Qualitative Properties of Positive Solutions to Non-Linear PDE and FBP

HENRIK SHAHGHOLIAN

The Royal Institute of Technology
Dept. of Math.
S-100 44 Stockholm 70, Sweden
Feb. 28, 1995

0. Background

In the last few years, the moving plane method of A.D. Alexandroff has been an important tool for studying qualitative properties of positive solutions to nonlinear elliptic partial differential equations. As a result, monotonicity and symmetry of the solutions have been obtained (see [BN], [GNN],[SE]).

Our objective here is to contribute in this direction. We begin by a slight generalization of the maximum principle in "small domains"; this is essentially done by [LU]. Then we recall the main theorem of [BN], and we generalize it to the doubly connected domains. In Theorem 1.2 using similar ideas to [BN] we establish starshapedness of the level sets in certain cases for solutions in ring-shaped domains, and we also prove that in domains like $\{x = (r, \theta) \in \mathrm{R}^2; 1 < r < 2, \pi/2 < \theta < 3\pi/2\}$ (polar coordinates) the solutions of $\Delta u = f(|x|, u, |Du|)$ with zero boundary condition are symmetric in $\{x_2 = 0\}$. As a result of these new methods, which simply are contraction or rotation of the original domains, we prove uniqueness theorems in domains such as truncated cones and almost ring-shaped domains for solutions to problems with mixed boundary conditions.

In section 2 we apply the methods of section 1 to free boundary problems and we obtain some interesting geometric properties. In some cases we can even establish the Lipschitz continuity of the free boundary.

1. Moving plane, Contraction and Rotation Methods in PDE

In this section Ω will in general be a bounded domain in $\mathrm{R}^N (N \geq 2)$. We will consider nonlinear second degree elliptic equations in divergence form

$$Qu = Q(x, u, Du, D^2u) = \nabla \cdot \overline{A}u + Bu = \nabla \cdot \overline{A}(x, u, Du) + B(x, u, Du),$$

and linear ones

$$Lu = \sum D_j(a_{ij}D_iu + a_iu) + \overline{b}\cdot Du + au.$$

The conditions that we need to impose on the coefficients are as follows. For each fixed $x \in \Omega$

(1) $A_i(x,u,p)$ are Lipschitz in u with Lipschitz constant $b(x)$;
(2) $B(x,u,p)$ is Lipschitz in (u,p) with Lipschitz constant $b(x)$;
(3) A_i are differentiable in p,

where $b(x) \in L^{q/2}(\Omega)$ for some $q > N$. Moreover we assume the uniform ellipticity condition:

$$\nu|\xi|^2 \le \sum \frac{\partial A_i}{\partial p_j}\xi_i\xi_j \le \mu|\xi|^2,$$

for some $\nu, \mu > 0$. We also assume that the coefficients of L are in $L^{q/2}(\Omega)$ for some $q > N$ and

$$\nu|\xi|^2 \le \sum a_{ij}\xi_i\xi_j \le \mu|\xi|^2.$$

Remark: The conditions on $\overline{A}$ and B seems somehow strange. However, if they are uniformly Lipschitz in $x \in \Omega$ then they satisfy this condition.

In the sequel $|E|$ will denote the volume of the domian E.

Lemma A. *Let $Lw \ge 0$ in Ω, $w \in W^{1,2}(\Omega)$ $w \le 0$ on $\partial\Omega$, where Ω is bounded. Then there is a constant $\delta > 0$ such that $w < 0$ in Ω provided $|\Omega| < \delta$.*

Proof: Suppose $G = \{w > 0\} \subset \Omega$ is nonempty. Then by [LU, Lemma 4.1] we have

$$\int_G \left[|\nabla w|^2 + \frac{4}{\nu}a^- w^2\right] dx \le$$
$$\frac{4}{\nu}\int_G \left[\sum(a_{ij}D_iw + a_iw)D_jw - (\overline{b}\cdot Dw + aw)w\right] dx + c_1\|w\|_2^2,$$

where c_1 depends on q, N, ν and $L^{q/2}(\Omega)$ norms of the coefficients. Here

$$a^- = -\inf\left(a, \frac{1}{|G|}\int_G a\right).$$

Since the first term to the right of the above inequality is non-positive ($Lw \geq 0$) and

$$a^- \geq -\frac{1}{|G|}\int a,$$

we obtain

$$\int_G |\nabla w|^2 \leq \left(c_1 + \frac{4}{\nu}\frac{1}{|G|}\int a\right)\|w\|_2^2 \leq \left(c_1 + \frac{4}{\nu}|G|^{-2/q}\|a\|_{q/2}\right)\|w\|_2^2.$$

