A PDE APPROACH TO REGULARITY OF SOLUTIONS TO FINITE HORIZON OPTIMAL SWITCHING PROBLEMS

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Abstract. We study optimal 2-switching and n-switching problems and the corresponding system of variational inequalities. By using PDE based methods we can extend the results of [DH] regarding existence of viscosity solutions of the 3-switching problem to cover some special cases when the cost of switching is non-deterministic. We also give regularity results for the solutions of the variational inequalities. The solutions are $C^{1,1}$-regular away for the free boundaries of the action sets.

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1. Introduction

We consider the problem discussed in [DH] where an investment manager faces the options to run the project when conditions are profitable, to temporarily shut it down when conditions are non-profitable and to permanently shut it down when the project is bankrupt. As mentioned in [DH] this problem has several industrial applications. The study of this class of problems, referred to as *optimal switching problems*, originate in the work of Brennan and Schwartz (1985) and Dixit (1989) who study different models for the life cycle of an investment in the natural resource industry. We refer to [DH] for a list of several extensions and related papers on this subject.

The special case when risk of default is not present was considered in Hamandène and Jeanblanc (2007). In the Markovian setting when the underlying commodity follows a diffusion process this problem corresponds to a double obstacle problem which is a well studied problem and existence of a solution follows by classical results. As is made clear in [DH] this representation is lost when adding the possibility of default. Instead the problem can be expressed in terms of a system of variational inequalities with inter-connected obstacles. In [DH] existence of viscosity solutions to this system is assured under the assumption that the cost of switching is constant.

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In the present paper we use PDE-based methods to establish uniqueness for some special cases when the cost of switching depends on the underlying diffusion process. In the same spirit we obtain $C^{1,1}$-regularity of the solutions away from the free boundary. More precisely, if we denote the solutions of the system of variational inequalities by v_1 and v_2 we will get, at most, two disjoint *action sets* on which the difference of the two solutions, $v_i - v_j$, equals the cost of switching from state j to state i, where $i, j \in \{1, 2\}$. $C^{1,1}$-regularity is in fact obtained everywhere except on the free boundaries of the action sets. Furthermore, we present an example where $C^{1,1}$-regularity is lost and the solution is only Lipschits continuous on the boundary of the action set.

The paper is organized as follows. In section 2 we define the problems of two- and multiple switching in stochastical terms using the notion of Snell envelopes. In section 3 we consider the corresponding variational inequalities and give regularity and partial uniqueness results. Section 4 includes some comments on the multiple switching problem. The last section is devoted to numerical treatment of systems of variational inequalities. We suggest an algorithm for solving the problem and illustrate with some examples.

2. Formulation of the problem and preliminary results

2.1. **Two-modes switching problem.** The two-modes switching problem can be formulated as follows. The production activity of the investment project, under a time interval $[0, T]$, can be either "on/open" indicated by 1, "off/closed" indicated by 0, or "definitely closed/defaulting" indicated by †. The management strategy of the project consists of

- An increasing sequence of stopping times $(\tau_n)_{n\geq 1}$ (i.e. $\tau_n \leq \tau_{n+1}$ and $\tau_0 = 0$) where the manager decides to switch the activity from a mode to another. Here, for any $n \geq 1$, τ_{2n} (resp. τ_{2n-1}) is the instant where the activity is switched to mode 1 "on/open" (resp. mode 0 "off/closed"). On $(\tau_{2n}, \tau_{2n+1}]$ the activity is in mode 1 and on $(\tau_{2n+1}, \tau_{2n+2}]$ it is on mode 0.

- A stopping time γ at which the manager decides to definitely stop the production. The activity is then switched to the mode †.

Let u_t be an indicator of the production activity being either on the mode 1 or 0, at time $t \in [0, T]$:

$$u_t = 1\!\!1_{[0,\tau_1]}(t) + \sum_{n\geq 1} 1\!\!1_{(\tau_{2n},\tau_{2n+1}]}(t), \quad t > 0, \qquad u_0 = 0. \tag{2.1}$$

Letting X_t denote the market price process of a set of underlying commodities at time t, the state of the whole economic system related to the project at time t is represented by the vector:

$$\begin{array}{ll} (t, X_t, u_t), & \text{if } \tau_n < t \leq \tau_{n+1}; \\ (\gamma, X_\gamma), & \text{if in mode †}. \end{array} \tag{2.2}$$

The sequence of stopping times $\delta := ((\tau_n)_{n\geq 1}, \gamma)$ is called a strategy for our starting and stopping problem.

Let $\ell^1 := (\ell_t^1)_{0\leq t\leq T}$ (resp. $\ell^2 := (\ell_t^2)_{0\leq t\leq T}$) be two positive continuous adapted stochastic processes. The closing (resp. opening) cost of the production at time τ_{2n-1} (resp. τ_{2n}) is given by $\ell^1_{\tau_{2n-1}}$ (resp. $\ell^2_{\tau_{2n}}$). $F_i(\gamma, X_\gamma),\ \ i=1,2$ stands for the cost of default at time γ, when the system defaults while it is in mode 1 resp. 0. The functions F_i are assumed non-positive. Moreover, let $\psi_1(t,x)$ be the profit per unit time when the system is in state $(t,x,1)$. This profit can be a loss (negative) as well. Such a situation occurs when the price has gone below the running costs. When the system is in state $(t,x,0)$, the profit is $\psi_2(t,x)$. Denote

$$\Phi(t,x,u) = \begin{cases} \psi_1(t,x), & \text{if } u=1; \\ \psi_2(t,x), & \text{if } u=0, \end{cases} \tag{2.3}$$

and

$$F(t,x,u) = \begin{cases} F_1(t,x), & \text{if } u=1; \\ F_2(t,x), & \text{if } u=0. \end{cases} \tag{2.4}$$

Then the expected total profit of running the system with the strategy $\delta := ((\tau_n)_{n\geq 1}, \gamma)$ is then given by:

$$J(\delta) = E\left[\int_0^\gamma \Phi(s,X_s,u_s)ds - \sum_{n\geq 1}\{\ell^1_{\tau_{2n-1}}1\!\!1_{[\tau_{2n-1}<\gamma]} + \ell^2_{\tau_{2n}}1\!\!1_{[\tau_{2n}<\gamma]}\} + F(\gamma, X_\gamma, u_\gamma)1\!\!1_{[\gamma<T]}\right].$$

Roughly speaking, solving an optimal switching problem with default risk consists in finding a strategy $\delta^* := ((\tau_n^*)_{n\geq 1}, \gamma^*)$ such that $J(\delta^*) \geq J(\delta)$ for any other strategies $\delta := ((\tau_n)_{n\geq 1}, \gamma)$.

We make the standard assumptions that $(\Omega, \mathcal{F}, P)$ is a fixed probability space on which is defined a standard n-dimensional Brownian motion $B = (B_t)_{0\leq t\leq T}$ whose natural filtration is $(\mathcal{F}_t^0 := \sigma\{B_s, s \leq t\})_{0\leq t\leq T}$. Let $\mathbf{F} = (\mathcal{F}_t)_{0\leq t\leq T}$ be the completed filtration of $(\mathcal{F}_t^0)_{0\leq t\leq T}$ with the P-null sets of $\mathcal{F}$. Furthermore, let $\mathcal{P}$ be the σ-algebra on $[0,T]\times\Omega$ of $\mathbf{F}$-progressively measurable sets and $\mathcal{S}^p$ is the set of $\mathcal{P}$-measurable, continuous processes $w := (w_t)_{0\leq t\leq T}$ such that $E[\sup_{0\leq t\leq T}|w_t|^p] < \infty$.

Let (Y_t^1, Y_t^2) be the value-function associated with the optimal switching problem, where the process Y_t^1 (resp. Y_t^2) stands for the optimal expected profit if, at time t, the production activity is on/open (resp. off/closed).

