A FIRST GLIMPSE AT THE MINIMAL MODEL PROGRAM

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CONTENTS

This article is an account of the afternoon sessions at Snowbird that were devoted to an introduction to the Minimal Model Program and led by Sándor Kovács. The sections were written by the participants based on the discussion there and then kneaded together into one article by the fifth named author. Despite trying to make the presentation cohesive, the reader may detect a different style of writing in each section. We hope that this will only entertain rather than annoy the reader.

Most participants had no prior knowledge of the Minimal Model Program other than a general (strong) background in algebraic geometry. Readers with similar preparation form our targeted audience. As the title suggests this article is only a glimpse at this beautiful and important area. If we succeed in raising the reader's interest, we suggest further investigation. A natural next step could be Kollár's introductory, but much more detailed article [**Kol87**]. We also recommend consulting other survey articles, such as [**Rei87**] and [**Mor87**], or the technically more demanding [**KMM87**] and [**Kol91**]. For

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the serious reader who wants to learn the details we recommend [**KM98**], [**Deb01**], and [**Mat02**].

For simplicity, unless otherwise stated, we will work over an algebraically closed field of characteristic zero.

DISCLAIMER. The modern approach to the theory of minimal models was initiated by Reid and Mori in the early 1980s. In the following decade it has grown into a whole new area of algebraic geometry, called the Minimal Model Program, or Mori Theory. The main architects of the Minimal Model Program were Benveniste, Kawamata, Kollár, Mori, Reid, Shokurov, Tsunoda, and Viehweg. Many others made important contributions, but in the spirit of this article being only a "glimpse" we will not make an attempt to sort out and give proper credit for all the contributors in the text. The interested reader can find this information in the aforementioned references. We hope that everyone whose name is omitted will graciously forgive us.

ACKNOWLEDGEMENT. Our basic references at the workshop as well as during the writing of this article have been [**KM98**] and [**Deb01**]. In particular, we have learnt from and made use of several examples included in [**Deb01**]. We would like to thank Olivier Debarre for allowing us to include these examples. We would also like to thank Karl Schwede and the referee for useful suggestions.

1. Minimal models of surfaces

We start by discussing minimal models of surfaces. This material predates the theory that today we call the 'Minimal Model Program' or 'Mori Theory'. Incidentally, a minimal surface according to the classical definition is not necessarily a minimal model according to the current usage. For traditional reasons those minimal surfaces are still called minimal. Therefore when speaking about surfaces one has to be careful about the meaning of *minimal*. Once we have both definitions available, we will point out the difference between them.

In this section we introduce the minimal and the canonical models of surfaces. We will first do that following the traditional approach and then analyze the definition to make it possible to generalize to higher dimensions. In particular, we explore the connections between the minimality of a surface and the *nef*-ness of its canonical bundle.

NOTATION 1.1. Throughout this section S will denote a smooth, projective surface and C will denote a reduced, irreducible curve.

Definition 1.2. A smooth rational curve C on a smooth projective surface S with self-intersection $C^2 = -1$ is called an *exceptional curve of the first kind* or a (-1)*-curve*.

A fundamental theorem of Castelnuovo asserts that (-1)-curves can be blown down. Moreover, when a (-1)-curve is blown down the resulting surface is still smooth. In other words the only way to produce a (-1)-curve is to blow up a smooth point on a surface.

Theorem 1.3 (Castelnuovo). *If C is a (-1)-curve on a smooth surface $\tilde{S}$, then there exists a morphism $f : \tilde{S} \to S$, where S is a smooth surface, $\tilde{S}$ is the blow-up of S at a point and the exceptional divisor of the blow-up is C.*

PROOF. [**Har77**, Theorem V.5.7] □

Castelnuovo's theorem motivates a naive definition of a minimal surface. Namely, a *minimal surface* is a surface which does not contain any (-1)-curves.

Proposition 1.4. *Every surface is birational to a minimal surface.*

PROOF. If the surface contains a (-1)-curve, we blow it down using Castelnuovo's theorem. The Picard number of a surface is a non-negative integer. Each time we blow down a (-1)-curve we decrease the Picard number by one. Therefore, this process must terminate to produce a minimal surface. □

REMARK 1.5. Observe that a surface may contain infinitely many (-1)-curves. Blow up the transverse intersection points of two general smooth cubics in $\mathbb{P}^2$. The automorphism group of this surface contains a copy of $\mathbb{Z}^8$ generated by translations by the differences of the nine sections of the elliptic fibration arising from the base points of the pencil of cubic curves. Taking the orbit of a (-1)-curve by the automorphism group we see that there are infinitely many (-1)-curves.

REMARK 1.6. Observe that the minimal model does not have to be unique for rational or ruled surfaces. For example, if we start with the blow-up of $\mathbb{P}^2$ at two points, we can either blow down the two exceptional divisors to get the minimal surface $\mathbb{P}^2$ or blow down the proper transform of the line joining the two points and get the minimal surface $\mathbb{P}^1 \times \mathbb{P}^1$. However, the minimal model is unique for non-rational and non-birationally ruled surfaces.

Our definition of a minimal surface is inconvenient for generalizing to higher dimensions. The definition of a (-1)-curve uses the fact that the ambient variety is a surface. We therefore reformulate our definition in terms of the canonical bundle K_S of the surface S. The new definition will not be equivalent to the previous one in the case of rational and ruled surfaces.

Definition 1.7. We will call a smooth projective surface S *minimal* if K_S is *nef*, i.e., its intersection with every effective curve is non-negative.