Now applying the Poincarés inequality we end up with

$$\int_G |\nabla w|^2 \leq |G|^{2/N}\left(c_1 + \frac{4}{\nu}|G|^{-2/q}\|a\|_{q/2}\right)\int_G |\nabla w|^2.$$

Hence if

$$|G| \leq |\Omega| < \delta := \min\left(1, c_1 + \frac{4}{\nu}\|a\|_{q/2}\right)^{Nq/2(N-q)},$$

then we obtain a contradiction, unless G is empty. This implies $w \leq 0$. Applying the weak Harnack inequality we obtain the result.

Remark: This lemma holds for general linear elliptic equations of second degree, provided the coefficients for D_iD_ju are in $W^{1,q/2} \cap L^\infty(\Omega)$, for some $q > N$. Also for the case of bounded coefficients the proof is direct and easier. Cf. also [Dan].

The condition that Q must satisfy in order to obtain x_1 monotonicity of u is

$$(*) \qquad \begin{aligned} &Q(x, u, p_i, p_{jk}) \leq \\ &Q(x_1', x_2, ..., x_N, u, -p_1, p_2, ...p_N, p_{11}, -p_{12}, ..., -p_{1N}, p_{22}, ...), \end{aligned}$$

for $x \in \Omega$ with $x_1 < x_1'$, $x_1 + x_1' < 0$ and all p_1. (in [BN] using certain parabolic methods they only assume this for $p_1 \geq 0$.)

Theorem 1.1. *Let Ω be convex in the x_1 direction and symmetric w.r.t. the plane $x_1 = 0$. Let u be a nonnegative solution to $Qu = 0$, $u \in W_o^{1,2}(\Omega) \cap C(\Omega)$. Assume moreover Q satisfies (*). Then $u(x)$ is increasing in the x_1 direction for $x_1 < 0$ and symmetric w.r.t. $x_1 = 0$.*

Proof: Define $-\lambda_0 = \inf_{x\in\Omega} x_1$. Set $\Sigma_\lambda = \{x \in \Omega : x_1 < \lambda\}$ and define $u^\lambda(x) = u(2\lambda - x_1, x_2, ...)$. Then by condition $(*)$ $Qu^\lambda \leq 0$ (in the weak sense). By the theorem of the mean (in integral form) for $\phi \in W_o^{1,2}(\Sigma_\lambda)$,

$$\begin{aligned} 0 &\geq \int (\overline{A}u - \overline{A}u^\lambda)\cdot\nabla\phi - (Bu - Bu^\lambda)\phi \\ &= \int \sum (a_{ij}D_iw^\lambda + a_iw^\lambda)D_j\phi + (\overline{b}\cdot Dw^\lambda + aw^\lambda)\phi, \end{aligned}$$

where $w^\lambda = u - u^\lambda$ and the linear operator appearing here has $L^{q/2}$ coefficients . Now for $0 < \lambda + \lambda_0$ small, the volume of Σ_λ is also small and $w^\lambda \le 0$ on $\partial\Sigma_\lambda$. Hence for such λ we can apply Lemma A to obtain $w^\lambda < 0$ in Σ_λ.