In terms of a Verification Theorem, it is shown in [DH] that the two processes $Y^1 := (Y_t^1)_{0\leq t\leq T}$ and $Y^2 := (Y_t^2)_{0\leq t\leq T}$ of $\mathcal{S}^p$ are continuous and uniquely solve the following system of Snell envelopes.

$$\begin{aligned} Y_t^1 &= \operatorname{ess\,sup}_{\tau\geq t} E\left[\left.\int_t^\tau \psi_1(s,X_s)ds + (-\ell^1_\tau + Y^2_\tau)\vee F(\tau, X_\tau, u_\tau)1\!\!1_{[\tau<T]}\right| \mathcal{F}_t\right] \quad (Y_T^1 = 0), \\ Y_t^2 &= \operatorname{ess\,sup}_{\tau\geq t} E\left[\left.\int_t^\tau \psi_2(s,X_s)ds + (-\ell^2_\tau + Y^1_\tau)\vee F(\tau, X_\tau, u_\tau)1\!\!1_{[\tau<T]}\right| \mathcal{F}_t\right] \quad (Y_T^2 = 0). \end{aligned} \tag{2.5}$$

where the essential sup is taken over $\mathbf{F}$-stopping times τ larger than t. Furthermore,

$$Y_0^1 = \sup_\delta J(\delta).$$

Consider the following $\mathbf{F}$-stopping times.

(i) The first time the activity defaults while in mode i:

$$\sigma_t^i := \inf\{s \geq t,\ Y_s^i = F_i(s, X_s)\} \wedge T, \qquad i = 1, 2,$$

(ii) The first time the activity is switched from mode 1 to mode 0:

$$\sigma_t^{12} := \inf\{s \geq t,\ Y_s^1 = -\ell_s^1 + Y_s^2\} \wedge T,$$

(iii) The first time the activity is switched from mode 0 to mode 1:

$$\sigma_t^{21} := \inf\{s \geq t,\ Y_s^2 = -\ell_s^2 + Y_s^1\} \wedge T,$$

Using the properties of the Snell envelop (see [DH] and the references therein), the $\mathbf{F}$-stopping times τ_t^i, $i = 1, 2,\ \ 0 \leq t \leq T$, defined by

$$\tau_t^1 = \sigma_t^{12} \wedge \sigma_t^1 \wedge T$$

and

$$\tau_t^2 = \sigma_t^{21} \wedge \sigma_t^2 \wedge T,$$

are optimal in the sense that

$$Y_t^1 = E\left[\int_t^{\tau_t^1} \psi_1(s, X_s) ds + ((-\ell_{\tau_t^1}^1 + Y_{\tau_t^1}^2) \vee F_1(\tau_t^1, X_{\tau_t^1}))1\!\!1_{[\tau_t^1 < T]} \middle| \mathcal{F}_t \right] \tag{2.6}$$

and

$$Y_t^2 = E\left[\int_t^{\tau_t^2} \psi_2(s, X_s) ds + ((-\ell_{\tau_t^2}^2 + Y_{\tau_t^2}^1) \vee F_2(\tau_t^2, X_{\tau_t^2}))1\!\!1_{[\tau_t^2 < T]} \middle| \mathcal{F}_t \right]. \tag{2.7}$$

In particular, on the continuation region $\{(\omega, t);\ \ \sigma_t^{12} > \sigma_t^1\}$ in which it is more profitable to run the activity, while at t the system is in mode 1, until default than switching to mode 0, we have

$$Y_t^1 = E\left[\int_t^{\sigma_t^1} \psi_1(s, X_s) ds + F_1(\sigma_t^1, X_{\sigma_t^1})1\!\!1_{[\sigma_t^1 < T]} \middle| \mathcal{F}_t \right]. \tag{2.8}$$

Similarly, on the set $\{(\omega, t);\ \ \sigma_t^{21} > \sigma_t^2\}$ in which it is more profitable to run the activity, while at t the system is in mode 0, until default than switching to mode 1, we have

$$Y_t^2 = E\left[\int_t^{\sigma_t^2} \psi_2(s, X_s) ds + F_2(\sigma_t^2, X_{\sigma_t^2})1\!\!1_{[\sigma_t^2 < T]} \middle| \mathcal{F}_t \right]. \tag{2.9}$$

When the market price process X of the commodity is an Itô diffusion with infinitesimal generator

$$\mathcal{A} = \frac{1}{2} \sum_{i,j=1,n} (\sigma \cdot \sigma^T)_{ij}(t, x) D_{ij} + \sum_{i=1,n} b_i(t, x) D_i, \tag{2.10}$$

and the switching cost processes ℓ^i are given by sufficiently smooth deterministic positive functions $a_i(t,x)$, i.e. $\ell^i_t = a_i(t, X_t)$, and provided that the functions $\psi_i(t,x)$ are smooth, the pair of the value-processes (Y^1, Y^2) associated to our optimal switching problem can be characterized (see [DH]) in terms of deterministic functions of the underlying price process X in the sense that $Y^1 = v^1(t, X_t)$ and $Y^2 = v^2(t, X_t)$ where the deterministic functions $v^1(t,x)$ and $v^2(t,x)$ are viscosity solutions of the following system of two partial differential equations with inter-connected obstacles:

$$
\begin{cases}
\min\{v_1(t,x) - (-a_1(t,x) + v_2(t,x)) \vee F_1(t,x), -\partial_t v_1(t,x) - \mathcal{A} v_1(t,x) - \psi_1(t,x)\} = 0, \\
\min\{v_2(t,x) - (-a_2(t,x) + v_1(t,x)) \vee F_2(t,x), -\partial_t v_2(t,x) - \mathcal{A} v_2(t,x) - \psi_2(t,x)\} = 0, \\
v_1(T,x) = 0, \;\; v_2(T,x) = 0.
\end{cases}
\tag{2.11}
$$

In particular, the set $\{(\omega,t);\;\; \sigma^{12}_t > \sigma^1_t\}$ on which Y^1 satisfies (2.8) corresponds to the complement of the *action set*

$$A_1 = \{(x,t);\;\; v_1(t,x) = v_2(t,x) - a_1(t,x)\}.$$

Correspondingly, on A_1^c, v_1 solves the following variational inequality:

$$\min\{v_1(t,x) - F_1(t,x), -\partial_t v_1(t,x) - \mathcal{A} v_1(t,x) - \psi_1(t,x)\} = 0, \qquad v_1(T,x) = 0.$$

Similarly, the set $\{(\omega,t);\;\; \sigma^{12}_t > \sigma^1_t\}$ on which Y^2 satisfies (2.9) corresponds to the complement of the action set

$$A_2 = \{(x,t);\;\; v_2(t,x) = v_1(t,x) - a_2(t,x)\}.$$

Furthermore, on A_2^c, v_2 solves the following variational inequality:

$$\min\{v_2(t,x) - F_2(t,x), -\partial_t v_2(t,x) - \mathcal{A} v_2(t,x) - \psi_2(t,x)\} = 0, \qquad v_2(T,x) = 0.$$

In fact, in [DH] (see also [DHP]), using purely probabilistic methods, the authors were only able to show existence of continuous functions v_1 and v_1 with polynomial growth in (t,x) such that $Y^1 = v^1(t, X_t)$ and $Y^2 = v^2(t, X_t)$ and are viscosity solutions of the System (2.11) under the following conditions on the involved coefficients:

Assumption [H]:

(1) The processes ℓ^1 and ℓ^2 are deterministic functions of the time parameter, *i.e.* $\ell^1_t(\omega) \equiv a_1(t)$ and $\ell^2_t(\omega) \equiv a_2(t)$ where a_1 and a_2 are positive deterministic functions.

(2) The non-positive functions $F_i(t,x) \equiv F(x)$ and ψ_i, $i = 1,2$ (of Subsection 2.1) are continuous, respectively, jointly continuous. Moreover, they are of polynomial growth, i.e., there exist some positive constants C and $\gamma \geq 1$ such that:

$$|\psi_1(t,x)| + |\psi_2(t,x)| + |F(x)| \leq C(1 + |x|^\gamma), \quad (t,x) \in [0,T] \times \mathbb{R}^n.$$

(3) The functions b and σ, with appropriate dimensions, satisfy the following standard conditions: There exits a constant $C \geq 0$ such that

$$|b(t,x)| + |\sigma(t,x)| \leq C(1+|x|) \quad \text{and} \quad |\sigma(t,x) - \sigma(t,x')| + |b(t,x) - b(t,x')| \leq C|x - x'| \tag{2.12}$$

for any $t \in [0,T]$ and $x, x' \in \mathbb{R}^n$.

2.2. **Multiple switching problem.** Suppose that besides running the production at full capacity or keeping it completely off (the two-modes switching model), there also exit a total of $q-2$ ($q \geq 3$) intermediate operating modes, corresponding to different subsets of production running.

