# EXISTENCE OF QUADRATURE SURFACES FOR POSITIVE MEASURES WITH FINITE SUPPORT

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Abstract. Let $\mu$ be a positive measure with finite support in $\mathbb{R}^n$ $(n \geq 2)$. Then we show that there is a bounded open set $\Omega$, containing the support of $\mu$, whose single-layer potential coincides with the potential of $\mu$ outside $\Omega$.

## PRELIMINARIES

Let $\Omega$ be a bounded open set in $\mathbb{R}^n$ $(n \geq 2)$ whose boundary has finite $(n-1)$-dimensional Hausdorff measure $(H^{n-1})$ [FE, page 171]. Then, for a fixed measure $\mu$ with support in $\Omega$, we say $\partial\Omega$ is a *quadrature surface* with respect to $\mu$ if the quadrature identity

$$\int_{\partial\Omega} h \; dH^{n-1} = \langle h, \mu \rangle,$$

holds for all functions $h$ harmonic on $\overline{\Omega}$. Unfortunately, for the moment, there are not many (explicit) examples of quadrature surfaces where both the measure and the corresponding quadrature surface are known. The simplest and the most classical example is a sphere, where the corresponding measure is a constant multiple of Dirac measure with support at the center of the sphere. There are plenty of positive measures for which there are no quadrature surfaces with $C^2$ boundaries [SH, theorem 2.4]. However, for a bounded domain $\Omega$ with analytic boundary $\partial\Omega$, one can show, using the Cauchy-Kovalevsky theorem, that $\partial\Omega$ is a quadrature surface with respect to an "unknown" (positive) measure.

Our aim here is to construct quadrature surfaces with respect to positive measures with finite support. For this purpose we use an outstanding result due to H.W. Alt and L.A. Caffarelli [AC] which states that for any bounded open set $D$ in $\mathbb{R}^n$ and any constant $c > 0$ there is a solution to the following free boundary problem

$$\begin{aligned} \Delta u &= 0 && \text{in } \Omega \setminus D, \\ u &= 0 && \text{on } \partial\Omega \quad \text{(free boundary)}, \\ |\nabla u| &= 1 && \text{on } \partial\Omega, \\ u &= c && \text{on } D. \end{aligned}$$

The main theorem (theorem 1) depends strongly on this result and it also uses the fact that any solution to this problem is a quadrature surface with respect to

$-\Delta u\chi_\Omega$, [SH, theorem 5.2]. In proving the main result we will (w.l.o.g.) assume $n = 3$, $\mu = c_1\delta_1 + c_2\delta_2$ ($\delta_j$ is the Dirac measure with support at $x^j$) and then take balls $B(x^j, m^{-1})$, for large $m$ and $j = 1, 2$, and solve the above free boundary problem, where the last condition will be replaced by $u = c_j m/4\pi$ on $B(x^j, m^{-1})$. For every $m$ we obtain a solution $u_m$ and $\Omega_m$. Then we prove that outside a neighborhood of supp $\mu$ these solutions are equicontinuous and therefore there is a subsequence converging to a function $u$. It will then remain to show that $\Omega := \{x : u(x) > 0\}$ solves our problem. We cope with this through several lemmas.

## NOTATION

$\nabla$ denotes the gradient. $\Delta$ is the Laplacian operator.

$C$ is a generic constant changing its value from one occurrence to another.

$\hat{\mu}(x) = \omega \int |x - y|^{2-n} d\mu$, where $\mu$ is a measure and $\omega$ is a normalization factor so that $\Delta\hat{\mu} = -\mu$ in the sense of distributions.

$C_c^\infty(A)$ is the space of infinitely differentiable functions with compact support in $A(\subset \mathbb{R}^n)$.

$H_o^{1,2}(A)$ is the closure of $C_c^\infty(A)$ in $H^{1,2}(A) = \{u \in L^2(A) : \partial u/\partial x_j \in L^2(A), j = 1, 2, ..., n\}$. For a positive integer $j$ $H^j$ is the $j$-dimensional Hausdorff measure. For a set $A \subset \mathbb{R}^n$ we define $H^j(A) := \int_A dH^j$.