Let $(-\lambda_0, \lambda_1)$ be the largest interval of those λ such that $w^\lambda < 0$. Then by weak Harnack inequality $w^{\lambda_1} < 0$ in Σ_{λ_1} unless $w^\lambda \equiv 0$. We claim $\lambda_1 = 0$. Suppose $\lambda_1 < 0$. Take a compact set K in Σ_{λ_1} such that $|\Sigma_{\lambda_1} \setminus K| < \delta/2$. Then, since $w^{\lambda_1} < 0$ on K (by compactness), we obtain by continuity that for all $\epsilon > 0$ and small enough

$$\begin{cases} |\Sigma_{\lambda_1+\epsilon} \setminus K| < \delta, & \\ \qquad w^{\lambda_1+\epsilon} < 0 & \text{on } K. \end{cases}$$

Now in $\Sigma_{\lambda+\epsilon} \setminus K$ we repeat our argument to see that $w^{\lambda_1+\epsilon}$ is a subsolution of a linear operator in divergence form with $L^{q/2}$ coefficients and it is nonpositive on $\partial(\Sigma_{\lambda+\epsilon} \setminus K)$. Hence, by Lemma A, $w^{\lambda_1+\epsilon} < 0$ in $\Sigma_{\lambda+\epsilon} \setminus K$ and therefore in $\Sigma_{\lambda+\epsilon}$. This is a contradiction to the maximality of the interval $(-\lambda_0, \lambda_1)$ and implies that u increases in the x_1-direction. To see that u is symmetric in $\{x_1 = 0\}$ let $\lambda \to 0$ then we'll have $w^0 \le 0$, i.e. $u \le u^0$ in Σ_0. Now changing the domain in consideration to $x_1 > 0$ we obtain $u \ge u^0$ and hence equal.

THEOREM 1.1$'$. *Let Ω be a bounded doubly connected domain. Denote by Ω^i the bounded connected component of the complement of Ω and Ω^e the unbounded. Suppose Ω^i and $\Omega \cup \Omega^i$ are convex in the x_1-direction and symmetric in $\{x_1 = 0\}$. Let $u \in W^{1,2}(\Omega) \cap C(\Omega)$ be a nonnegative solution to*

$$\begin{cases} Qu = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega^e \\ u = c & \text{on } \partial\Omega^i \\ u \le c & \text{in } \Omega, \end{cases}$$

where $c > 0$ and Q satisfies $(*)$. Then u is increasing in the x_1 direction for $x_1 < 0$ and symmetric in $x_1 = 0$.

PROOF: The proof is similar to that of Theorem 1.1. However in this case we should apply the maximum principle to $u - u^\lambda$ in $\Sigma_\lambda \setminus \tilde{\Omega}^i_\lambda$, where $\tilde{\Omega}^i_\lambda$ is the reflection of Ω^i in $\{x_1 = \lambda\}$, and use the fact that $u \le c$.

REMARK: Similar generalizations as in Theorem 1.1 can be made here as well.

THOEREM 1.2. *Let Ω be a bounded doubly connected domain, and Ω^i denote the bounded connected component of the complement and Ω^e*

the unbounded. Suppose Ω^i and $\Omega \cup \Omega^i$ are starshaped w.r.t. the origin. Let now $u \in W^{1,2}(\Omega) \cap C(\Omega)$ be a solution to

$$Qu = 0 \quad in\ \Omega, \qquad u = h \quad on\ \partial\Omega,$$

where Q satisfies

$$(**) \qquad t^2 Q(x, u, Du, D^2u) \geq Q(x/t, u, tDu, t^2D^2u).$$

Assume moreover $h(x) \geq u(x/t)$ for $x \in \partial\Omega^i$, $x/t \in \Omega$ and $h(x) \leq u(tx)$ for $x \in \partial\Omega^e$, $tx \in \Omega$ where $0 < t \leq 1$. Then for $t \in (0,1)$

$$u(x/t) < u(x) \qquad \forall x, x/t \in \Omega,$$

and consequently the level sets of u are starshaped w.r.t. the origin. Moreover u is the unique solution.

Proof: Define $u_t(x) = u(x/t)$ in $\Omega_t := \{tx; x \in \Omega\}$. Then by $(**)$

$$Qu_t \geq 0,$$

in Ω_t. Hence as in the proof of Theorem 1.1 $u - u^\lambda$ will be a subsolution to a linear elliptic equation with $L^{q/2}$ coefficients. Now for t small enough $|\Omega_t \cap \Omega|$ is small, and the maximum principle of Lemma A can be applied. The rest of the proof follows the line of the proof of Theorem 1.1. For two different solutions u and v we may compare u by v_t and v by u_t and obtain uniqueness.