Let ℓ_{ij} denote the switching costs from state i to state j, to cover the required extra fuel and various overhead costs. Furthermore, let $X = (X_t)_{t\geq 0}$ denote a vector of stochastic processes that stands for the market price of the underlying commodities and other financial assets that influence the production of power. The payoff rate in mode i, at time t, is then a function $\psi_i(t, X_t)$ of X_t. $F_i(\gamma, X_\gamma)$ stands for the cost of default (definitely stop the production) at time γ, when in mode i and denote $F(\gamma, X_\gamma, u_\gamma) := F_i(\gamma, X_\gamma)$ when $u_\gamma = i$. The functions F_i are assumed non-positive.

A management strategy for our model is a combination of the following sequences:

(i) a nondecreasing sequence of stopping times $(\tau_n)_{n\geq 0}$, where, at time τ_n, the manager decides to switch the production from its current mode to another one;

(ii) A stopping time γ at which the manager decides to definitely stop the production. The activity is then switched to the mode †.

(iii) a sequence of indicators $(\xi_n)_{n\geq 1}$ taking values in $\{1, \ldots, q\}$ of the state the production is switched to. At τ_n the station is switched from its current mode ξ_{n-1} to ξ_n.

When the power plant is run under a strategy $(\delta, u) = (((\tau_n)_{n\geq 1}, \gamma), (\xi_n)_{n\geq 1})$, over a finite horizon $[0,T]$, the total expected profit up to T for such a strategy is

$$J(\delta, u) = E\left[\int_0^T \psi_{u_s}(s, X_s)ds - \sum_{n\geq 1} \ell_{u_{\tau_{n-1}}, u_{\tau_n}}(\tau_n) 1\!\!1_{[\tau_n < \gamma]} + F(\gamma, X_\gamma, u_\gamma) 1\!\!1_{[\gamma < T]}\right].$$

The optimal switching problem we will investigate is to find a management strategy $(\delta^*, u^*) = (((\tau_n^*)_{n\geq 1}, \gamma*), (\xi_n^*)_{n\geq 1}))$ such that

$$J(\delta^*, u^*) = \sup_{(\delta,u)} J(\delta, u).$$

In the same fashion as in the two-modes switching model, it can be shown that (see [DHP]) that the value-processes $(Y^1, \ldots, Y^q)$ associated to the optimal multiple switching problem are continuous and satisfy the following system of Snell envelops.

$$Y_t^i = \text{ess sup}_{\tau \geq t} E\left[\int_t^\tau \psi_i(s, X_s)ds + \left(\max_{j\neq i}(-\ell_{ij}(\tau) + Y_\tau^j) \vee F_i(\tau, X_\tau)\right) 1\!\!1_{[\tau < T]} \middle| \mathcal{F}_t\right], \qquad i \in \{1, \ldots, q\}.$$

Furthermore,

$$Y_0^1 = \sup_\delta J(\delta).$$

Consider the following **F**-stopping times.

(*i*) The first time the activity defaults while in mode i:

$$\sigma_t^i := \inf\{s \geq t,\ Y_s^i = F_i(s, X_s)\} \wedge T, \qquad i = 1, 2, \ldots, q$$

(*ii*) The first time the activity is switched from mode i to any of the other modes $j \neq i$:

$$\tilde{\sigma}_t^i := \inf\{s \geq t,\ Y_s^i = \max_{j \neq i}(-\ell_s^{ij} + Y_s^j)\} \wedge T,$$

Using the properties of the Snell envelop (see [DH] and [DHP] etc..), the **F**-stopping times τ_t^i, $i = 1, 2, \ldots, q$, $0 \leq t \leq T$, defined by

$$\tau_t^i = \sigma_t^i \wedge \tilde{\sigma}_t^i \wedge T,$$

are optimal in the sense that

$$Y_t^i = E\left[\int_t^{\tau_t^i} \psi_i(s, X_s)ds + (\max_{j \neq i}(-\ell_{\tau_t^i}^{ij} + Y_{\tau_t^i}^j) \vee F_i(\tau_t^i, X_{\tau_t^i}))1\!\!1_{[\tau_t^i < T]} \middle| \mathcal{F}_t\right]. \tag{2.13}$$

In particular, on the set $\{(\omega, t);\ \tilde{\sigma}_t^i > \sigma_t^i\}$ in which it is more profitable to run the activity, while at time t the system is in mode i, until default than switching to another mode $j \neq i$, we have

$$Y_t^i = E\left[\int_t^{\sigma_t^i} \psi_i(s, X_s)ds + F_i(\sigma_t^i, X_{\sigma_t^i})1\!\!1_{[\sigma_t^i < T]} \middle| \mathcal{F}_t\right]. \tag{2.14}$$

When the process X is an Itô diffusion, with infinitesimal generator $\mathcal{A}$, and the switching cost processes ℓ^{ij} are given by sufficiently smooth deterministic functions $a_{ij}(t, x)$, i.e. $\ell_t^{ij} = a_{ij}(t, X_t)$, and provided that the functions $\psi_i(t, x)$ are smooth, the q-ple of the value-processes $(Y^1, Y^2, \ldots, Y^q)$ associated to our optimal switching problem can be characterized (see [DHP]) in terms of deterministic functions of the underlying price process X in the sense that $Y^i = v^i(t, X_t)$, $i = 1, 2, \ldots, q$ where the deterministic functions $v^i(t, x)$, $i = 1, 2, \ldots, q$ are viscosity solutions of the following system of q partial differential equations with inter-connected obstacles:

$$\min\left\{\phi_i(t, x) - \left(\max_{j \neq i}(-a_{ij}(t) + \phi_j(t, x)) \vee F_i(t, x)\right), -\partial_t\phi_i(t, x) - \mathcal{A}\phi_i(t, x) - \psi_i(t, x)\right\} = 0,$$

with

$$\phi_i(T, x) = 0, \quad i \in \{1, \ldots, q\}.$$

In particular, the sets $\{(\omega, t);\ \tilde{\sigma}_t^i > \sigma_t^i\}$ on which Y^i satisfy (2.13) correspond to the complement of the sets

$$A_i = \{(x, t);\ v_i(t, x) = \max_{j \neq i}(v_j(t, x) - a_{ij}(t, x))\}.$$

Correspondingly, on A_i^c, v_i solve the following variational inequality with fixed obstacle:

$$\min\{v_i(t,x) - F_i(t,x), -\partial_t v_i(t,x) - \mathcal{A}v_i(t,x) - \psi_i(t,x)\} = 0, \qquad v_i(T,x) = 0.$$

In fact, using similar assumptions as Assumption **[H]** on the involved coefficients, the results in [DHP], corresponding only to the default-free model i.e. when $F_i(t,x) = -\infty$, insure only existence of q deterministic continuous functions $v^1(t,x),\dots,v^q(t,x)$, with polynomial growth, such that for any $i \in \{1,\dots,q\}$, $Y_t^i = v^i(t,X_t)$ and for which the vector $(v^1,\dots,v^q)$ is a viscosity solution of the following system of q variational inequalities with inter-connected obstacles.

$$\min\left\{\phi_i(t,x) - \max_{j\neq i}(-a_{ij}(t) + \phi_j(t,x)), -\partial_t\phi_i(t,x) - \mathcal{A}\phi_i(t,x) - \psi_i(t,x)\right\} = 0,$$

with

$$\phi_i(T,x) = 0, \quad i \in \{1,\dots,q\}.$$

3. Regularity of the solution of system of variational inequalities

3.1. **Notation and Definitions.** We start this section by giving some necessary notations and definitions from the standard parabolic PDE theory.

Let $X = (t,x) = (t,x_1,...,x_n) \in \mathbb{R}\times\mathbb{R}^n$ and $r > 0$. We define the lower cylinder with the center X and radius r as

$$Q_r^-(X) = (t-r^2,t] \times B_r(x),$$

where $B_r(x) = \{y \in \mathbb{R}^n : ||y-x||_{\mathbb{R}^n} < r\}$ is the open ball in $\mathbb{R}^n$ with center x and radius r.

