} \cap Q_{r_1/2}(0)$  and  $\{u < 0\} \cap Q_{r_1/2}(0)$  are sub/supergraphs of Lipschitz continuous functions. Therefore  $\{u = 0\} \cap Q_\delta(s, y)$  satisfies the thickness condition required for [2, Theorem 15.1] and the statement follows.

**Step 5 (Existence of space normals in  $Q_{r_1/2}(0)$ ):**  $\partial\{u > 0\} \cap Q_{r_1/2}(0)$  and  $\partial\{u < 0\} \cap Q_{r_1/2}(0)$  are graphs of Lipschitz continuous functions which are differentiable with respect to the space variables.

*Proof:* Let  $(s, y) \in Q_{r_1/2}(0)$  be a free boundary point. We have to prove existence of a tangent plane at  $(s, y)$ .

First, if  $(s, y)$  is a one-phase point, i.e. if the solution  $u$  is non-negative or non-positive in  $Q_\delta(s, y)$ , then the statement holds at  $(s, y)$  by the result of Step 4. Second, if  $|\nabla u(s, y)| \neq 0$ , the statement holds by Lemma 7.1. Last, if  $|\nabla u(s, y)| = 0$  and  $(s, y)$  is the limit point of both phases  $\{u > 0\}$  and  $\{u < 0\}$ , then Step 3 applies.

**Step 6 (Equicontinuity of the space normals):** It remains to prove that the space normals are equicontinuous on  $Q_{r_1/2}(0) \cap \partial\{u > 0\}$  and on  $Q_{r_1/2}(0) \cap \partial\{u < 0\}$  for  $u$  in the class of solutions specified in the statement of the main theorem.

*Proof:* By Step 2 we know already that the spatial Lipschitz norms of  $\partial\{u > 0\} \cap Q_{r_1/2}(0)$  and  $\partial\{u < 0\} \cap Q_{r_1/2}(0)$  are less than 1. We prove that the space normals are equicontinuous on  $Q_{r_1/2}(0) \cap \partial\{u > 0\}$ .

We may assume that  $\nu(0)$  points in the direction of the  $x_1$ -axis and that  $x_1 = f(t, x_2, \dots, x_n)$  is the representation of  $\partial\{u > 0\} \cap Q_{r_1/2}(0)$ . Besides we have  $|\nabla f(t, x')| < 1$  for  $(t, x) = (t, x_1, x') \in \partial\{u > 0\} \cap Q_{r_1/2}(0)$ . We claim that for  $\varepsilon > 0$  there is  $\delta_\varepsilon > 0$  depending only on the parameters in the statement such that for any pair of free boundary points  $(s^1, y^1), (s^2, y^2) \in \partial\{u > 0\} \cap Q_{r_1/2}(0)$ ,

$$(12.1) \quad \text{pardist}((s^1, y^1), (s^2, y^2)) \leq \delta_\varepsilon \quad \Rightarrow \quad |\nu(s^1, y^1) - \nu(s^2, y^2)| \leq 2\varepsilon.$$

In what follows let  $\rho_\varepsilon := \sigma_\varepsilon r_\varepsilon / 2 \leq r_1/2$ .

Suppose first that  $u$  is non-negative in  $Q_{\rho_\varepsilon}(s^1, y^1)$ . Here we may as in Step 4 apply [2, Theorem 15.1] to the scaled function  $w(t, x) := u(s^1 + \rho_\varepsilon^2 t, y^1 + \rho_\varepsilon x) / \rho_\varepsilon^2$ ; since the  $C^{1,\alpha}$ -norm of the free boundary normal of  $w$  is on  $Q_{c_2} \cap \partial\{w > 0\}$  bounded by a constant  $C_3$ , where  $c_2 > 0$  and  $C_3 < +\infty$  depend only on the parameters in the

statement, we may choose

$$\delta_\varepsilon := \min\left(\frac{\varepsilon^{\frac{1}{\alpha}}}{C_3^{\frac{1}{\alpha}}}, c_2\right)\rho_\varepsilon$$

to obtain (12.1).