REMARK 1.8. When we define the notion of a minimal model for higher dimensional varieties. We will allow certain mild singularities, called "terminal". Surfaces with terminal singularities are smooth, so in the case of surfaces this definition is equivalent to the general one.

REMARK 1.9. If a surface contains a (-1)-curve C, then $K_S \cdot C = -1$ by the adjunction formula. Therefore, K_S is not nef. Hence, a surface S that is minimal according to our new definition is also minimal according to the old one. We also have the converse for surfaces which are not rational or ruled.

Proposition 1.10. *If a smooth projective surface contains no (-1)-curves and it is not rational or ruled by rational curves, then its canonical bundle is nef.*

PROOF. Suppose S is a surface that satisfies the hypotheses of the theorem, but K_S is not nef. Then there exists a curve B for which $K_S \cdot B < 0$. By (2.3) there exists a rational curve C also satisfying $K_S \cdot C < 0$. Since the surface does not contain any (-1)-curves, we conclude that $K_S \cdot C < -1$, or, by the adjunction formula, $C^2 \geq 0$. By deformation theory, C admits non-trivial deformations. Hence the surface has to be rational or ruled. □

In order to place our discussion in better context we need to introduce a closely related concept, namely, that of the canonical model. Classically, the classification of surfaces was carried out by studying their Kodaira dimensions.

Definition 1.11. The *Kodaira dimension* of a variety is the growth rate of the dimension of sections of the pluricanonical series $h^0(X, \mathcal{O}_X(mK_X))$ as m tends to infinity. Alternatively, the Kodaira dimension is equal to the dimension of the image of the rational map one obtains from a sufficiently high and divisible multiple of the canonical bundle. Finally, it can also be reformulated as one less than the transcendence degree of the canoncial ring $\oplus_{m\geq 0} H^0(X, \mathcal{O}_X(mK_X))$.

REMARK 1.12. If the canonical ring $\oplus_{m\geq 0} H^0(X, \mathcal{O}(mK_X))$ is finitely generated, then we get a canonically defined map $X \to \mathbb{P}(H^0(X, \mathcal{O}(lK_X)))$ for $l \gg 0$. The image is isomorphic to $\mathrm{Proj}(\oplus_{m\geq 0} H^0(X, \mathcal{O}(mK_X)))$, and it is called the *canonical model* of X.

REMARK 1.13. Depending on what one chooses to call the dimension of the empty set, one may end up with different values in the case when $H^0(X, \mathcal{O}(mK_X)) = 0$ for all $m > 0$. In fact, both $\kappa(X) = -1$ and $\kappa(X) = -\infty$ are used in the literature. Depending on the situation, one or the other makes formulas more convenient, so there is no clear better choice. On the bright side, this is the only case when $\kappa(X) < 0$, so the statement "negative Kodaira dimension" is unambiguous.

An important conjecture asserts that the canonical ring of a smooth variety is always finitely generated. The conjecture is known to be true for surfaces and threefolds. It is also known when the canonical bundle is ample (or more generally when it is big and nef). Note that finite generation, if true, would be a special property of the canonical bundle. Nagata has constructed examples where the ring $\oplus_{m\geq 0} H^0(X, \mathcal{O}_X(mL))$ is not finitely generated for suitable line bundles L on X.

A conjecture closely related to the finite generation of the canonical ring is the abundance conjecture. The abundance conjecture asserts that if X is a minimal variety (i.e. K_X is nef and X has terminal singularities), then mK_X is basepoint-free for a large and divisible enough m. In that case the m-th pluricanonical map is actually a morphism and its image is the canonical model. It is easy to see that in this case the canonical ring $\oplus_{m\geq 0} H^0(X, \mathcal{O}(mK_X))$ is finitely generated.

Going back to the case of surfaces we note that even if S is a surface of general type, the canonical model and the minimal model do not have to coincide. For example, take a quintic surface in $\mathbb{P}^3$ which is smooth except for a single ordinary double point. If we blow up the singular point, we obtain a smooth surface with nef canonical bundle. This blow-up is a minimal model. However, the canonical model is the singular surface in $\mathbb{P}^3$. More generally, the canonical models of surfaces of general type do not have to be smooth. They can have rational double points, also known as Du Val singularities.

We conclude this section by reformulating the theorem guaranteeing the existence of minimal models of surfaces.

Theorem 1.14. *Let S be a smooth projective surface. Then there exists a morphism, $f : S \to S'$, which is a composition of blowing down (-1)-curves and a morphism $g : S' \to Z$ such that one of the following holds:*

(1) *$S' \simeq Z$ is a smooth surface with K_Z nef;*
(2) *Z is a smooth curve and S' is a minimal ruled surface over Z; or*
(3) *Z is a point and S' is isomorphic to $\mathbb{P}^2$.*

After repeating the process finitely many times we either end up with a surface with nef canonical bundle or with cases 2 or 3. In the first case we have to study the variety by other means. In the latter two cases we have reduced the study of the variety to smaller dimensional varieties.

This theorem sums up the goal of the minimal model program. Given a variety X, the task is to find extremal K_X negative curves which can be contracted. The hope is that by repeating the procedure one ends up either with a variety where K_X is nef (a minimal model) or with a variety of smaller dimension. The rest of the notes will be a discussion of how far the program can be carried out for higher dimensional varieties and the various difficulties one encounters along the way.

2. Bend and Break

The main goal of this section is to show that if K_X is not nef, then there exists a K_X-negative rational curve on X. In particular, we are going to discuss one of the central techniques of the theory, Mori's 'Bend & Break' trick.

Start with a smooth projective variety X. The ultimate goal is to find a minimal model for X, that is a birational model X_{min} such that $K_{X_{\text{min}}}$ is nef.