For $X = (t,x), Y = (\tau,y) \in \mathbb{R}\times\mathbb{R}^n$ we define the **parabolic distance** between X and Y to be

$$d(X,Y) = \sqrt{||x-y||^2_{\mathbb{R}^n} + |t-\tau|}.$$

Let $\Omega \subset \mathbb{R}\times\mathbb{R}^n$ be an open subset and $0 < \alpha \leq 1$. As usual, we denote by $C(\Omega) = C^0(\Omega)$ the set of continuous functions f on Ω with finite

$$||f||_{C(\Omega)} = \sup_{X\in\Omega} |f(X)|$$

sup-norm. The parabolic Hölder space $C^{0,\alpha}(\Omega)$ is defined as the subset of $C(\Omega)$ consisting of functions f such that the norm

$$||f||_{C^{0,\alpha}(\Omega)} := ||f||_{C(\Omega)} + \sup_{X,Y\in\Omega, X\neq Y} \frac{|f(X)-f(Y)|}{\big(d(X,Y)\big)^\alpha}$$

is finite. Next, the parabolic space $C^k(\Omega)$ for positive integer k is defined as the space of continuous functions f for which the derivatives $D_x^i D_t^j f$ are continuous for all multi-indices i and all nonnegative integers j with $|i| + 2j \leq k$ and for which the norm

$$||f||_{C^k(\Omega)} := \sum_{|i|+2j\leq k} ||D_x^i D_t^j f||_{C(\Omega)}$$

is finite. For positive integer k, $0 < \alpha \leq 1$ and $f \in C(\Omega)$ let

$$[f]_{k,\alpha}(X) := \inf_{P_k} \sup_{r>0} \frac{1}{r^\alpha} ||f - P_k||_{C(Q_r^-(X)\cap\Omega)},$$

where $P_k(X) = \sum_{|i|+2j\leq k} a_{ij}x^i t^j$ are polynomials of parabolic order $\leq k$. Then the parabolic Hölder space $C^{k,\alpha}(\Omega)$ is defined as the subspace of $C^k(\Omega)$ consisting of functions f such that the norm

$$||f||_{C^{k,\alpha}(\Omega)} := ||f||_{C^k(\Omega)} + \sup_{X\in\Omega} [f]_{k,\alpha}(X)$$

is finite.

For $a, b \in \mathbb{R}$ we denote $a^+ = \max(a, 0)$ and $a \vee b := \max\{a, b\}$.

Recall the operator $\mathcal{A}$ given by (2.10) and let $\mathcal{H}$ be the following operator

$$\mathcal{H} = \frac{\partial}{\partial t} + \mathcal{A}. \tag{3.1}$$

Now consider the following system of variational inequalities in $(0, T) \times \mathbb{R}^n$:

$$\begin{cases} \min\Big\{v_1(t,x) - \big(v_2(t,x) - a_1(t,x)\big) \vee F_1(t,x); -\mathcal{H}v_1(t,x) - \psi_1(t,x)\Big\} = 0, \\ \min\Big\{v_2(t,x) - \big(v_1(t,x) - a_2(t,x)\big) \vee F_2(t,x); -\mathcal{H}v_2(t,x) - \psi_2(t,x)\Big\} = 0, \\ v_1(T,x) = v_2(T,x) = 0 \quad \text{in } \mathbb{R}^n. \end{cases} \tag{3.2}$$

Definition 3.1. *Let $v_1, v_2 \in C([0,T] \times \mathbb{R}^n)$ are real-valued functions with $v_1(T,x) = v_2(T,x) = 0$, $x \in \mathbb{R}^n$. The pair (v_1, v_2) is called a **viscosity supersolution** of the system (3.2) in $[0,T] \times \mathbb{R}^n$ if for any $(t_0, x_0) \in [0,T] \times \mathbb{R}^n$ and any pair of functions $\varphi_1, \varphi_2 \in C^2([0,T] \times \mathbb{R}^n)$ satisfying*

(i) *$\varphi_i(t_0, x_0) = v_i(t_0, x_0)$ for $i = 1, 2$,*

(ii) *(t_0, x_0) is a local maximum point of $\varphi_i - v_i$ for $i = 1, 2$,*

we have

$$\begin{cases} \min\Big\{v_1(t_0,x_0) - \big(v_2(t_0,x_0) - a_1(t_0,x_0)\big) \vee F_1(t_0,x_0); -\mathcal{H}\varphi_1(t_0,x_0) - \psi_1(t_0,x_0)\Big\} \geq 0, \\ \min\Big\{v_2(t_0,x_0) - \big(v_1(t_0,x_0) - a_2(t_0,x_0)\big) \vee F_2(t_0,x_0); -\mathcal{H}\varphi_2(t_0,x_0) - \psi_2(t_0,x_0)\Big\} \geq 0. \end{cases} \tag{3.3}$$

*Respectively, the pair (v_1, v_2) is called a **viscosity subsolution** of the system (3.2) in $[0,T] \times \mathbb{R}^n$ if for any $(t_0, x_0) \in [0,T] \times \mathbb{R}^n$ and any pair of functions $\varphi_1, \varphi_2 \in C^2([0,T] \times \mathbb{R}^n)$ satisfying*

(i) *$\varphi_i(t_0, x_0) = v_i(t_0, x_0)$ for $i = 1, 2$,*

(ii) *(t_0, x_0) is a local minimum point of $\varphi_i - v_i$ for $i = 1, 2$,*

we have

$$\begin{cases} \min\Big\{v_1(t_0,x_0) - \big(v_2(t_0,x_0) - a_1(t_0,x_0)\big) \vee F_1(t_0,x_0); -\mathcal{H}\varphi_1(t_0,x_0) - \psi_1(t_0,x_0)\Big\} \leq 0, \\ \min\Big\{v_2(t_0,x_0) - \big(v_1(t_0,x_0) - a_2(t_0,x_0)\big) \vee F_2(t_0,x_0); -\mathcal{H}\varphi_2(t_0,x_0) - \psi_2(t_0,x_0)\Big\} \leq 0. \end{cases} \tag{3.4}$$

*The pair (v_1, v_2) is called a **viscosity solution** of the system (3.2) in $[0,T] \times \mathbb{R}^n$ if it is both a viscosity subsolution and supersolution.*

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3.2. **Regularity of the solution.** The following regularity result for parabolic obstacle problems has been proved in [PS]

Theorem 3.2. *Let u be the viscosity solution to the following obstacle problem in Q_1^-:*

$$\min\Big\{u-\psi, \mathcal{H}(u)\Big\} = 0$$

where $\mathcal{H}(u) = \dfrac{\partial u}{\partial t} + \mathcal{F}(D^2u, Du, u, t, x)$ and F is a fully nonlinear uniformly elliptic operator with certain homogeneity properties (see [S]). Then up to the $C^{1,1}$, the function u is as regular as ψ is. More precisely, if $\psi \in C^{k,\alpha}(Q_1^-)$ with $k=0$ or $k=1$ and $0<\alpha\le 1$, then $u \in C^{k,\alpha}_{loc}(Q_1^-)$.

This result allows us to analyze the question of the regularity of the viscosity solutions of (3.2). For this, let us introduce the following notations:

$$A_1 = \Big\{(t,x) : v_1(t,x) = v_2(t,x) - a_1(t,x)\Big\},$$

$$A_2 = \Big\{(t,x) : v_2(t,x) = v_1(t,x) - a_2(t,x)\Big\}.$$

It is clear, that the sets A_1 and A_2 are closed subsets of $[0,T]\times\mathbb{R}^n$, and since $a_i>0$ for $i=1,2$ we have $A_1\cap A_2 = \emptyset$.

By the first equation of the system (3.2), on the open set A_1^c the function v_1 is the solution to the following obstacle problem:

$$\min\Big\{v_1(t,x) - F_1(t,x); -\mathcal{H}v_1(t,x) - \psi_1(t,x)\Big\} = 0.$$

Let us assume that $F_i, a_i \in C^{1,1}([0,T]\times\mathbb{R}^n)$ for $i=1,2$. It follows from Theorem 3.2 that $v_1 \in C^{1,1}(A_1^c \cap ([0,T)\times\mathbb{R}^n))$. In the same way, $v_2 \in C^{1,1}(A_2^c\cap([0,T)\times\mathbb{R}^n))$, and, in particular, $v_2\in C^{1,1}(A_1)$. Since $v_1 = v_2 - a_1$ on A_1, then it follows that $v_1 \in C^{1,1}(A_1)$. Also, $v_2 \in C^{1,1}(A_2)$. So we have proved the following

Theorem 3.3. *Assume that $F_i, a_i \in C^{1,1}([0,T]\times\mathbb{R}^n)$ and $a_i(t,x)>0$ for all $(t,x)\in[0,T]\times\mathbb{R}^n$ and $i=1,2$. Let (v_1,v_2) be a viscosity solution of (3.2). Then $v_i \in C^{1,1}([0,T)\times\mathbb{R}^n\setminus\partial A_i)\cap C^{0,1}([0,T]\times\mathbb{R}^n)$, $i=1,2$.*

The following example shows that we can loose $C^{1,1}$-regularity on ∂A_i.