A simple example satisfying condition $(**)$ is $\Delta u - f(u)$ where $f \geq 0$ is Lipschitz. The case f is monotone nondecreasing is well studied. The same method of proof works for our next theorem.

Theorem 1.2$'$. *Let Ω be such that for any $x \in \Omega$ the set of points $\{tx; \quad 0 < t\} \cap \Omega$ is a "connected" segment. Let moreover $u \in W^{1,2}(\Omega) \cap C(\Omega)$ be a solution to*

$$Qu = 0 \quad in\ \Omega,$$

where Q satisfies $(**)$. Assume moreover that for $x^1, x^2 \in \partial\Omega$ and $x \in \Omega$ lying on the same line emanating from the origin with $|x^1| \leq |x| \leq |x^2|$ $h(x^1) \geq u(x)$ and $h(x^2) \leq u(x)$. Then

$$u(x/t) < u(x), \qquad \forall x, x/t \in \Omega,$$

where $0 < t < 1$.

In the rest of this section Ω is in R^2. We call Ω convex w.r.t circles if the intersection of any circle centered at the origin with Ω is connected. We also assume in the following that $\Omega \subset \{x = (r, \theta) \in \mathrm{R}^2; 0 < \theta < 2\pi\}$.

THEOREM 1.3. *Let Ω be as above and let $u \in W^{1,2}(\Omega) \cap C(\Omega)$ be a solution to*

$$\Delta u = f(|x|, u, |Du|),$$

which satisfies

$$u(r, \theta^1) \leq u(r, \theta) \leq u(r, \theta^2),$$

where f is Lipschitz in u, $\theta^1 \leq \theta \leq \theta^2$ and (r, θ^1) and (r, θ^2) lie on $\partial\Omega$. Then

$$u(r, \theta) < u(r, \theta'),$$

for $0 < \theta < \theta' < 2\pi$ with $(r, \theta), (r, \theta') \in \Omega$

PROOF: Let $\theta_0 = \min\{\theta; (r, \theta) \in \Omega\}$ $\theta_1 = \max\{\theta; (r, \theta) \in \Omega\}$ and define Ω_λ to be the rotation of Ω with angle λ and set

$$u^\lambda(x) = u^\lambda(r, \theta) = u(r, \theta + \lambda), \qquad w^\lambda = u - u^\lambda.$$

Then in $\Omega \cap \Omega_\lambda$ we'll have $\Delta w^\lambda + c w^\lambda = 0$ and $w^\lambda \geq 0$ on $\partial(\Omega \cap \Omega_\lambda)$, where c is bounded. Hence by Lemma A $w^\lambda > 0$ there provided $0 < \theta_1 - \theta_0 - \lambda$ is small enough. The rest of the proof is similar to that of Theorem 1.1.

Next we prove a symmetric version of this theorem. The method will be by reflection and rotation.

THEOREM 1.4. *Let Ω in R^2 be convex w.r.t. circles and symmetric w.r.t. $\{x_2 = 0\}$. Suppose $u \in W_o^{1,2}(\Omega) \cap C(\Omega)$ is a nonnegative solution to*

$$\Delta u = f(|x|, u, |Du|) \quad \text{in } \Omega,$$

where f is Lipschitz in u.

Then

$$u(r, \theta) < u(r, \theta'),$$

for $\theta < \theta'$, $\theta + \theta' < 2\pi$ and $\inf_{x \in \Omega} \theta_x < \theta < \pi$. Moreover u is symmetric w.r.t. $\{x_2 = 0\}$.

PROOF: Set $\Sigma_\lambda = \{x \in \Omega; \theta_x < \lambda\}$, $u^\lambda(r, \theta) = u(r, 2\lambda - \theta)$ and $w^\lambda = u - u^\lambda$ in Σ_λ. We take first λ such that $0 < \lambda - \inf_{x \in \Omega} \theta_x$ is small enough in order for Lemma 1 to be applicable. The rest of the proof follows the line of the proof of Theorem 1.1.