If K_X is nef, we are done, so we may assume that K_X is not nef. We would like to obtain a statement similar to (1.3). In other words we would like to identify simple obstructions that prevent K_X from being nef. In particular, we would like to get rid of those curves on which K_X is negative. As in (1.3), we want to contract such curves. In order to do that we want to identify the simplest K_X-negative curves. They should be rational curves and their degree should be bounded by a constant only depending on the dimension of X.

If K_X is not nef, then there exists a curve $C \subset X$ such that $-K_X \cdot C > 0$. The idea of 'Bend & Break' is that we deform ("bend") this curve inside X until it has to degenerate to a reducible or non-reduced curve, that is, it "breaks" up. This way we produce rational curves or if we start with rational curves, then we produce rational curves of lower degree.

To produce the desired deformation of our curve, it is more advantageous to deform the morphism from its normalization to X. Consider, $f : C \to X$, where C is a smooth projective curve. First, we want to deform the morphism f without changing C or X, so the appropriate parameter space to consider is the open subscheme $\operatorname{Hom}(C, X)$ of the Hilbert scheme $\operatorname{Hilb}(C \times X)$ near $[f]$. The local dimension of this space is at least ([**Kol96**, II.1.2]):

$$\deg f^* T_X + (1 - g) \dim X.$$

However, we actually want to deform the curve keeping a point fixed. Fixing a point imposes $\dim X$ conditions, so we obtain that the curve can be deformed with fixing a point if

$$\deg f^* T_X - g \dim X > 0.$$

Hence we are in business, if we manage to make $\deg f^* T_X = -K_X \cdot_f C$ big enough.

EXAMPLE 2.1. If C is an elliptic curve, we may compose f with an isogeny $\iota : C \to C$.

$$f \circ \iota : C \xrightarrow{\iota} C \xrightarrow{f} X$$

It is easy to see that

$$-K_X \cdot_{f \circ \iota} C = \deg \iota \cdot (-K_X \cdot_f C),$$

so if we choose ι with sufficiently high degree, $-K_X \cdot_{f \circ \iota} C$ will be large enough.

Unfortunately this only works as long as the curve admits a high degree endomorphism, so for curves with genus $g > 1$, we need a different method.

This is done by the 'reduction mod p' technique. Let's assume that everything that we encounter is defined over $\mathbb{Z}$. In other words, we consider the problem over $\operatorname{Spec} \mathbb{Z}$. Then if we reduce at the prime $(0) \in \operatorname{Spec} \mathbb{Z}$, we get our original problem, while reducing at the prime $(p) \in \operatorname{Spec} \mathbb{Z}$ we get the equivalent problem over $\bar{\mathbb{F}}_p$. Over $\bar{\mathbb{F}}_p$, applying the Frobenius map allows us to make $-K_X \cdot_f C$ as large as desired, so the deformation space will be big enough. We will soon see that then one can produce rational curves of bounded degree. On the other hand it is relatively easy to see that the existence of rational curves of bounded degree for infinitely many primes implies the existence of such a curve in characteristic 0.

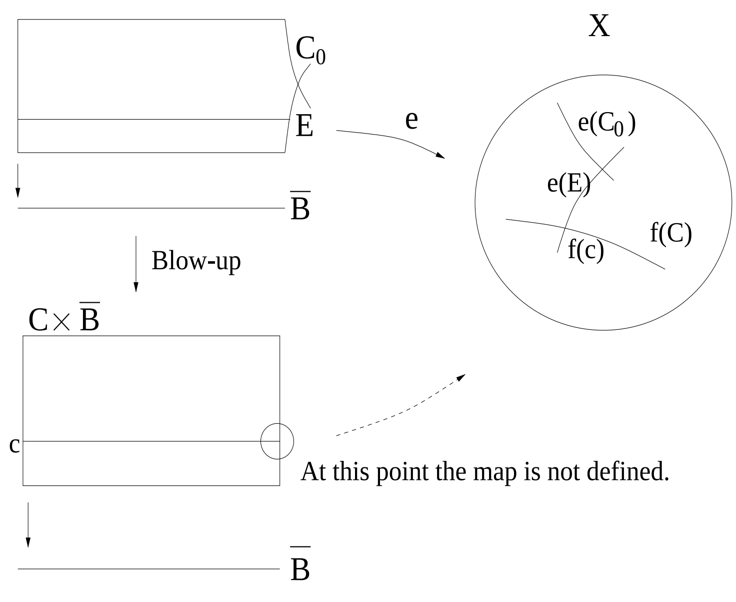
We should remark that to do this in the generality needed one has to deal with the case when not everything is defined over $\operatorname{Spec}\mathbb{Z}$. In that case one ends up with a family over $\operatorname{Spec} A$ for some finitely generated $\mathbb{Z}$-algebra A and has to reduce at a prime $\mathfrak{p} \in \operatorname{Spec} A$ and work with the problem over $\overline{k(\mathfrak{p})}$ instead of over $\overline{\mathbb{F}}_p$. For a more detailed discussion of this step see [**KM98**, 1.10].

In the following we will make this a little bit more precise. The first ingredient is the Rigidity Lemma.