Example : Let $T=1$, $F_1=F_2\equiv 0$, $a_1=a_2\equiv 1$, $\psi_1(t,x) = x^2 - 2(1-t)$, $\psi_2(t,x) = -2$. It is easy to check, that the pair $(v_1,v_2) = \Big(x^2(1-t), \big[x^2(1-t)-1\big]^+\Big)$ is a solution to system (3.2) (in fact, it is the unique solution to (3.2), as it follows from the next paragraphs). But the function v_2 is only $C^{0,1}$ along the boundary of $v_2 = v_1 - 1$, that is, along $\{(t,x) : x^2(1-t) = 1\}$ (see Figures 1).

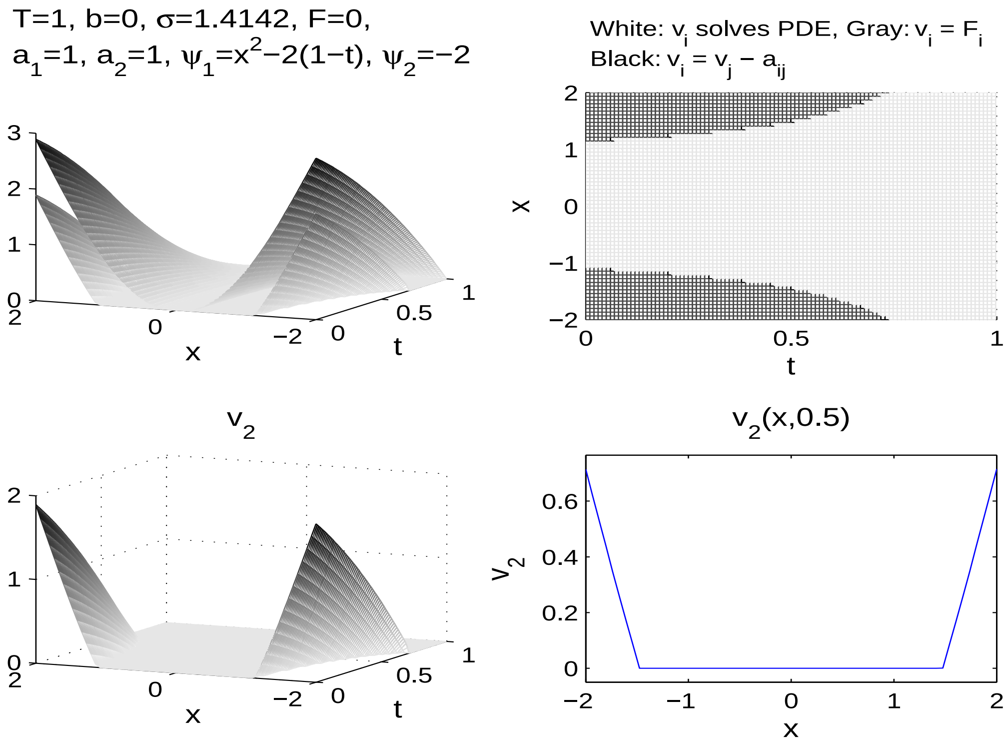

FIGURE 1. $v_1 = x^2(1-t)$, $v_2 = [x^2(1-t)-1]^+$. $C^{1,1}$-regularity is lost on ∂A_2

.

3.3. **Partial uniqueness results.** Here we prove uniqueness of the solution of the system (3.2) in following three cases: when $F_1 = F_2 = -\infty$, that is, in language of finance, in case of absence of default; in the case when $\psi_1(t,x) \neq \psi_2(t,x)$ for every $(t,x) \in [0,T] \times \mathbb{R}^n$, $F_1 \equiv F_2$ and $a_1, a_2 \equiv \text{const}$; and in the case when $\psi_1 \equiv \psi_2$, $F_1 \equiv F_2$ and $a_1, a_2 \equiv \text{const}$.

The uniqueness of the solution of the system (3.2) in the general case still remains open.

Case 1: $F_1 = F_2 = -\infty$**.**

In this case, the system (3.2) takes the following form:

$$\begin{cases} \min\Big\{v_1(t,x) - \big(v_2(t,x) - a_1(t,x)\big); -\mathcal{H}v_1(t,x) - \psi_1(t,x)\Big\} = 0, \\ \min\Big\{v_2(t,x) - \big(v_1(t,x) - a_2(t,x)\big); -\mathcal{H}v_2(t,x) - \psi_2(t,x)\Big\} = 0, \\ v_1(T,x) = v_2(T,x) = 0 \quad \text{in } \mathbb{R}^{\mathrm{n}}. \end{cases} \tag{3.5}$$

It is easy to show that, after letting $\psi := \psi_1 - \psi_2$, the function $v := v_1 - v_2$ is a viscosity solution of the following double obstacle variational inequality (see [HJ]):

$$\begin{cases} \min\Big\{v(t,x) + a_1(t,x); \max\big\{v(t,x) - a_2(t,x), -\mathcal{H}v(t,x) - \psi(t,x)\big\}\Big\} = 0, \\ v(T,x) = 0 \quad \text{in } \mathbb{R}^{\mathrm{n}}. \end{cases} \tag{3.6}$$

In a standard way one can show that the solution to (3.6) is a solution to the following problem:

$$v \in \mathbb{K}: \quad a(t; v, u - v) \geq \langle \psi, v - u \rangle, \quad \forall u \in \mathbb{K} \tag{3.7}$$

where

$$a(u,v) := \int \frac{1}{2} \sum_{ij}^{n} \tilde{\sigma}_{ij} u_i v_j dx + \int \left(u_t + \sum_{i}^{n} \tilde{b}_i u_i \right) v dx$$

is a coercive bilinear form, where $\tilde{\sigma}$ and $\tilde{b}$ are such that $\mathcal{H}$ is given in divergence form by

$$\mathcal{H}u = u_t + \frac{1}{2} \sum_{i,j=1}^{n} D_i(\tilde{\sigma}_{ij}(t,x) D_j u) + \sum_{i=1}^{n} \tilde{b}_i(t,x) D_i u,$$

and

$$\mathbb{K} := \{u \in W^2([0,T] \times \mathbb{R}^n) : a_1(t,x) \le u(t,x) \le a_2(t,x)\}.$$

So the uniqueness of the solution of (3.6) is a consequence of well-known results (see [F],[BL]).

As soon as we get the unique solution v to the problem (3.6), the system (3.5) can be rewritten in the following form:

$$\begin{cases} \min\Big\{ v(t,x) + a_1(t,x); -\mathcal{H}v_1(t,x) - \psi_1(t,x) \Big\} = 0, \\ \min\Big\{ -v(t,x) + a_2(t,x); -\mathcal{H}v_2(t,x) - \psi_2(t,x) \Big\} = 0, \\ v_1(T,x) = v_2(T,x) = 0 \quad \text{in } \mathrm{R^n}. \end{cases} \tag{3.8}$$

In the region $v(t,x)+a_1(t,x) > 0$, the function v_1 is the solution to $-\mathcal{H}v_1(t,x) - \psi_1(t,x) = 0$, $v_1(T,x) = 0$, so on this region the function v_1 is determined in a unique way . In the same way, in the region $-v(t,x) + a_2(t,x) > 0$, the function v_2 can be determined in a unique way. And for the uniqueness of the solution it is enough to ensure that these two regions are disjoint and use the fact that $v = v_1 - v_2$.

ed some condition

Case 2: $\psi_1(t,x) \ne \psi_2(t,x)$, $\forall (t,x) \in [0,T] \times \mathbb{R}^n$, $F_1 \equiv F_2$ **and** $a_1, a_2 \equiv$ const.

Since the functions ψ_1, ψ_2 are continuous, then it follows that $\psi_1(t,x) > \psi_2(t,x)$, for all $(t,x) \in [0,T] \times \mathbb{R}^n$ or $\psi_1(t,x) < \psi_2(t,x)$, for all $(t,x) \in [0,T] \times \mathbb{R}^n$.

Let us assume, that $\psi_1(t,x) > \psi_2(t,x)$, for all $(t,x) \in [0,T] \times \mathbb{R}^n$. Then it follows, that the set $A_1 := \big\{(t,x) : v_1(t,x) = v_2(t,x) - a_1\big\}$ has no interior points. Indeed, let $(t_0,x_0) \in intA_1$. From (3.2) we have $v_1(t_0,x_0) = v_2(t_0,x_0) - a_1 \ge F_1(t_0,x_0)$ and $-\mathcal{H}v_1(t_0,x_0) \ge \psi_1(t_0,x_0)$. Since $v_2(t_0,x_0) = a_1 + v_1(t_0,x_0) \ge a_1 + F_1(t_0,x_0)$, it follows from the second line of the system (3.2) that $-\mathcal{H}v_2(t_0,x_0) = \psi_2(t_0,x_0)$. But from $(t_0,x_0) \in intA_1$ we get $-\mathcal{H}v_1(t_0,x_0) = -\mathcal{H}v_2(t_0,x_0) = \psi_2(t_0,x_0) \ge \psi_1(t_0,x_0)$, which contradicts our assumption.