In the following we let Ω be defined as

$$\Omega := \{x = (r, \theta); 0 < r_1 < r < r_2, \quad 0 < \theta_1 < \theta < \theta_2 < 2\pi\}.$$

THEOREM 1.5. *Let $u \in C^1(\overline{\Omega})$ be a solution to*

$$\begin{cases} \Delta u = f(|x|, u, |Du|) & \text{in } \Omega \\ u(r, \theta_j) = g_j & r_1 < r < r_2 \\ u_r(r_j, \theta) = 0 & \theta_1 < \theta < \theta_2 \end{cases}$$

where $j = 1, 2$, f is Lipschitz in u and $g_1(r, \theta_1) \le u(r, \theta) \le g_2(r, \theta_2)$ for $\theta \in (\theta_1, \theta_2)$. Then

$$u(r, \theta') < u(r, \theta''),$$

for $r_1 \le r \le r_2$ and $\theta_1 \le \theta' < \theta'' \le \theta_2$. Moreover if $v \ge 0$ is another solution then $u = v$.

The proof is again the same. However we need to use Lemma A in combination with Hopf's lemma. For mixed boundary problem see [BP].

REMARK: Of course the same technique works for domains in R^N. One can also prove existence results in such domains by modifying the proof in [BN, Theorem 7.1].

2. Applications to FBP

The method of section 1 can be applied to obtain some geometrical properties and in some cases Lipschitz continuity of free boundaries. For instance a direct application of Theorem 1.1 is to the problem of reaction diffusion equation. We consider the simplest case

$$\Delta u + f(u) = 0 \qquad \text{in } \Omega$$

with boundary value $u = 1$ on $\partial\Omega$, where f is Lipschitz and $f(0) = 0$. With certain assumptions on f (see [D]) the set $\{u = 0\}$ (the dead core) is not empty. Assuming Ω to be convex in the x_1 direction and symmetric w.r.t. $x_1 = 0$ then the method of the proof of Theorem 1.1 shows that the dead core inherits this property. Similar results hold when Ω is as in Theorem 1.4. The first result in this section is concerned with exterior problem of the above kind.

THEOREM 2.1. *Let $\Omega^- = \Omega \cap \{x_1 < 0\}$ be bounded and set $-\lambda_0 = \inf_{x \in \Omega} x_1$. Suppose Ω^- admits a nonnegative solution $u \in W^{1,2}(\Omega^-) \cap C(\Omega^-)$ to the following problem:*

$$\begin{cases} \Delta u = f(u) + g(x)\chi_{\Omega^-} & \text{in } \mathrm{R}^N_- \\ u = 0 & \text{on } \mathrm{R}^N_- \setminus \Omega^-, \end{cases}$$

where $f(0) = 0$, f is Lipschitz and $g(\in L^\infty(\mathrm{R}^N_-))$ is nonnegative and nonincreasing in the x_1 direction in Ω^-. Then u is nondecreasing in the x_1 direction in $\{x_1 < -\lambda_0/2\}$.

PROOF: Set $\Sigma_\lambda = \Omega \cap \{x_1 < \lambda\}$, $u^\lambda(x)(x) = u(2\lambda - x_1, x_2, ..., x_n)$ and $w^\lambda(x) = u(x) - u^\lambda(x)$ in Σ_λ. Then

$$\Delta w + cw \geq 0, \qquad \text{in } \Sigma_\lambda,$$

and $w \leq 0$ on $\partial\Sigma_\lambda$. Hence by Lemma A $w^\lambda < 0$ in Σ_λ, provided $0 < \lambda + \lambda_0$ is small enough. The rest of the proof follows as in the proof of Theorem 1.1.

REMARK: Theorem 2.1 is true for much more general equations of type $(*)$ with $u \in C^{1,\alpha}(\mathrm{R}^N_-)$ and $Q(x, 0, 0, 0) = g(x)\chi_{\Omega^-}$ and g as above

Using the above theorem we can prove results for a special case of the Pompei problem.

COROLLARY 2.2. *Let Ω admit a nonnegative solution $u \in W^{1,2}_o(\Omega) \cap C(\Omega)$ to*

$$\Delta u + \beta u = \chi_\Omega \qquad \text{in } \Omega,$$

for some β. Then Ω is a ball.