Lemma 2.2 (Rigidity Lemma). [**KM98**, 1.6] *Let Y be an irreducible variety and let $h : Y \to Z$ be a proper surjective morphism. Assume that every fiber of h is connected and of dimension n. Let $g : Y \to W$ be a morphism such that $g(h^{-1}(z_0))$ is a single point for some $z_0 \in Z$. Then $g(h^{-1}(z))$ is a single point for all $z \in Z$.*

Proposition 2.3 (Bend & Break I). [**KM98**, 1.7, 1.13] *Let X be a smooth projective variety, $f : C \to X$ a morphism from a smooth projective curve, $c \in C$ an arbitrary closed point. If $-K_X \cdot_f C > 0$, then there exists a rational curve C' on X through $f(c)$ such that $-K_X \cdot_f C' > 0$.*

SKETCH OF PROOF. We may assume that C is not a rational curve. Let B be the normalization of a 1-dimensional subvariety of $\operatorname{Hom}(C, X; f|_c)$ passing through $[f]$ and let $\overline{B}$ be a smooth compactification of B. One uses the above sketched 'reduction mod p' technique to ensure that such a B exists.



Applying the rigidity lemma (2.2) with $Y = C \times \overline{B}$, $Z = C$, and $W = X$, one concludes that the induced map is not well-defined on the whole of $C \times \overline{B}$. Therefore one needs to blow-up to resolve the indeterminacies of the map. The image of (at least) one of the exceptional curves produces a K_X-negative rational curve passing through $f(c)$. $\square$

Proposition 2.4 (Bend & Break II). [**KM98**, 1.9] *Let X be a projective variety and let $f : \mathbb{P}^1 \to X$ be a morphism from a smooth rational curve. Finally let $p, q \in \mathbb{P}^1$ be two* *closed points. Assume that f can be deformed in a 1-parameter family, leaving X, and the images of p, q fixed. More precisely, assume that there exists a smooth connected pointed curve $(B, 0_B \in B)$ and a morphism $F : \mathbb{P}^1 \times B \to X$ such that*

- $F|_{\mathbb{P}^1 \times \{0_B\}} = f$,
- $F(\{p\} \times B) = f(p)$, $F(\{q\} \times B) = f(q)$, *and*
- $F(\mathbb{P}^1 \times B)$ *is a surface.*

Then the 1-cycle $f_*\mathbb{P}^1$ is numerically equivalent to a reducible or non-reduced chain of rational curves passing through the two fixed points, $f(p)$ and $f(q)$.

REMARK 2.5. Any degeneration of a rational curve is a rational tree.

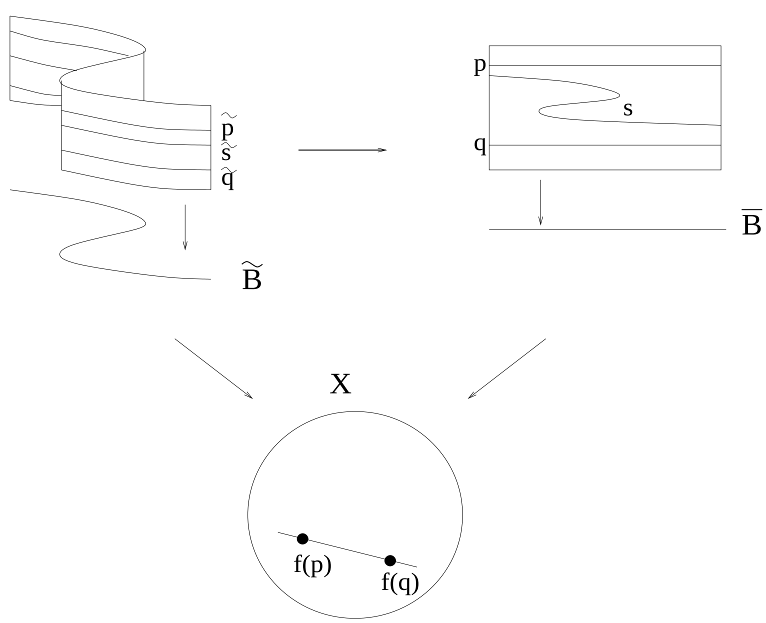
SKETCH OF PROOF. Let $\bar{B}$ be a smooth projective closure of B and $Y = \mathbb{P}^1 \times \bar{B}$. Let $\pi : Y \to B$ denote the projection to the second factor. F contracts $\{p\} \times \bar{B}$ to a point, but its image is two dimensional, so it cannot contract $\{p'\} \times \bar{B}$ for all $p' \in \mathbb{P}^1$. Therefore, by the rigidity lemma (2.2) F cannot be extended to the entire Y. Let

$$\Xi = \{b \in \bar{B} \mid F \text{ is not defined at some point of } Y_b = \pi^{-1}(b)\}.$$

Hence, in order to extend F over $\bar{B}$, one has to pass to a blow-up of Y, denoted by $\bar{Y}$. Let $\bar{F} : \bar{Y} \to X$ denote the extension and $\bar{\pi} : \bar{Y} \to \bar{B}$ the composition of π with the blow-ups. Note that the exceptional curves created by the blow ups lie over the points of Ξ and that at least one of them will not be contracted by $\bar{F}$ as otherwise F would extend to Y.

First assume that there is a $b \in \Xi$ such that at least two components of $\bar{Y}_b = \bar{\pi}^{-1}(b)$ are not contracted to a point by $\bar{F}$. Then the images of these components give the required chain of rational curves.