It follows, that in this case the function v_1 is the solution to

$$\begin{cases} \min\Big\{ v_1(t,x) - F_1(t,x); -\mathcal{H}v_1(t,x) - \psi_1(t,x) \Big\} = 0, \\ v_1(T,x) = 0 \quad \text{in } \mathrm{R^n}, \end{cases}$$

so it is unique. As soon as we have the values of the function v_1, the function v_2 can be found by solving an obstacle problem with **fixed** obstacle $\big(v_1(t,x) - a_2\big) \vee F_2(t,x)$, so it is also can be found by a unique way.

Case 3: $\psi_1 \equiv \psi_2$, $F_1 \equiv F_2$ and $a_1, a_2 \equiv$ const.

Let (v_1, v_2) and (u_1, u_2) be solutions of the system (3.2). If $(v_1, v_2) \not\equiv (u_1, u_2)$, then, without loss of generality, we can assume, that the set $\{(t,x) : v_1(t,x) > u_1(t,x)\}$ is not empty. Let Ω be a connected component of the open set $\{(t,x) : v_1(t,x) > u_1(t,x)\}$. It is clear, that on $\partial\Omega$ we have $v_1(t,x) = u_1(t,x)$. Next, in Ω, $v_1(t,x) > u_1(t,x) \geq F_1(t,x)$, so by the first line of the (3.2), either $v_1(t,x) = v_2(t,x) - a_1$ or $-\mathcal{H}v_1(t,x) - \psi_1(t,x) = 0$. Let $\Omega_1 \subset \Omega$ be the set where $v_1(t,x) = v_2(t,x) - a_1$. In Ω_1 we have $v_2(t,x) = v_1(t,x) + a_1 > F_2(t,x)$, so by the second line of the system (3.2), $-\mathcal{H}v_2(t,x) = \psi_2(t,x)$ in Ω_1, hence, $-\mathcal{H}v_1(t,x) = -\mathcal{H}v_2(t,x) = \psi_2(t,x) = \psi_1(t,x)$ in Ω_1. As a consequence we obtain that $-\mathcal{H}v_1(t,x) = \psi_1(t,x)$ in the whole Ω.

On the other hand, $-\mathcal{H}u_1(t,x) - \psi_1(t,x) \geq 0$ for all (t,x), and particularly, in Ω, so we get $\mathcal{H}v_1(t,x) \geq \mathcal{H}u_1(t,x)$ in Ω and $v_1(t,x) = u_1(t,x)$ on $\partial\Omega$. Applying the maximum principle, we get $u_1(t,x) > v_1(t,x)$ in Ω, which is a contradiction.

4. Some issues concerning multiple switching problem

In [DHP] the authors consider the following system of variational inequalities

$$\begin{cases} \min\Big\{v_i(t,x) - \max\limits_{j\neq i}\{v_j(t,x) - a_{ij}(t,x)\}; -\mathcal{H}v_i(t,x) - \psi_i(t,x)\Big\} = 0, \\ v_i(T,x) = 0 \quad \text{in } \mathbb{R}^n, \quad i = 1, ..., m \end{cases} \tag{4.1}$$

where $\mathcal{H}$ is the operator defied in (3.1), the functions ψ_i, a_{ij} are continuous and $a_{ij}(t,x) \geq a_0 = const > 0$ for all $i, j = 1, ..., m$, $i \neq j$ and for all $(t,x) \in [0,T] \times \mathbb{R}^n$. They prove, under some assumptions, the existence of viscosity solution of this system.

Here we consider a slightly more general system of variational inequalities:

$$\begin{cases} \min\Big\{v_i(t,x) - \max\limits_{j\neq i}\{v_j(t,x) - a_{ij}(t,x)\} \vee F_i(t,x); -\mathcal{H}v_i(t,x) - \psi_i(t,x)\Big\} = 0, \\ v_i(T,x) = 0 \quad \text{in } \mathbb{R}^n, \quad i = 1, ..., m \end{cases} \tag{4.2}$$

with $F_i, a_{ij} \in C^{1,1}([0,T] \times \mathbb{R}^n)$.

We can define the notion of viscosity solution for this problem in the same way as in the 2-dimensional case. In this section we give several remarks on the nature of this solutions.

Denote

$$A_i = \Big\{(t,x) \in [0,T] \times \mathbb{R}^n : v_i(t,x) = \max_{j\neq i}\{v_j(t,x) - a_{ij}(t,x)\}\Big\}.$$

First of all, it is easy to see, that $\bigcap\limits_{i=1}^{m} A_i = \emptyset$. Indeed, assume that $(t_0, x_0) \in \bigcap\limits_{i=1}^{m} A_i$. Let j_0 be such that $v_{j_0}(t_0, x_0) = \max\limits_{j=1,...,m} v_j(t_0, x_0)$. Then $v_{j_0}(t_0, x_0) \geq v_j(t_0, x_0)$ for all $j = 1, ..., m$. But since $(t_0, x_0) \in A_{j_0}$, we have

$$v_{j_0}(t_0, x_0) = \max_{j\neq j_0}\{v_j(t_0, x_0) - a_{j_0 j}(t_0, x_0)\}.$$

which means that for some j_1, $v_{j_0}(t_0, x_0) = v_{j_1}(t_0, x_0) - a_{j_0 j_1}(t_0, x_0) < v_{j_1}(t_0, x_0)$.

Next, we show by an example that the sets A_i, $i = 1, ..., m$ are not necessarily pairwise disjoint.

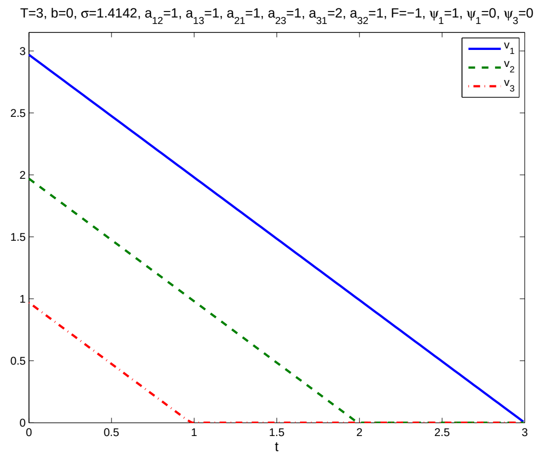

FIGURE 2. v_1 (solid) , v_2 (dashed), v_3 (dash-dot)

Example: Let $n = 1$, $T = 3$, $m = 3$, $a_{12} = a_{21} = a_{23} = a_{32} = a_{31} \equiv 1$, $a_{13} \equiv 2$, $\mathcal{H} = \frac{\partial}{\partial t} + \frac{d^2}{dx^2}$, $\psi_1(t,x) \equiv 1$, $\psi_2(t,x) = \psi_3(t,x) \equiv 0$ and $F_1 = F_2 = F_3 \equiv -1$.

Then, it is easy to see that the triple (v_1, v_2, v_3) is a solution for the system (4.2), where

$$v_1(t,x) = v_1(t) = 3 - t, \quad v_2(t,x) = v_2(t) = (2-t)^+,$$

$$v_3(t,x) = v_3(t) = (1-t)^+$$

for all $(t,x) \in [0,3] \times \mathbb{R}^1$ (see Fig. 2).

In this case, $A_1 = \emptyset$, $A_2 = [0,2] \times \mathbb{R}^1$ and $A_3 = [0,1] \times \mathbb{R}^1$, so $A_2 \cap A_3 \neq \emptyset$.

The next issue is addressed to the regularity of solutions of the system (4.2). As in the case of (3.2), in the complement of the set A_i, the functions v_i is $C^{1,1}$, because it is a solution to the following obstacle problem:

$$\begin{cases} \min\Big\{v_i(t,x) - F_i(t,x); -\mathcal{H}v_i(t,x) - \psi_i(t,x)\Big\} = 0, \\ v_i(T,x) = 0 \quad \text{in } \mathbb{R}^n. \end{cases}$$

The set A_i can be represented in the form $A_i = \bigcup_{j \neq i} A_{ij}$, where

$$A_{ij} = \Big\{(t,x) \in A_i : v_i(t,x) = v_j(t,x) - a_{ij}(t,x)\Big\}.$$

Then there exists a finite family of disjoint sets $\{D_k\}$, $D_k \subset A_i$, such that every intersection $A_{ij} \cap A_{pq}$ can be represented as a union of some subfamily of $\{D_k\}$. Let us show, that the function v_i is $C^{1,1}$ in the interior of every set D_k.