THEOREM 2.3. *Let u and Ω be as in Theorem 2.1 and suppose g is strictly decreasing in the x_1-direction and nonnegative. Then $\partial\Omega \cap \partial\Omega^-$ is Lipschitz continuous.*

PROOF: Let $x \in \partial\Omega \cap \partial\Omega^-$ and denote by C_x the part of the solid cone, generated by x and $\partial\Omega \cap \{x_1 = 0\}$, which lies in R^N_-. Define moreover

$$C^e_x = \{y : (y - x) \cdot (z - x) < 0, \quad \forall z \in C_x\}$$
$$C^i_x = \{y : (y - x) \cdot (z - x) > 0, \quad \forall z \in C_x\}.$$

Applying the method of moving plane it is not hard to show that

(1) for $y \in C^i_x$, $u(t(y - x)) > 0$ provided $0 < t$ is small enough,
(2) for $z \in C^e_x$, $u(z) = 0$.

This proves the theorem.

THEOREM 2.4. *Let Ω be doubly connected. Denote by Ω^i the bounded connected component of the complement and Ω^e the unbounded. Suppose Ω admits a nonnegative solution $u \in W^{1,2}(\Omega) \cap C(\Omega)$ to the following overdetermined Cauchy problem:*

$$\begin{cases} \Delta u = f(u, |Du|) + \chi_\Omega & \text{in } \mathrm{R}^N \\ u = 0 & \text{on } \Omega^e \\ u = c & \text{on } \Omega^i \\ 0 \leq u \leq c & \text{on } \Omega, \end{cases}$$

where f is Lipschitz, $f(0) = 0$. Then Ω is an annulus.

PROOF: The proof is similar to that of Theorem 2.1. However, we must apply Lemma A in the domain $\Sigma_\lambda \setminus \tilde{\Omega}^i$, where ~ (tilde) means reflection in $\{x_1 = \lambda\}$. Since the direction x_1 is not essential we can replace it by any other direction.

The method in the proof of the above theorems requiers the solutions to have continuous derivatives across the boundary. This, however, is included in data. In several free boundary problems this is not fulfilled. Since most of these free boundaries are arising in minimization problems, it makes it easier to work directly with minimizing functional. Our aim in the following will be to deal with a minimization problem arising in potential flow.

Let $G \subset \mathrm{R}^N$ be a bounded open set and consider the problem of minimizing

$$J(v) = J_g(v) = \int_{\{v>0\}} |\nabla v|^2 + g^2\chi_{\{v>0\}},$$

over the set $\mathrm{K} := \{v; v, |\nabla v|^2 \in L^1(\mathrm{R}^N); v = 1 \text{ on } G\}$, where $g \geq c > 0$ is bounded. This functional admits a minimizer and it is not necessary unique.(This functional and the local minimizers have been studied by [AC].) As the functional is not convex there may well be local minima, which are defined as follows: A function u says to be a local minimum to J if there is an $\epsilon > 0$ such that if

$$\int (|\nabla(u - v)|^2 + g^2|\chi_{\{u>0\}} - \chi_{\{v>0\}}|) < \epsilon,$$

then $J(u) \leq J(v)$, for $v \in \mathrm{K}$. It is also known [AC, Lemma 2.8] that local minima have compact support.

THEOREM 2.5. *Let u be a local minimizer of J. Define $-\lambda_0 = \inf_{x\in\Omega} x_1$ and suppose g is nonincreasing in the x_1 direction for $x_1 < \lambda_0$. Let $G^\lambda := \{x \in G; x_1 < \lambda\}$ and suppose $\tilde{G}^\lambda \in G$ for $\lambda < 0$ (where ~ (tilde) means reflection in $\{x_1 = \lambda\}$). Then for $\lambda \in (-\lambda_0, 0)$*

$$u^\lambda(x) > u(x), \qquad \forall x \in \Sigma_\lambda := \{x \in \mathit{supp}\ u; x_1 < \lambda\},$$

where $u^\lambda(x) = u(x^\lambda)$ and x^λ is the reflection of x in $\{x_1 = \lambda\}$.