$\chi_A$ is the characteristic function of $A$, i.e. $\chi_A = 1$ on $A$ and zero elsewhere.

$\chi_{(F(x)>0)}$ is the characteristic function of the set $\{x : F(x) > 0\}$, where $F$ is a function defined in $\mathbb{R}^n$.

$B(x, \rho) = \{y \in \mathbb{R}^n : |x - y| < \rho\}$.

For $A$, an open set in $\mathbb{R}^n$, the reduced boundary $\partial_{red}(A)$ [FE, page 477] is defined as follows

$$\partial_{red}(A) := \{x \in \mathbb{R}^n : |\nu(x)| = 1\},$$

where $\nu$ is a unique unit vector so that

$$\int_{B(x,\rho)} |\chi_A - \chi_{(y:(y-x).\nu(x)<0)}| dH^n = o(\rho^n),$$

else $\nu = 0$.

## MAIN RESULT

**THEOREM 1.** *Let $\mu$ be a positive measure with finite support in $\mathbb{R}^n$, i.e. $\mu = \sum_{j=1}^k c_j\delta_j$ where $c_j$ is a positive constant, $k \geq 1$ is an integer and $\delta_j$ is the Dirac measure with support at $x^j$. Then, there is a bounded open set $\Omega$ ($\supset \{x^j\}$) such that*

$$\int_{\partial\Omega} h(x)\ dH^{n-1} = \sum_{j=1}^k c_j h(x^j),$$

*for all functions $h$ harmonic on $\overline{\Omega}$. Moreover $\partial_{red}\Omega$ (reduced boundary) is analytic and the rest of the boundary has $H^{(n-1)}$-measure zero. If in addition $n = 2$, then the boundary is real analytic.*

We prove this theorem in the special case $n = 3$ and $k = 2$. The general case is similar with small changes in the proof. We will, however, remark these after proving the special case. Let us recall a result due to [AC].

**THEOREM 2.** *Let $x^1$, $x^2$ be two different points in $\mathbb{R}^3$. Let moreover $a_j > 0$ ($j = 1, 2$), $D_m = \cup B(x^j, m^{-1})$ and $m_0 > 2|x^1 - x^2|^{-1}$. Then for any integer $m > m_0$ there is a bounded open set $\Omega_m$ and a Lipschitz function $u_m$, such that*

$$\begin{aligned}
\Delta u_m &= 0 && \text{in } \Omega_m \setminus D_m,\\
u_m &= 0 && \text{on } \mathbb{R}^3 \setminus \Omega_m,\\
u_m &= a_j m && \text{on } B(x^j, m^{-1}), \qquad (j = 1,2)\\
\lim_{x\to y\in\partial\Omega_m} |\nabla u_m|(x) &\leq 1 && x \in \Omega_m,
\end{aligned}$$

*where the last equation holds with equality for $H^2$-a.a. $y \in \partial\Omega_m$. Moreover $\partial_{red}\Omega_m$ is analytic and the rest of the boundary has $H^2$ measure zero.*

The following lemma, whose simple proof is omitted, will frequently be used in this paper.

**LEMMA 1.** *Let $h \geq 0$ be continuous in $U$ (open set) and subharmonic in $U \cap \{h > 0\}$. Then $h$ is subharmonic in $U$.*

To prove theorem 1 we will need several lemmas to take care of technical details. The proof of the next lemma uses classical Lavrentiev principle.