Next, suppose that  $u$  changes its sign at  $Q_{\rho_\varepsilon}(s^1, y^1)$ . If there is a point  $(s, y) \in Q_{\rho_\varepsilon}(s^1, y^1) \cap \partial\{u > 0\}$  such that  $|\nabla u(s, y)| \leq \rho_\varepsilon$  then we are in the situation of Step 1. By Step 2 the free boundary  $\partial\{u > 0\} \cap Q_{r_\varepsilon/2}(s, y)$  is Lipschitz with spatial Lipschitz norm not greater than  $\varepsilon$ . Hence (12.1) follows in this case with  $\delta_\varepsilon := r_\varepsilon/2$ . Last, if  $|\nabla u(s, y)| \geq \rho_\varepsilon$  for all points  $(s, y) \in Q_{\rho_\varepsilon}(s^1, y^1) \cap \partial\{u > 0\}$ , we proceed as follows: from the equation  $u(t, f(t, x'), x') = 0$  we infer that  $\nabla' u + \partial_1 u \nabla' f = 0$  on  $\partial\{u > 0\} \cap Q_{r_1/2}(0)$ . Hence we obtain

$$\begin{aligned} |\nabla' f(s^1, (y^1)') - \nabla' f(s^2, (y^2)')| &= \left| \frac{\nabla' u(s^1, y^1)}{\partial_1 u(s^1, y^1)} - \frac{\nabla' u(s^2, y^2)}{\partial_1 u(s^2, y^2)} \right| \\ &\leq \frac{|\nabla' u(s^2, y^2) - \nabla' u(s^1, y^1)|}{|\partial_1 u(s^1, y^1)|} \\ &\quad + \left| \frac{\nabla' u(s^2, y^2)}{\partial_1 u(s^2, y^2)} \right| \frac{|\partial_1 u(s^2, y^2) - \partial_1 u(s^1, y^1)|}{|\partial_1 u(s^1, y^1)|} \\ &\leq 4M\rho_\varepsilon^{-1} \text{pardist}((s^1, y^1), (s^2, y^2)), \end{aligned}$$

where  $M = \|\nabla u\|_{C^{1/2,1}(Q_{1/2}(0))}$ . In particular we may choose

$$\delta_\varepsilon := \frac{\varepsilon}{4M}\rho_\varepsilon$$

to arrive at (12.1).

**Part II:** We conclude the proof of the main theorem by pointing out a counter-example to  $C^1$ -regularity.

Consider the one-phase counter-example  $u : [-r^2, r^2] \times [0, r] \rightarrow [0, +\infty)$  from [1, p. 376] satisfying the following:  $\sup_{[-r^2, r^2] \times [0, r]} \max(|\partial_t u|, |\partial_{xx} u|) < +\infty$ ,  $u(t, 0) = 0$  for  $-r^2 \leq t \leq r^2$ , and the free boundary touches the lateral boundary at the origin in a non-tangential way (for the sake of completeness we repeat the construction of [1] below). Thus we may reflect  $u$  to a solution

$$v(t, x) := \begin{cases} u(t, x), & x \geq 0 \\ -u(t, -x), & x < 0 \end{cases}$$

and obtain that  $v$  is a solution of our two-phase problem (1.1) in  $Q_r$  for  $\lambda_+ = \lambda_- = 1$ . As the free boundary  $\partial\{v > 0\}$  is only Lipschitz at the origin, we conclude that differentiability with respect to the time variable is in general not true.  $\square$

**Construction of the counter-example** (cf.[1, p. 376]):

let  $u : [-1, 1] \times [0, 1] \rightarrow [0, +\infty)$  be any solution of the one-phase obstacle problem

$$\partial_t u - \partial_{xx} u = -\chi_{\{u > 0\}} \text{ in } (-1, 1) \times (0, 1)$$