A similar result in a different problem has been obtained by [GS] and the proof of this theorem is much the same as that in [GS] and therefore we only give a sketch of the proof.

PROOF: (Sketch) By the hypothesis it is easy to see that

$$w^\lambda(x) = \begin{cases} \max(u(x), u^\lambda(x)) & x_1 > \lambda \\ \min(u(x), u^\lambda(x)) & x_1 \leq \lambda \end{cases}$$

is admissible and minimizes the functional locally, provided $0 < \lambda_0 + \lambda$ is small enough. This implies, (see [K; Theorem 2.29]) that $w = u$. Hence by the maximum principle applied to $u - u^\lambda$ in Σ_λ we must have $u - u^\lambda < 0$ there. This proves the theorem for small $0 < \lambda_0 + \lambda$. To complete we take the largest interval $(-\lambda_0, \lambda_1)$ such that $u^\lambda > u$ for $\lambda \in (-\lambda_0, \lambda_1)$. If $\lambda_1 < 0$ then we proceed as in the proof of Theorem 1.1, by taking a compact sets $K \subset \Sigma_{\lambda_1}$ with $\tilde{K} \subset \tilde{\Sigma}_{\lambda_1}$ which absorbs the most part of the volume and such that $u < u^{\lambda_1}$ on K and $u > u^{\lambda_1}$ on $\tilde{K}$. This will imply that $w^{\lambda_1+\epsilon}$ will be a local variation of u provided ϵ is small enough; and the above argument works again. However we must require that $\{x_1 = \lambda_1 + \epsilon\}$ doesn't meet $\tilde{K}$, which is fulfilled if ϵ is arbitrary small. This Contradicts the maximality of $(-\lambda_0, \lambda_1)$ and the theorem is proved.

A combination of the methods in Theorems 2.5 and 2.3 gives the following.

THEOREM 2.6. *Let G, u and g be as above, where we in addition assume that g is strictly decreasing in the x_1 direction for $x_1 < \lambda_0$ and closure$(\tilde{G}_\lambda) \subset G$ for $\lambda < 0$. Then $\partial\{u > 0\} \cap \{x_1 < 0\}$ is Lipschitz.*

REMARK: The method of this section works for much more general functionals $\int F(x, u, Du)$ where certain kind of monotonicity assumptions are imposed on F.

REFERENCES

[AC] H.W. Alt, L.A. Caffarelli, *Existence and regularity for a minimum problem with free boundary*, J. Reine Angew. Math. **325** (1981), 434–448.

[BN] H. Berestycki, L. Nirenberg, *On the method of moving planes and the sliding method*, Bol. da Soc. Brasileira de Mathematica **22** (1991), 1–37.

[BP] H. Berestycki, F. Pacella, *Symmetry properties in semilinear elliptic equations with mixed boundary conditions*, J. Func. Anal. **87** (1989), 177–211.

[Dan] E. N. Dancer, *Some notes on the method of moving planes*, Bull. Austral. Math. Soc. **46** (1992), 425–434.

[D] I, Diaz, "Nonlinear partial differential equations and free boundaries vol I (Elliptic equations)," Research Notes Math., Pitman, Boston, 1985.

[GNN] G. Gidas, Wei-Ming Ni and L. Nirenberg, *Symmetry and related properties via the maximum principle*, Commun. Math. Phys. **68** (1979), 209–243.

[GS] B. Gustafsson, H. Shahgholian, *In preparation.*

[K] B. Kawohl, "On rearrangements and Convexity of Level Sets in PDE," Lecture Notes in Math. (1150), Springer Verlag, 1985.

[LU] O.A. Ladyzhenskaya, N.N. Ural'tseva, "Linear and Quasilinear Elliptic Equations," Academic Press, New York, 1968.

[SE] J. Serrin, *A symmetry problem in potential theory*, Arch. Rat. Mech. Anal. **43** (1971), 304–318.

Keywords. elliptic pde and fbp, maximum principle, monotonicity, symmetry, star-shapedness, uniqueness
1980 *Mathematics subject classifications*: 35J60