Assume $D_k \subset A_{ij}$. Then $v_i = v_j - a_{ij}$ in D_k. If $D_k \cap A_j = \emptyset$, then the function v_j is $C^{1,1}$ in D_k, and so is v_i. In the case of $D_k \cap A_j \neq \emptyset$, there exists k such that $D_k \subset A_{js}$. It follows that $v_j = v_s - a_{js}$ in D_k, so $v_i = v_s - a_{ij} - a_{js}$ in D_k. If $D_k \cap A_s = \emptyset$, then the function v_s is $C^{1,1}$ in D_k, and so is v_i. In the case of $D_k \cap A_s \neq \emptyset$, there exists p such that $D_k \subset A_{sp}$ and so on. At the end we'll arrive to

$v_i = v_q - a_{ij} - a_{js} - ... - a_{lq}$ in D_k and $D_k \cap A_q = \emptyset$, so the function v_q is $C^{1,1}$ in D_k, and we can conclude that v_i is $C^{1,1}$ in D_k.

As a conclusion, we get the following theorem for the regularity of viscosity solutions of (4.2):

Theorem 4.1. *Let $F_i, a_{ij} \in C^{1,1}([0,T] \times \mathbb{R}^n)$ for all $i,j = 1,...,m$ $(i \neq j)$ and the operator $\mathcal{H}$ given by (3.1). If $(v_1, v_2, ..., v_m)$ is a viscosity solution of the system (4.2), then*

$$v_i \in C^{0,1}\Big([0,T] \times \mathbb{R}^n\Big) \cap C^{1,1}\Big(([0,T] \times \mathbb{R}^n) \setminus (\cup_{j \neq i} \partial A_{ij})\Big), \quad i = 1, ..., m.$$

Remark 4.2. *Here we want to emphasize that by $C^{0,1}$ and $C^{1,1}$ we denote the* ***parabolic*** *Hölder spaces defined in the section 3.1. So, in particular, the last result states that in $[0,T] \times \mathbb{R}^n$ the function $v_i(t,x)$ is Lipschitz continuous in time variable t as well as in spatial variable x, and in the set $([0,T] \times \mathbb{R}^n) \setminus (\cup_{j \neq i} \partial A_{ij})$ the function $v_i(t,x)$ has Lipschitz continuous partial derivatives in x variable (but it can fail to have partial derivative in t at some points).*

This kind of situation naturally arises in parabolic obstacle problems, where even for infinite differentiable obstacles the solution can have Lipschitz continuous partial derivatives in spatial variable and can be only Lipschitz continuous in time variable.

5. Numerical results

In this section we describe a numerical solution of the optimal switching problem (3.2). The same algorithm applies to the multiple switching problem (4.2).

We solve the system (3.2) iteratively and in each step we treat the two variational inequalities separately. This is done by assuming that v_2 is fixed in the first variational inequality and v_1 is fixed in the second. Hence v_1 is treated as the solution to the classical obstacle problem with the fixed obstacle $(v_2(t,x) - a_1(t,x)) \vee F_1(t,x)$ and vice verse for v_2. We initiate the procedure with $v_1 = v_2 = 0$, which agrees with the terminal condition $v_1(T,x) = v_2(T,x) = 0$. The iterations are continued until the norm of the difference of the solutions between two iteration steps, $\|v_i^{k+1} - v_i^k\|$, is smaller than some prespecified tolerance. For our application we chose the L^∞-norm.

We use finite differences to solve the variational inequalities. The derivatives in the PDEs are approximated by the Crank-Nicolson finite difference scheme. Denoting the time index m and space index n this scheme gives the following approximations

$$\begin{aligned} \frac{\partial}{\partial t} v &\sim \frac{1}{\Delta t}(v_n^{m+1} - v_n^m) + \mathcal{O}(\Delta t^2) \\ \frac{\partial^2}{\partial x^2} v &\sim \frac{1}{2\Delta x^2}(v_{n+1}^{m+1} - 2v_n^{m+1} + v_{n-1}^{m+1} + v_{n+1}^m - 2v_n^m + v_{n-1}^m) + \mathcal{O}(\Delta x^2). \end{aligned}$$

A naive approach to solving the variational inequality would be, for each time step, to solve the PDE with this scheme and apply the obstacle condition to the solution of the PDE. We note however that the Crank-Nicolson scheme gives us an expression for v_n^{m+1} which depends on v_{n-1}^{m+1} and v_{n+1}^{m+1}. Thus there is no guarantee that the result will satisfy the PDE part of the variational inequality if we follow the naive approach.

Instead we apply the iterative method of projected successive over relaxation (SOR). Introduce the over relaxation parameter $\omega \in (1,2)$. Suppose the Crank-Nicolson scheme applied to the PDE gives us $v_n^{m+1} = \Phi(v_{n-1}^{m+1}, v_{n+1}^{m+1}, v_{n-1}^{m}, v_n^{m}, v_{n+1}^{m})$. In the k'th step of the iterative process we set $y = \Phi(v_{n-1}^{m+1,k}, v_{n+1}^{m+1,k-1}, v_{n-1}^{m,k}, v_n^{m,k}, v_{n}$
We set

$$v_n^{m+1,k} = \max(v_n^{m+1,k-1} + \omega(y - v_n^{m+1,k-1}), F_n^m),$$

where $F = v_i - a_j \vee F_i$ is the obstacle (recall that F is known since we have fixed the involved function v_i). This procedure is iterated until the norm of $v^k - v^{k-1}$ is smaller than some prespecified tolerance. Convergence is guaranteed for $\omega \in (1,2)$ (see [WDH]). The algorithm counts the number of iterations required to obtain the tolerance condition and value of ω is adjusted in order to minimize the number of iterations at each time step.

5.1. **Transformation to the Heat operator.** In order to reduce the number of calculation and get faster convergence the PDE given by the operator (3.1) is transformed to the heat equation. This is straight forward if the coefficients σ and b are constants. We introduce the new variable $(\tau, x) = (\frac{1}{2}\sigma^2(T-t), x)$ and set

$$\tilde{v}_i(\tau, x) = \exp\left(-\frac{b}{\sigma^2}x - \frac{3b^2}{2\sigma^2}\tau\right) v_i(T, x - \frac{\sigma^2}{2}\tau). \tag{5.1}$$

The system (3.2) becomes

$$\begin{cases} \min\Big\{\tilde{v}_1(\tau,x) - \big(\tilde{v}_2(\tau,x) - \tilde{a}_1(\tau,x)\big) \vee \tilde{F}_1(\tau,x); -\tilde{\mathcal{H}}\tilde{v}_1(\tau,x) - \tilde{\psi}_1(\tau,x)\Big\} = 0, \\ \min\Big\{\tilde{v}_2(\tau,x) - \big(\tilde{v}_1(\tau,x) - \tilde{a}_2(\tau,x)\big) \vee \tilde{F}_2(\tau,x); -\tilde{\mathcal{H}}\tilde{v}_2(\tau,x) - \tilde{\psi}_2(\tau,x)\Big\} = 0, \\ \tilde{v}_1(0,x) = \tilde{v}_2(0,x) = 0 \quad \text{in } \mathbb{R}^{\mathrm{n}}. \end{cases} \tag{5.2}$$

where $\tilde{\mathcal{H}} = -\frac{\partial}{\partial \tau} + \frac{1}{2}\frac{\partial^2}{\partial x^2}$ and all involved functions are transformed as in (5.1).

Applying the Crank-Nicolson scheme to the heat equation and setting $\alpha = \Delta t / \Delta x^2$ gives us the following expression

$$v_n^{m+1} = \frac{1}{1+\alpha}\Big((1-\alpha)v_n^m + \frac{\alpha}{2}(v_{n+1}^{m+1} + v_{n-1}^{m+1} + v_{n+1}^m + v_{n-1}^m)\Big) + \Delta t\ \psi_n^{m+1}.$$

5.2. **Boundary values.** For numerical treatment of the problem we can only calculate the solution on a bounded domain. This introduces a problem since the numerical methods require the solution to be known on the boundaries of the domain. We only know the values of $\tilde{v}_i$ at initial time $t = 0$, but we have no information on the values at x_0 and x_{N+1}, where x_k, $k = 0, \ldots, N+1$ is our spacial mesh. We overcome this by linear interpolation at the boundaries. Hence we set $v_i(t, x_0) = 2v_i(t, x_1) - v_i(t, x_2)$ and $v_i(t, x_{N+1}) = 2v_i(t, x_N) - v_i(t, x_{N-1})$.