Next assume that all but one curve in each fiber of $\bar{\pi}$ is contracted by $\bar{F}$. For $b \in \Xi$, let this curve be denoted by $E_b \subset \bar{Y}_b$. Take a multisection s of $\bar{\pi}$ by pulling back a divisor from X that misses $f(p)$, $f(q)$ and $f(\bar{Y}_b \setminus E_b)$ for all $b \in \Xi$ and take a finite base change $g : \widetilde{B} \to \bar{B}$ so that the pull-back $\tilde{s}$ of s is an actual section over $\widetilde{B}$. Let



$\tilde{\pi} : \widetilde{Y} = \widetilde{B} \times_{\bar{B}} \bar{Y} \to \widetilde{B}$, $\widetilde{\Xi} = g^{-1}(\Xi)$, $\widetilde{Y}_b = \tilde{\pi}^{-1}(b)$ and for $b \in \widetilde{\Xi}$, $\widetilde{E}_b$ the intersection of $\widetilde{Y}_b$ and the preimage of $E_{g(b)}$. Further let the proper transforms of $\{p\} \times B$ (resp. $\{q\} \times B$) on $\widetilde{Y}$ be denoted by $\tilde{p}$ (resp. $\tilde{q}$). Finally let $\widetilde{F} : \widetilde{Y} \to X$ denote the obvious composition.

Observe that $\tilde{s}$, $\tilde{p}$, and $\tilde{q}$ form a set of three pairwise disjoint sections such that for $b \in \widetilde{\Xi}$ they all intersect $\widetilde{Y}_b$ in $\widetilde{E}_b$ and $\tilde{s}$ does not intersect $\widetilde{Y}_b \setminus \widetilde{E}_b$.

The sections $\tilde{s}$, $\tilde{p}$, and $\tilde{q}$ define a surjective morphism $\widetilde{Y} \to \hat{Y} = \mathbb{P}^1 \times \widetilde{B}$ such that $\widetilde{E}_b$ maps isomorphically onto $\mathbb{P}^1 \times \{b\}$ and $\widetilde{Y}_b \setminus \widetilde{E}_b$ is contracted to a set of points (this follows from the fact that $\tilde{s}$ avoids these curves). It follows that $\widetilde{F}$ factors through $\hat{Y}$. However, we have already seen above that this is impossible, so we are done. $\square$

3. The cone of curves

Definition 3.1. Let X be a proper scheme, and let $N_1(X)_{\mathbb{Z}}$ denote the group of 1–cycles modulo numerical equivalence (two 1–cycles C and C' are numerically equivalent if they have the same intersection number with every Cartier divisor). Also set

$$N_1(X)_{\mathbb{Q}} = N_1(X)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q} \qquad N_1(X)_{\mathbb{R}} = N_1(X)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}$$

Let

$$NE(X) = \left\{ \sum a_i [C_i] \mid C_i \subseteq X \text{ proper curve}, a_i \in \mathbb{R}_{>0} \right\} \subset N_1(X)_{\mathbb{R}}$$

and let $\overline{NE}(X)$ denote the closure of $NE(X)$ in the Euclidean topology. $NE(X)$ is often called the *effective cone* or the *cone of curves*.

We have our first result relating the geometry of $NE(X)$ to the ampleness of line bundles.

Theorem 3.2 (Kleiman's Criterion). *Let X be a projective variety and D a Cartier divisor on X. Then*

$$D \text{ is ample} \iff D \cdot \sigma > 0, \forall\, \sigma \in \overline{NE}(X), \sigma \neq 0.$$

In other words, D is ample if and only if D is positive on $\overline{NE}(X) \backslash \{0\}$.

REMARK 3.3. It is not enough that D has positive intersection with every effective curve. There are examples of Cartier divisors, due to Mumford, having this property which still fail to be ample cf. §8.1. In order to be ample D must remain positive on all "limits" of curves.

The next result describes the geometry of the cone $\overline{NE}(X)$. This is the Cone Theorem for smooth varieties cf. (4.9).

Theorem 3.4 (Cone Theorem). [**Mor82**] *Let X be a smooth projective variety. Then there exists a countable set of rational curves $\{\Gamma_i\}$, $\Gamma_i \subset X$ such that*

$$0 < -K_X \cdot \Gamma_i \leq \dim(X) + 1$$

and

$$\overline{NE}(X) = \overline{NE}(X)_{K_X \geq 0} + \sum \mathbb{R}_{\geq 0}[\Gamma_i].$$

The rays $\mathbb{R}_{\geq 0}[\Gamma_i]$ are locally discrete in the half-space $(K_X < 0)$.

SKETCH OF PROOF. Mori invented his famous 'Bend & Break' trick (2.3), (2.4) in order to prove this theorem. The idea of the proof is very simple. Given any curve $C \subset X$ with $K_X \cdot C < 0$, using 'Bend & Break' one finds that the class of C is a linear combination of rational curves with the same property and possibly other curves on which K_X is non-negative. This is basically the content of the Cone Theorem. For details see [**KM98**, 1.24] $\square$

REMARK 3.5. See [**Deb01**, p. 144] for a picture of the typical geometry of $\overline{NE}(X)$. The theorem says that $\overline{NE}(X)$ is divided into two halves by the sign of K_X. It makes no specific statement about the K_X–positive half. It does say that the K_X–negative portion is generated by countably many *extremal rays*, and that moreover these can only accumulate on the hyperplane $K_X = 0$. In particular, $\overline{NE}(X)$ need not be polyhedral. For example, consider the case of $\mathbb{P}^2$ blown up at the nine points of intersection of two cubics. On the other hand, if X is Fano, i.e., $-K_X$ is ample, then $\overline{NE}(X)$ is always polyhedral cf. (4.10).

3.6. $\overline{NE}(X)$ **in the case of surfaces.** In this section, we examine $\overline{NE}(X)$ in the case where X is a smooth projective surface. This is a very special case, since a curve is also a divisor. $N_1(X)_{\mathbb{R}}$ carries a quadratic form defined by the intersection pairing. Fix an ample divisor H, and let

$$Q(X) = \{\sigma \in N_1(X) | \sigma^2 \geq 0,\ \sigma \cdot H \geq 0\}$$

This is also a convex cone, and one may ask how it is related to $\overline{NE}(X)$. It follows from the next exercise that

$$Q(X) \subset \overline{NE}(X)$$

EXERCISE 3.6.1. Let D be a divisor on X such that $D^2 > 0$. Prove that either $|mD|$ or $|-mD|$ is non-empty for $m \gg 0$.