**LEMMA 2.** *Let $u_m$ be a solution to the above free boundary problem and for $j = 1, 2$ and $m > m_0$ set*

$$v_{m,j}(x) = \begin{cases} \dfrac{r_{m,j}^2}{|x - x^j|} - r_{m,j}, & x \notin B(x^j, m^{-1}),\\[2ex] m a_j, & x \in B(x^j, m^{-1}),\end{cases}$$

*where*

$$r_{m,j} = \frac{1}{2m} + \sqrt{\frac{1}{4m^2} + a_j}.$$

*Then $B(x^j, r_{m,j}) \subset \Omega_m$ and consequently (for $j = 1, 2$)*

(1) $v_{m,j} \leq u_m$, *(by the maximum principle);*
(2) $\Delta v_{m,1} + \Delta v_{m,2} \leq \Delta u_m$ *in* $\cup B(x^j, 2r)$, *(by (1) and lemma 1);*
(3) $B(x^j, 2r) \subset \Omega_m$;

*where* $2r = \inf_{i=1,2} \sqrt{a_i} \le r_{m,j}$.

*PROOF.* Let $\epsilon > 0$ be arbitrarily small and define $v^\epsilon_{m,j}$ by

$$v^\epsilon_{m,j}(x) = \begin{cases} \dfrac{(r^\epsilon_{m,j})^2}{|x - x^j|} - r^\epsilon_{m,j}, & x \notin B(x^j, m^{-1}), \\ m a_j - 2\epsilon, & x \in B(x^j, m^{-1}), \end{cases}$$

where $j = 1, 2$, $m > m_0$ and

$$r^\epsilon_{m,j} = \frac{1}{2m} + \sqrt{\frac{1}{4m^2} + a_j - \frac{2\epsilon}{m}}.$$

Suppose $B(x^j, r_{m,j}) \setminus \Omega_m$ is non-empty for some $m$ and $j$. Letting this $m$ and $j$ be fixed we define $\tilde{v}^\epsilon(x) = v^\epsilon_{m,j}(\alpha x + (1-\alpha)x^j)$, where $\alpha > 1$ is the smallest value such that $B(x^j, \frac{r_{m,j}}{\alpha}) \subset \Omega^\epsilon_m := \{x : u_m(x) > \epsilon\}$. We also remark that $\alpha$ depends on $\epsilon$, i.e. $\alpha = \alpha(\epsilon)$. Since we have assumed $B(x^j, r_{m,j}) \setminus \Omega_m$ is non-empty we obtain

$$\alpha(0) = \lim \alpha(\epsilon) > 1.$$

Now the function $w = \tilde{v}^\epsilon - u_m$ is subharmonic in $B(x^j, \frac{r_{m,j}}{\alpha}) \setminus \overline{B(x^j, \frac{1}{m\alpha})}$, which, by the maximum principle, implies that the supremum of $w$ is attained on the boundary. On $B(x^j, \frac{1}{m\alpha})$ $w = -2\epsilon$ and on $\partial B(x^j, \frac{r_{m,j}}{\alpha})$ $w \le -\epsilon$, with equality only at $x^\epsilon \in \partial B(x^j, \frac{r_{m,j}}{\alpha}) \cap \partial\Omega^\epsilon_m$. Thus the supremum of $w$ in $B(x^j, \frac{r_{m,j}}{\alpha}) \setminus \overline{B(x^j, \frac{1}{m\alpha})}$ is attained at $x^\epsilon$ on the boundary. Hence the derivative of $w$ in the inward normal direction $\nu$ at $x^\epsilon$ is non-positive, i.e.

$$\frac{\partial \tilde{v}^\epsilon}{\partial \nu}(x^\epsilon) \le \frac{\partial u_m}{\partial \nu}(x^\epsilon).$$

Since the inward normal derivatives of these functions at $x^\epsilon$ coincide with the magnitude of their gradient vectors, we obtain

$$|\nabla \tilde{v}^\epsilon| \le |\nabla u_m|,$$

at $x^\epsilon$. And by direct calculations we arrive at

$$\alpha = |\nabla \tilde{v}^\epsilon|(x^\epsilon) \le |\nabla u_m|(x^\epsilon).$$

Now as $\epsilon \to 0$ we obtain $\alpha(0) \le 1$, which is a contradiction. Thus the lemma is proved.