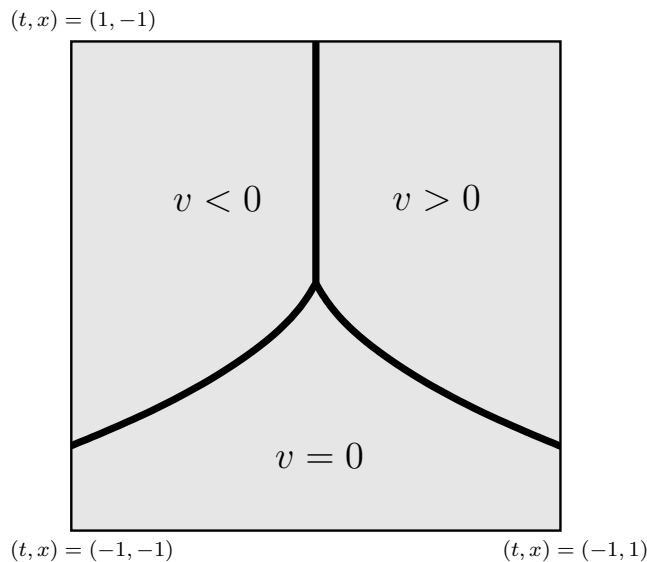


FIGURE 2. A counter-example to  $C^1$ -regularity

such that  $\sup_{[-1,1] \times [0,1]} \max(|\partial_t u|, |\partial_{xx} u|) < +\infty$ ,  $\partial_t u \geq 0$  in  $Q_1$ ,  $u(t, 0) = 0$  for  $-1 \leq t \leq 1$ ,  $\{u = 0\}^\circ$  contains  $(-\delta, 0) \times \{0\}$  for some  $\delta > 0$  and the free boundary touches the lateral boundary at the origin. Such a solution surely exists as  $\partial_x u(0, x)$  depends continuously on the boundary data on  $(\{0\} \times [0, 1]) \cup ([-1, 1] \times \{1\})$ . By Corollary 10.2 (applied to the reflected solution)  $\partial_t u$  is continuous on the closure of some smaller cylinder  $Q_r$ , and by Theorem 1.1 (applied to the reflected solution) the free boundary  $\partial\{u > 0\} \cap ([-r^2, r^2] \times [0, r])$  is the graph of a Lipschitz function of the time variable, say  $f(t)$ . As  $\partial_t u \geq 0$ ,  $f$  is a non-increasing function in  $[-r^2, r^2]$ . From our construction we also obtain that  $f(t) > 0$  in  $-\delta < t < 0$ . Choosing  $r$  even smaller if necessary we may assume that  $0 < f(t) < r/2$  in  $-r^2 < t < 0$ . Consider now the continuous function  $w := \partial_t u$  with the change of variables  $y = x - f(t)$ . In  $C := \{0 < y < r/2, -r^2 < t < r^2\}$  the function  $w(t, y)$  is a non-negative solution of the equation

$$\partial_t w(t, y) - f'(t) \partial_y w(t, y) - \partial_{yy} w(t, y) = 0 .$$

Since  $w(t, 0) = 0$  in  $-r^2 < t < 0$ , the Hopf principle implies that  $w(t, y) \geq \beta y$  in  $\{(t, y) : -r^2 < t < 0, 0 < y < \rho\}$  for some positive  $\beta$  and  $\rho$ . It follows that  $\partial_t u(t, x) \geq \beta(x - f(t))$  in  $C_\rho := \{f(t) < x < f(t) + \rho, -r^2 < t < 0\}$ . On the other hand  $\partial_x u = 0$  on  $\{x = f(t)\}$  implies that  $|\partial_x u(t, x)| \leq (x - f(t)) \sup_{C_\rho} |\partial_{xx} u|$  in  $C_\rho$ . Consequently for any  $e = (a, b) \in \partial B_1$  such that  $a \leq 0$  and  $b > 0$ ,

$$\partial_e u \geq a \sup_{C_\rho} |\partial_{xx} u| + b\beta \text{ in } C_\rho .$$

But then  $u$  is in  $C_\rho$  increasing in every direction  $e$  satisfying  $-a \sup_{C_\rho} |\partial_{xx}u| < b\beta$ . As  $u$  is non-negative and  $u(0,0) = 0$  we obtain that  $\{u = 0\} \cap \{t < 0\}$  contains a cone of positive measure around the  $t$ -axis.  $\square$

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