SOLUTION 3.6.2. Using Riemann-Roch, we obtain:

$$\begin{aligned} h^0(X, \mathscr{O}_X(mD)) + h^2(X, \mathscr{O}_X(mD)) &\geq \chi(X, \mathscr{O}_X(mD)) \\ &= \frac{1}{2} mD \cdot (mD - K_X) + \chi(\mathscr{O}_X) \\ &= \frac{1}{2} m^2 D^2 + o(m^2) \end{aligned}$$

and similarly with $\mathscr{O}_X(-mD)$. Because $D^2 > 0$, the right-hand side goes to $+\infty$ as m does. It follows that

$$\lim_{m\to\infty} \big(h^0(X, \mathscr{O}_X(mD)) + h^2(X, \mathscr{O}_X(mD))\big) = +\infty$$

$$\lim_{m\to\infty} \big(h^0(X, \mathscr{O}_X(-mD)) + h^2(X, \mathscr{O}_X(-mD))\big) = +\infty$$

Suppose now that $h^0(X, \mathscr{O}_X(mD)) = 0$ and $h^0(X, \mathscr{O}_X(-mD)) = 0$ for all m. This implies that

$$\lim_{m\to\infty} h^2(X, \mathscr{O}_X(mD)) = \lim_{m\to\infty} h^0(X, \mathscr{O}_X(K_X - mD)) = \infty$$

and the same with $-m$, which in turn implies that $h^0(X, \mathscr{O}_X(2K_X)) = \infty$, yielding a contradiction.

The following lemma allows us to check whether curves lie on the boundary of $\overline{NE}(X)$, and whether they span an extremal ray (the latter is very difficult to check in general).

Lemma 3.6.3. *Let X be a smooth projective surface. Then*

(3.6.3.1) *the class of an irreducible curve $C \subset X$ such that $C^2 \leq 0$ lies in $\partial\overline{NE}(X)$, and*
(3.6.3.2) *the class of an irreducible curve $C \subset X$ such that $C^2 < 0$ spans an extremal ray of $\overline{NE}(X)$.*

PROOF. (3.6.3.1) The statement follows easily from (3.6.3.2) if $C^2 < 0$, and so we may assume that $C^2 = 0$. Then, since C is irreducible, it follows that for every

$\sigma \in \overline{NE}(X)$, $C \cdot \sigma \geq 0$. Let H be an ample divisor on X and assume that $[C]$ is in the interior of $\overline{NE}(X)$. Then so is $[C] - t[H]$ for small $t > 0$. It follows, that

$$0 \leq C \cdot (C - tH) \leq -tC \cdot H.$$

However, this contradicts the fact that H is ample and so $C \cdot H \geq 0$.

For the proof of (3.6.3.2), see [**Deb01**, p. 145]. □

REMARK 3.6.4. (3.6.3.2) may be used to justify the assertion that the cone of curves is not polyhedral in the case where X is $\mathbb{P}^2$ blown up at the nine basepoints of a general pencil of cubics cf. [**Deb01**, 6.6].

EXERCISE 3.6.5. Consider the case where X is an abelian surface. In that case,

$$Q(X) = \overline{NE}(X)$$

SOLUTION 3.6.6. The inclusion $Q(X) \subset \overline{NE}(X)$ always holds and was treated earlier. For the reverse inclusion, suppose $C \in NE(X)$. Since every curve in an abelian surface can be moved using the group law, it will have non-negative intersection with every other effective curve. Therefore, $C^2 \geq 0$, and $C \cdot H \geq 0$ is automatic.

4. Introduction to the Minimal Model Program

Classically minimal surfaces were defined as those without a (-1)-curve. This is, however, a definition strictly restricted to surfaces. On the other hand, a non-uniruled surface, or in other words, a surface of non-negative Kodaira dimension is minimal if and only if K_X is nef cf. (1.10). For this reason we define *minimal varieties* by this property cf. (1.7), (5.4).

The Minimal Model Program starts with a variety X and is searching for a minimal variety which is birational to X, i.e., for the minimal model of X. The natural thing to do is to try to remove those curves $C \subset X$ for which $K_X \cdot C < 0$. In the surface case this is achieved by Castelnuovo's Theorem (1.3). The higher dimensional analogue is the Contraction Theorem (4.13). The purpose of this section is to sketch the major steps needed to prove this theorem. This is a significant part of the MMP, but we will only state the main theorems without proof.

One important aspect of the the Minimal Model Program is that one must allow the varieties to acquire singularities. (It's often worthwhile to consider the more general setup of singularities of pairs (cf. [**Kol97**]), but for the sake of simplicity here we will not do that).

The first step of the Minimal Model Program is to contract (some) K_X-negative curves. This is an analogue of contracting (-1)-curves on surfaces. In higher dimensions some of these lead to singular varieties. In order to run the MMP, we want our target variety to be in the same category as the one we start with. Hence we have to allow singularities. On the other hand, our techniques will not work on all singularities, so we need to find the class that's big enough so contractions do not lead out of it, but small enough so our methods will still work.