**REMARK.** *Lemma 2 controls that points on $\partial\Omega_m$ do not degenerate to $\{x^1, x^2\}$. It also gives a barrier for $u_m$ from below. This will imply that the limit function $u = \lim u_m$ (whose existence we prove later) satisfies $\Delta u \ge -4\pi(a_1\delta_1 + a_2\delta_2)$.*

Before proceeding with our next lemma we find it convenient to introduce new notation. Namely, $\hat{H}_m(x) = \omega \int_{\partial\Omega_m} |x - y|^{-1}\, dH^2$.

**LEMMA 3.** *Set* $\mu_m := -\Delta u_m \chi_{\Omega_m}$. *Then the following hold:*

(1) $u_m = \hat{\mu}_m - \hat{H}_m \quad in\ \Omega_m$;
(2) $\sup u_m \le C$, *where sup is taken over* $\partial B(x^j, r_0)$ *for* $j = 1, 2$;
(3) $H^2(\partial\Omega_m) \le C$;
(4) $|\nabla u_m| \le C = (C_{r_0})$ *on* $\partial B(x^j, r_0)$ *for* $j = 1, 2$, $m$ *large and* $r_0 < r$.

*PROOF.* (1) is obvious.

Since $\hat{H}_m > 0$ we will have $u_m < \hat{\mu}_m$. Therefore

$$\sup_{|x-x^j|=r_0} u_m < \sup_{|x-x^j|=r_0} \hat{\mu}_m \le \omega \sup_{\substack{|x-x^j|=r_0\\ |y-x^j|=m^{-1}}} |x-y|^{-1} \int d\mu_m \le C \int d\mu_m.$$

By lemma 2 part (2)

$$-\mu_m \ge \Delta v_{m,1} + \Delta v_{m,2},$$

which gives

$$\int d\mu_m \le -\int_{\cup B(x^j,2r)} d(\Delta v_{m,1} + \Delta v_{m,2}).$$

Now using the divergence theorem and the fact that the inward normal derivative of $v_{m,j}$ on the boundary of $B(x^j, r_{m,j})$ is 1, we obtain

$$\int d\mu_m \le C \int_{|x-x^1|=r_{m,1}} dH^2 + \int_{|x-x^2|=r_{m,2}} dH^2$$
$$\le C((r_{m_j,1})^2 + (r_{m_j,2})^2),$$

As $r_{m,j} \le r_{m_0,j}$ for all $m$. (2) follows. (3) is almost included in the proof of (2). However, by divergence theorem

$$H^2(\partial\Omega_m) = \int_{\partial\Omega_m} dH^2 = \int_{\partial\Omega_m} \nu.\nabla u_m = \int d\mu_m,$$

where $\nu$ is the unit normal vector on $\partial\Omega$ which points into $\Omega_m$. By the proof of part (2) the last term is uniformly bounded from above.

(4) is proved similarly, by using the representation $u_m = \hat{\mu} - \hat{H}_m$. For $x$ on $\partial B(x^1, r_0) \cup \partial B(x^2, r_0)$ and $y$ on $\partial\Omega_m$ we have, by part (3) of lemma 2, $|x-y| > r$ for all $m$. Therefore for $x$ on $\partial B(x^1, r_0) \cup \partial B(x^2, r_0)$ we obtain $|\nabla \hat{H}_m|(x) \le CH^2(\partial\Omega_m)$, which, by part (3) of this lemma, is uniformly bounded. Similarly one proves that $|\nabla\hat{\mu}_m|$ is uniformly bounded on $\partial B(x^1, r_0) \cup \partial B(x^2, r_0)$ for $m$ large. Hence, so is $|\nabla u_m|$.

**LEMMA 4.** *There is an integer* $m_1$ *such that for* $m > m_2 := m_1/\inf a_j$ *the level sets* $W_m := \{x : u_m(x) > m_1\}$ *consists of two non-intersecting open sets* $W_{m,1}$, $W_{m,2}$, *where* $B(x^j, m^{-1}) \subset W_{m,j} \subset B(x^j, r_1)$ *for* $j = 1, 2$.