5.3. **The algorithm.** We summarize the above discussion by presenting the following pseudo-code for solving the optimal switching problem

```
v1 = zeros;
v2 = zeros;

while max( ||v1-v1Old||, ||v2-v2Old|| ) > tolerance
  v1Old = v1;
  G1 = max(v2 - a1, F1);
  v1 = SOR_CrankNicolson(v1, G, ...);

  v2Old = v2;
  G2 = max(v1 - a2, F2);
  v2 = SOR_CrankNicolson(v2, G, ...);
end



v SOR_CrankNicolson(v, G, ...)
{
  alpha = dt/dx^2;
  loops = 0;

  for m=[1:M+1]
    v(:,m) = v(:,m-1);

    while err > tolerance
      err = 0;

      for n=[1:N+1]
        y = 1/(1+alpha) * ( (1-alpha) * v(n,m-1) ...
            + alpha/2*(v(n+1,m) + v(n-1,m) + v(n+1,m-1) + v(n-1,m-1)) ...
            + dt * psi(n,m) );

        y = max( v(n,m) + omega*(y - v(n,m)), G(n,m) );

        err = err + (y - v(n,m))*(y - v(n,m));
        v(n,m) = y;
```

```
      end

      v(1,m) = 2*v(2,m) - v(3,m);
      v(N+1,m) = 2*v(N,m) - v(N-1,m);

      loops++;
    end
  end;

  if loops >= loopsOld
    domega = -domega;
  end

  omega = min(max(omega + domega,1),2);
  loopsOld = loops;

  return v;
}
```

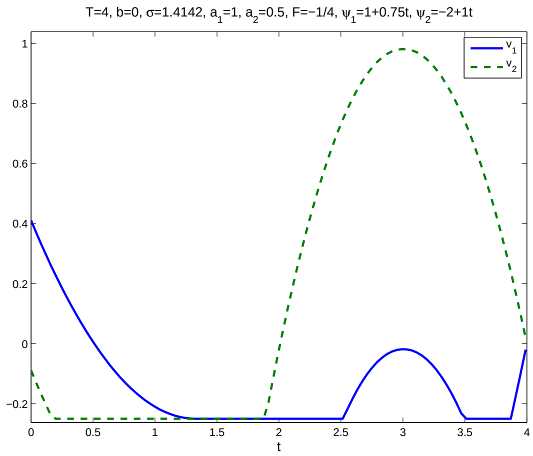

FIGURE 3. Smooth fit occurs on $\partial\{v_i = F_i\}$ but not on $\partial\{v_i = v_j - a_{ij}\}$.

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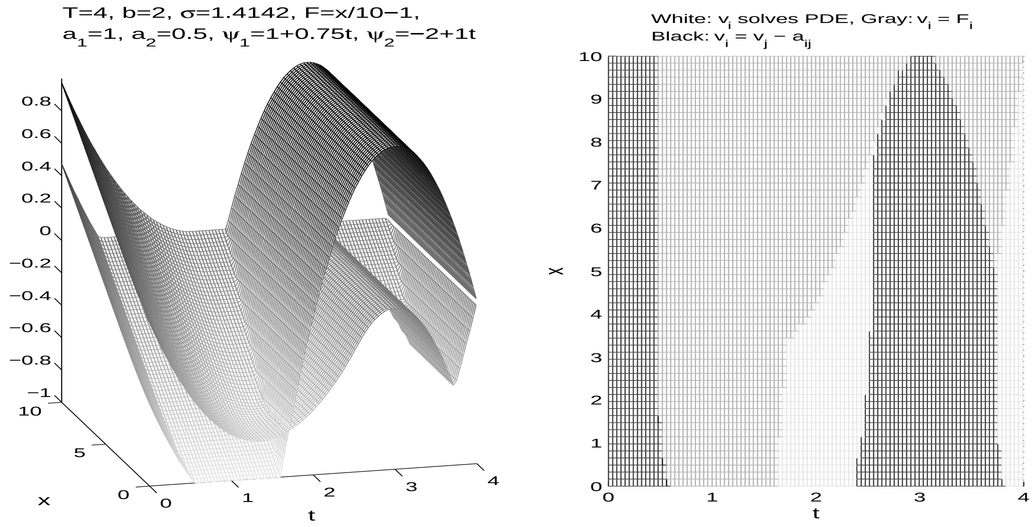

FIGURE 4. Same as Fig. 3 including dependence on the spacial variable.

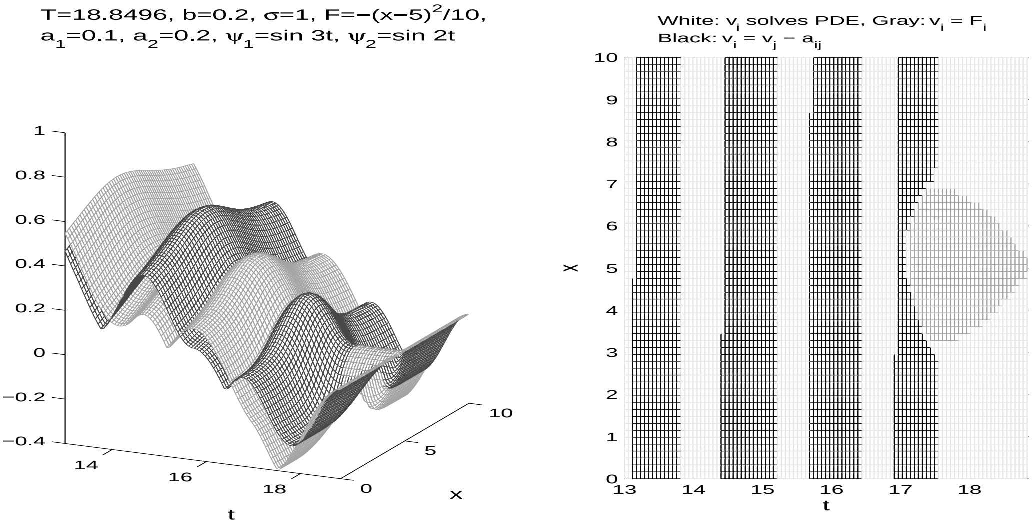

FIGURE 5. Periodic zeroth order term in the PDEs.

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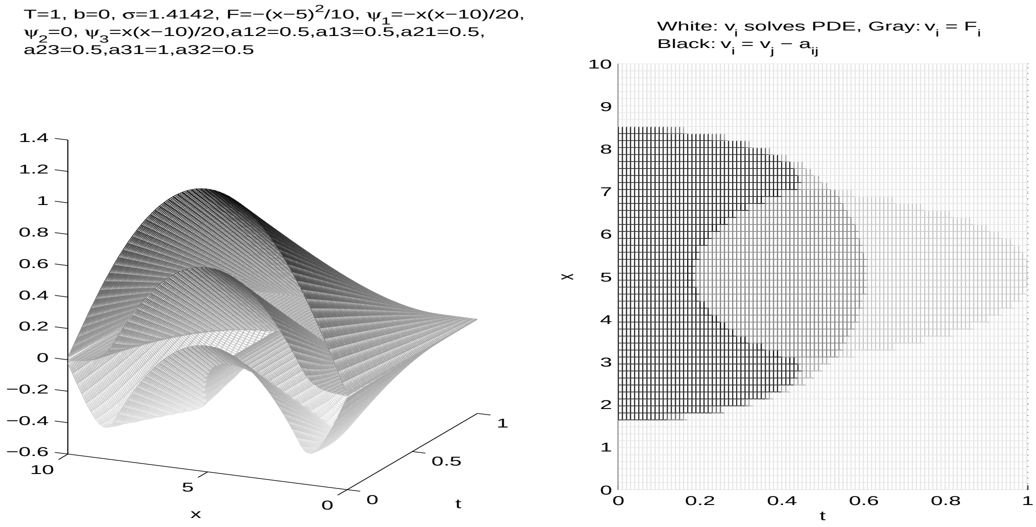

Figure 6. Three switching problem.

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