EXAMPLE 4.1. Let $\phi : X \to Y$ be a morphism between projective varieties that contracts an irreducible divisor $E \subseteq X$ to a point $P \in Y$ and that the restriction of ϕ gives an isomorphism $X \backslash E \cong Y \backslash P$. Assume that K_Y is a $\mathbb{Q}$-Cartier divisor. Then we can write

$$K_X \equiv \phi^* K_Y + aE$$

for some $a \in \mathbb{Q}$. Further assume that $-K_X|_E$ and $-E|_E$ are ample (we will see later that this is a commonly encountered situation). Then it follows that $-K_E$ is also ample and

$$-K_E \equiv -(\phi^* K_Y + (a+1)E)|_E \equiv -(a+1)E|_E,$$

and this is only possible if $a > -1$.

These kind of examples motivate the following definitions of types of singularities that are allowed in the Minimal Model Program.

Definition 4.2. [**KM98**, 2.34] Let X be a variety. Assume, that K_X is $\mathbb{Q}$-Cartier. Let $f : Y \to X$ be a resolution and E_i the irreducible components of the exceptional locus of f. Then there exists a unique collection $a_i \in \mathbb{Q}$ for $i = 1, 2, \dots, s$ such that

$$K_Y \equiv f^* K_X + \sum_{i=1}^{s} a_i E_i.$$

For any i, the a_i is called the *discrepancy* of E_i with respect to X. X is said to have

$$\left.\begin{array}{l} \textit{terminal} \\ \textit{canonical} \\ \textit{log terminal} \\ \textit{log canonical} \end{array}\right\} \text{ singularities, if } \left\{\begin{array}{l} a_i > 0 \\ a_i \geq 0 \\ a_i > -1 \\ a_i \geq -1 \end{array}\right\} \begin{array}{l} \text{for all } f : Y \to X \\ \text{and} \\ \text{for all } i. \end{array}$$

Note that instead of requiring the above conditions for all resolutions, it is enough if it holds for a *good resolution*, that is, one whose exceptional set is a simple normal crossing divisor.

These singularities are rather mild. Recall that a normal variety X has rational singularities if for one (hence every) resolution of singularities $\phi : \widetilde{X} \to X$, $R^i \phi_* \mathscr{O}_{\widetilde{X}} = 0$ for $i > 0$. Rational singularities are often the simplest singularities to deal with. For example, one can use vanishing theorems (e.g., (4.3)) for varieties with rational singularities.

Terminal, canonical, and log terminal singularities are rational cf. [**Elk81**], [**Kov00a**]. Log canonical singularities are not always rational. For instance, a cone over a smooth elliptic curve is log canonical, but not rational. However, it is expected that log canonical singularities are still rather "mild". Kollár conjectured [**Kol92**, 1.13] that log canonical singularities form the natural class for Kodaira type vanishing theorems. For more details on this conjecture and for results related to it see [**DB81**], [**Ste83**], [**Ish85**], [**Kov99**], [**Kov00b**].

Before stating the main theorems of the MMP let us point out that vanishing theorems play a very important role in the theory. However, a fair treatment would require more resources than that we have. The reader is referred to [**KM98**, §2.4-5] for details on the kind of generalizations of the Kodaira Vanishing Theorem that are needed. Here we only state a weaker version that is still very useful in numerous applications.

Theorem 4.3 (Kawamata-Viehweg Vanishing Theorem (weak form)). [**KM98**, 2.64] *Let X be a smooth projective variety and $\mathscr{L}$ a nef and big line bundle on X. Then $H^i(X, \mathscr{L}^{-1}) = 0$ for $i < \dim X$.*

REMARK 4.4. The original version of Kodaira's theorem required $\mathscr{L}$ to be ample and the stronger version allows for a fractional part as well.

Without further ado, let us now quote the promised list of theorems that make the MMP work.

Theorem 4.5 (Non-vanishing Theorem). [**KM98**, 3.4] *Let X be a proper variety with log terminal singularities, D a nef Cartier divisor such that $aD - K_X$ is $\mathbb{Q}$-Cartier, nef and big for some $a > 0$. Then*

$$H^0(X, mD) \neq 0 \quad \forall\, m \gg 0.$$

Theorem 4.6 (Basepoint-free Theorem). [**KM98**, 3.3] *Let X be a projective variety with log terminal singularities, let D be a nef Cartier divisor and suppose that $aD - K_X$ is nef and big for some $a > 0$. Then $|mD|$ is basepoint-free for every $m \gg 0$.*

Corollary 4.7. *Let X be a smooth projective variety. If K_X is nef and big, then $|mK_X|$ is free for every $m \gg 0$. In particular, the canonical ring*

$$R(X, K_X) = \bigoplus_{m \geq 0} H^0(X, \mathcal{O}(mK_X))$$

is finitely generated and hence the canonical model of X exists and is isomorphic to $\operatorname{Proj} R(X, K_X)$.

Theorem 4.8 (Rationality Theorem). *Let X be a proper variety with log terminal singularities such that K_X is not nef. Let $a(X)$ be an integer such that $a(X)K_X$ is Cartier. Let H be a nef and big Cartier divisor and define*

$$r = r(H) = \max\{t \in \mathbb{R} : H + tK_X \textit{ is nef}\}.$$

Then r is a rational number of the form u/v ($u, v \in \mathbb{Z}$) where

$$0 < v \leq a(X)(\dim X + 1).$$

Theorem 4.9 (Cone Theorem). [**KM98**, 3.7] *Let X be a projective variety with log terminal singularities. Then*

(4.9.1) *there exists a countable collection of rational curves $\{\Gamma_i\}$, $\Gamma_i \subset X$ such that*

$$0 < -K_X \cdot \Gamma_i \leq 2\dim(X)$$

and

$$\overline{NE}(X) = \overline{NE}(X)_{K_X \geq 0} + \sum \mathbb{R}_{\geq 0}[\Gamma_i].$$

(4.9.2) *The rays $\mathbb{R}_{\geq 0}[\Gamma_i]$ are locally discrete in the half-space $(K_X < 0)$.*

REMARK 4.0.1. Observe that in the Cone Theorem for smooth varieties (3.4), the statement is slightly stronger. Namely, the bound on the degrees of the generating rational curves in that statement is $\dim X + 1$ instead of the $2\dim X$ here.