*PROOF.* Recall the definition of $r$ (lemma 2) and set $r_1 = \inf(r, m_0^{-1})$. Then a similar argument as in lemma 3 part (2) shows that on $|x-x^j| = r_1$ $u_m$ is uniformly bounded by $C$, say. Let $m_1 > C$. Then for $m > m_2$ and $x \in B(x^j, m^{-1})$ we have $u_m(x) = a_j m > a_j m_2 = a_j m_1 (\inf a_j)^{-1} \ge m_1$ and by continuity on a neighborhood of the closure of $B(x^j, m^{-1})$. Now set $W_{m,j} := W_m \cap B(x^j, r_1)$. Then by the above argument $W_{m,j}$ is non-empty. It also follows from the maximum principle, since $u_m \le C$ on $|x-x^j| = r_1$, that $W_{m,j} \subset B(x^j, r_1)$. Now observing the fact that $B(x^1, r_1) \cap B(x^2, r_1) = \emptyset$ we arrive at $W_{m,1} \cap W_{m,2} = \emptyset$. Thus the lemma is proved.

**LEMMA 5.** *Let $m_1$, $m_2$ and $W_{m,j}$ $(j=1,2)$ be as in lemma 4 and for $m > m_2$ set*

$$w_{m,j} = \begin{cases} \dfrac{\beta_{m,j}}{|x-x^j|} + m_1 - r_1^{-1}\beta_{m,j}, & x \notin B(x^j, m^{-1}), \\ ma_j & x \in B(x^j, m^{-1}), \end{cases}$$

*where*

$$\beta_{m,j} = \frac{ma_j - m_1}{m - r_1^{-1}}.$$

*Then $w_{m,j} \geq u_m$ in $W_{m,j}$ and $\Delta w_{m,j} \geq \Delta u_m$ in $W_{m,j}$.*

*PROOF.* Define $u_{m,j}$ to be $u_m$ in $W_{m,j}$ and $m_1$ in $B(x^j, r_1) \setminus W_{m,j}$. By lemma 1 $w_{m,j} - u_{m,j}$ is superharmonic in the interior of $B(x^j, r_1) \setminus B(x^j, m^{-1})$. It also follows from the definition (since by lemma 4 $W_{m,j} \subset B(x^j, r_1)$) that this function is zero on the boundary of $B(x^j, r_1) \setminus B(x^j, m^{-1})$. Thus by the maximum (minimum) principle $w_{m,j} - u_{m,j} \geq 0$ on this domain. Hence, by lemma 1, $\Delta w_{m,j} \geq \Delta u_{m,j} = \Delta u_m$ in $W_{m,j}$.

**REMARK.** *Lemma 5 controls $\Delta u_m$ from above, and as one can easily see it leads to the fact that a limit function $u = \lim u_m$ will satisfy $\Delta u \leq -4\pi(a_1\delta_1 + a_2\delta_2)$. This together with the remark following lemma 2 implies $\Delta u = -4\pi(a_1\delta_1 + a_2\delta_2)$.*

**LEMMA 6.** *The domains $\Omega_m$, $m > m_0$ are uniformly bounded, i.e. $\cup\Omega_m$ is bounded.*

*PROOF.* To prove this we need a smoothing process. Let $h$ denote a positive radial $C^\infty$ function with support in $B(0,1)$ and $\int h = 1$. Now set

$$f_m(x) = \begin{cases} u_m(x), & x \notin W_m, \\ m_1, & x \in W_m, \end{cases}$$

and define $f_m^\epsilon = \hat{\nu}_m^\epsilon - \hat{H}_m$, where $\nu_m^\epsilon = \nu_m * h_\epsilon$ (convolution), $\nu_m = -\Delta u_m$ $(\geq 0$ by lemma 1) in $\Omega_m$ and $h_\epsilon = \epsilon^{-n} h(x/\epsilon)$. By lemmas 4 and 2 (part 3)

$$\mathrm{supp}(\nu_m^\epsilon) \subset \{\cup_x \overline{B(x,\epsilon)}; x \in W_{m,j}, j = 1,2\} \subset B(x^j, 2r) \subset \Omega_m$$

provided $\epsilon < r$. Moreover $f_m^\epsilon = f_m$ outside $\cup B(x^j, 2r)$, and $\Delta f_m^\epsilon = -\nu_m^\epsilon$.

Now for $\epsilon = r/2$ set $C = \sup_m \sup_x \nu_m^\epsilon(x)$, then it is easy to see that $C < \infty$. Indeed $C \leq H^2(\partial\Omega_m) \sup h_\epsilon$, which by lemma 3 (part 3) is uniformly bounded. Let $B^1$ be the smallest ball containing $\cup B(x^j, 2r)$ and choose a large positive number $C^1$ $(> C)$ such that there is a Lipschitz function $g^1$ with properties:

$$\begin{aligned} \Delta g^1 &= -C^1 \chi_{B^1} && \text{in } B, \\ g^1 &= 0 && \text{on } \mathbb{R}^3 \setminus B, \\ \lim_{x \to \partial B} |\nabla g^1(x)| &= 1 && x \in B, \end{aligned}$$

where $B(\supset B^1)$ is a ball concentric with $B^1$. In the rest of the paper we omit the lim sign for the values of the gradient of a function on the boundary. It is, however, tacitly understood that by the values of the gradient of a function on the boundary of a domain we mean the limit from the interior of that domain.

We claim now that $\cup\Omega_m \subset B$. Suppose not; let $\Omega_m$ be such that $\Omega_m \setminus B$ is not empty. Then let $g^2$ be a solution to the following Dirichlet problem

$$\begin{aligned} \Delta g^2 &= -\nu_m^\epsilon \qquad && \text{in } B \cap \Omega_m, \\ g^2 &= 0 && \text{on } \partial(B \cap \Omega_m). \end{aligned}$$

It is easy to see that $g^2 \leq g^1$ in $\mathbb{R}^3$ and $|\nabla g^2| \leq |\nabla g^1|$ on $(\partial B) \cap \Omega_m$. Similarly we obtain $0 \leq g^2 \leq f_m^\epsilon$ in $\mathbb{R}^v3$ and $|\nabla g^2| \leq |\nabla f_m^\epsilon|$ on $\partial\Omega_m \cap B$. As $f_m^\epsilon$ is Lipschitz it follows that $g^2$ is Lipschitz.

Applying this we obtain

$$\begin{aligned} H^2((\partial B) \cap \Omega_m) &= \int_{\partial B \cap \Omega_m} |\nabla g^1| dH^2 \geq \int_{\partial B \cap \Omega_m} |\nabla g^2| dH^2 \\ &= \int_{\Omega_m} (\Delta g^2 + \nu_m^\epsilon) dH^3 = \int_{\partial\Omega_m} (-|\nabla g^2| + |\nabla f_m^\epsilon|) dH^2 \\ &\geq \int_{\partial\Omega_m \setminus B} (-|\nabla g^2| + |\nabla f_m^\epsilon|) dH^2 = \int_{\partial\Omega_m \setminus B} dH^2 \\ &= H^2(\partial\Omega_m \setminus B), \end{aligned}$$

since $|\nabla f_m^\epsilon| \geq |\nabla g^2|$ on $\partial\Omega_m \cap B$, $|\nabla g^2| = 0$ on $\partial\Omega_m \setminus B$ and $|\nabla f_m^\epsilon| = 1$ a.e. on $\partial\Omega_m$. Thus we end up with

$$H^2(\partial(B \cup \Omega_m)) \leq H^2(\partial B).$$

Now project $\partial(B \cup \Omega_m)$ radially onto $\partial B$. Then it is known that the surface area decreases under this projection. Hence

$$H^2(\partial(B \cup \Omega_m)) > H^2(\partial B).$$

Thus a contradiction is obtained and the lemma is proved.

**REMARK.** *If the inequality in the last boundary condition in theorem 2 could be replaced with an equality (this is the case for $C^1$ boundaries), then the proof of lemma 6 would be much easier. Indeed the classical Lavrentiev principle is applicable in this case as well.*

*PROOF OF THEOREM 1.* By lemma 3 part (4) for $r_0 > 0$ and small $|\nabla u_m| \leq C(= C_{r_0})$, uniformly on $\partial B(x^1, r_0) \cup \partial B(x^2, r_0)$. Since $|\nabla u_m| \leq 1$ on $\partial\Omega_m$ and it is subharmonic in the interior of $\Omega_m \setminus (B(x^1, r_0) \cup B(x^2, r_0))$ we have $|\nabla u_m| \leq \sup(C, 1)$. It thus follows that $u_m$ is uniformly Lipschitz continuous outside any neighborhood of $\{x^1, x^2\}$ and consequently $\{u_m\}$ is equicontinuous there.

Now by Ascoli's theorem there is a subsequence (again) labeled $u_m$ which, outside any neighborhood of $\{x^1, x^2\}$, converges uniformly to a limit function $u$. Define now $\Omega := \{x : u(x) > 0\} \cup \{x^1, x^2\}$. By lemma 2 $(B(x^1, r) \cup B(x^2, r)) \subset \Omega$, and by lemma 6 $\Omega$ is bounded. Now we assert:

**ASSERTION.**

(1) *$\partial_{red}\Omega$ is analytic;*
(2) *$H^2(\partial\Omega \setminus \partial_{red}\Omega) = 0$;*
(3) *$|\nabla u| \leq 1$ on $\partial\Omega$;*
(4) *$|\nabla u| = 1$ $H^2$-a.e. on $\partial\Omega$;*
(5) *if $n = 2$, then $\partial\Omega$ is analytic,*

for all functions $h$ harmonic on $\overline{\Omega}$.

Assume this for the moment. Then by Green's theorem and the Cauchy data of $u$ on $\partial\Omega$ it follows that

$$\int_{\partial\Omega} h dH^2 = \langle -\Delta u \chi_\Omega, h\rangle.$$

By lemma 2 part (2) and lemma 5 it follows that $-\Delta u\chi_\Omega = 4\pi a_1\delta_1 + 4\pi a_2\delta_2$. Letting $a_j = \frac{c_j}{4\pi}$ we obtain the desired result.

To complete the proof we have to prove the assertion.

*PROOF OF THE ASSERTION.* By [AC, theorem 8.4] it suffices to show that locally near $\partial\Omega$ $u$ minimizes $\int |\nabla\psi|^2 + \chi_{(\psi>0)}$ i.e. for $x \in \partial\Omega$

$$\int_{B_x} (|\nabla u|^2 + \chi_{(u>0)}) \leq \int_{B_x} (|\nabla\psi|^2 + \chi_{(\psi>0)}),$$

where $B_x$ is a small ball with center $x$ and $\psi - u \in H^{1,2}_o(B_x)$. This however, is just lemma 5.4 in [AC]. For the reader's convenience we include the proof of this lemma.

Let $\eta \in C^\infty_c(B_x)$, $0 \leq \eta \leq 1$ and define $\psi_m := \psi + (1-\eta)(u_m - u)$, where $\psi - u \in H^{1,2}_o(B_x)$. Now for large $m$

$$\int_{B_x} (|\nabla u_m|^2 + \chi_{(u_m>0)}) \leq \int_{B_x} (|\nabla\psi_m|^2 + \chi_{(\psi_m>0)}),$$

which can be written as

$$0 \geq \int_{B_x} \nabla(u - \psi + \eta(u_m - u).\nabla(u + \psi + (2-\eta)(u_m - u)) + \chi_{(u_m>0)} - \chi_{(\psi_m>0)}$$

or

$$\begin{aligned} 0 \geq \int_{B_x} (&|\nabla u|^2 - |\nabla\psi|^2 + \eta(2-\eta)|\nabla(u_m - u)|^2 + (u_m - u)\varphi_m \\ &+ \chi_{(u_m>0)} - \chi_{(\psi_m>0)}), \end{aligned}$$

where $\{\varphi_m\}$ are uniformly bounded.

Since $\eta(2-\eta) \geq 0$, $\chi_{(\psi_m>0)} \leq \chi_{(\psi>0)} + \chi_{(\eta<1)}$ and $u_m \to u$ uniformly in $B_x$ we obtain

$$\begin{aligned} 0 &\geq \int_{B_x} (|\nabla u|^2 - |\nabla\psi|^2 - \chi_{(\psi>0)} - \chi_{(\eta<1)}) + \lim \int_{B_x} \chi_{(u_m>0)} \\ &\geq \int_{B_x} (|\nabla u|^2 - |\nabla\psi|^2 + \chi_{(u>0)} - \chi_{(\psi>0)} - \chi_{(\eta<1)}) \\ &\quad - \lim \int_{B_x\setminus\Omega_m} \chi_{(u>0)}. \end{aligned}$$

It remains to prove that $\lim \int_{B_x\setminus\Omega_m} \chi_{(u>0)} = 0$. But this is obvious as for any point $x$ with $u(x) > 0$ $u_m$ will also be positive in a small neighborhood of $x$, i.e. $\chi_{((u>0)\setminus(u_m>0))} \to 0$ $H^3-$a.e..

Summing up we arrive at

$$\int_{B_x} (|\nabla u|^2 + \chi_{(u>0)}) \leq \int_{B_x} (|\nabla\psi|^2 + \chi_{(\psi>0)} + \chi_{(\eta<1)}),$$

and for an appropriate choice of functions $\eta$ we obtain the desired result.

## CONCLUDING REMARKS

As we mentioned earlier the proofs given in this paper may be slightly different for the case $n \neq 3$. The first thing to note is the boundary condition $u_m = ma_j$ in theorem 2. For $n \geq 3$ we should take $u_m = m^{n-2}a_j$ and for $n = 2$ $u_m = \log ma_j$. The reason for being exact here is that for other values than these we can't control (see lemmas 2 and 5) $-\Delta u_m$ from above or below. If e.g. we let $u_m = d_m$ on $B(x^1, m^{-1})$ in $\mathbb{R}^3$, say, where $d_m/m \to \infty$ as $m \to \infty$, then by lemma 2 $B(x^1, r_{m,1}) \subset \Omega_m$ where

$$r_{m,1} = \frac{1}{2m} + \sqrt{\frac{1}{4m^2} + \frac{d_m}{m}} \to \infty \qquad \text{as } m \to \infty.$$

Hence $\Omega = \mathbb{R}^3$. Several places need attention when $n \neq 3$. The Newtonian kernel appearing in this paper is, of course, dimensional dependent. For $n \geq 3$ the kernel is $|\,.\,|^{2-n}$ and for $n = 2$ we should use $-\log|\,.\,|$. In the proof of theorem 1 we have $-\Delta u \chi_\Omega = 4\pi(a_1\delta_1 + a_2\delta_2)$; for the general case the constant $4\pi$ should be replaced by $\omega_n/(n-2)$ where $\omega_n$ is the surface area of the $(n-1)$-dimensional unit sphere. Lemma 6 is trivial when $n = 2$. For other values of $n$ the proof is the same.

The proof of lemma 3 (part 2) is the same when $n \geq 3$. It is however different for the case $n = 2$. In this case we can't use the positivity of the kernel as we did for the case $n = 3$. We can fortunately estimate the kernel from each side as follows. For $x \in \cup \partial B(x^j, r_0)$ and $y \in \partial\Omega_m$ we have, by lemma 3 part 3 and the fact that $n = 2$, the following estimate

$$|x - y| < H^2(\partial\Omega_m) < C.$$

This implies $\log|x - y| < \log C$. Hence

$$-\hat{H}_m(x) < \log C \int_{\partial\Omega_m} dH^{n-1} < C.$$

Now for $x \in \cup\partial B(x^j, r_0)$ and $y \in \cup\partial B(x^j, m^{-1})$ we have $-\log|x-y| < -\log(r_0 - m^{-1}) < C$. Hence $\hat{\mu}_m(x) < C \int d\mu_m$ and the rest of the proof is similar.

## References


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