The logical order of proof of these theorems is the following:

Non-vanishing $\Rightarrow$ Basepoint-free $\Rightarrow$ Rationality $\Rightarrow$ Cone

For the basic strategy see [**KM98**, 3.9] and for the entire proof see [**KM98**, Chapter 3].

Corollary 4.10. *Let X be a smooth Fano variety, i.e., such that $-K_X$ is ample. Then $\overline{NE}(X)$ is a (finite) polyhedral cone.*

PROOF. Because $-K_X$ is ample, $-K_X \cdot C > 0$ for every $C \in \overline{NE}(X)$, and hence $\overline{NE}(X)$ is contained in the half-space $(K_X < 0)$. The statement follows from (4.9.2). $\square$

REMARK 4.11. In general $\overline{NE}(X)$ is not a finite polyhedral cone already for surfaces. For instance, the cone of curves of a Kummer surface is never a finite polyhedral cone cf. [**Kov94**]. See (3.6.4) for another example.

EXERCISE 4.12. What conclusion can you draw from comparing (4.10) and (3.6.4)?

We finish this section with a sketch of the final steps in finding our desired contractions. Suppose that Γ is an extremal ray given to us by the Cone Theorem. We would like to find $\phi : X \to Y$ such that:

(1) ϕ contracts a curve if and only if it is numerically equivalent to some rational multiple of Γ, and
(2) Y is normal and ϕ has connected fibers.

Theorem 4.13 (Contraction Theorem). [**KM98**, 3.7] *Let X be a projective variety with log terminal singularities, and $F \subseteq \overline{NE}(X)$ a K_X-negative extremal face. Then there is a unique morphism $\mathrm{cont}_F : X \to Y$ to a projective variety such that $(\mathrm{cont}_F)_* \mathscr{O}_X \simeq \mathscr{O}_Y$ and an irreducible curve $C \subset X$ is mapped to a point by cont_F if and only if $[C] \in F$. This morphism, cont_F, is called the* contraction *of F.*

REMARK 4.0.1. The Contraction Theorem is often stated as part of the Cone Theorem [**KM98**, 3.7]. Here we wanted to emphasize it as the statement that we were looking for in this section.

SKETCH OF PROOF. Let F be a K_X-negative extremal face. Then we can find a divisor M such that $M \geq 0$ on $\overline{NE}(X)$ and

$$\{M = 0\} \cap \overline{NE}(X) = F.$$

We observe that we may choose M so that it is a Cartier divisor and that M is nef and $aM - K_X$ is ample for $a \gg 0$ by construction. In particular $aM - K_X$ is nef and big and hence by the Basepoint-free Theorem (4.6) $|bM|$ is basepoint-free for all b sufficiently large. This gives us a morphism $\mathrm{cont}_F : X \to Y$ which contracts F, with Y normal and cont_F having connected fibers. $\square$

5. Running the Minimal Model Program

We start this section with a definition.

Definition 5.1. A variety X of dimension n is called uniruled if there exist a variety Y of dimension $n-1$ and a dominant rational map $\mathbb{P}^1 \times Y \dashrightarrow X$. Equivalently, X is uniruled if for a general $x \in X$, there exists a rational curve through x.

REMARK 5.2. By [**MM86**], the above condition can be replaced by the following: for a general $x \in X$, there exists a curve $C \subset X$ such that $-K_X \cdot C > 0$.

Thus if X is uniruled, then $\kappa(X) = -1$. In fact, the two conditions are conjectured to be equivalent. Uniruled surfaces are exactly the ones with negative Kodaira dimension. In the context of the MMP it is clear why these cases behave differently with respect to finding a minimal model in the classical sense, i.e., by contracting (-1)-curves. According to the modern definition, uniruled varieties do not admit a minimal model, but we will see below that we still get a pretty good description of them as Fano fiber spaces.

Next we analyze what happens after we apply the Contraction Theorem.

5.3 CASES OF CONTRACTIONS. Let X be a smooth projective variety. If it is not minimal, then by the Cone Theorem (4.9) there is a K_X-negative extremal ray on $\overline{NE}(X)$ and by the Contraction Theorem (4.13) there is a morphism $\phi : X \to Y$ with connected fibers onto a normal projective variety Y. In order to successfully run the MMP, we need to repeat this procedure and hence make sure that Y also satisfies the conditions of the Contraction Theorem.

CASE I. $\dim Y < \dim X$. In this case, X is uniruled and ϕ is a *Fano fibration*. To see this, let F denote the class of a fiber of the map ϕ. Then $-K_X|_F$ is positive on every curve and $\rho(F) = 1$ because all curves on F are rationally proportional, so $-K_F = -K_X|_F$ is ample, implying that F is Fano. Understanding the geometry of X is now reduced to understanding lower dimensional varieties. Note that in this case X does not admit a minimal model.

CASE II. $\dim Y = \dim X$, and there is an exceptional divisor, whose image via ϕ has codimension at least two. In this case ϕ is called a *divisorial contraction*. In this case, Y may be mildly singular and so we have our first reason to enlarge our category of varieties that we run the Minimal Model Program on beyond the smooth ones. It turns out that if X has log terminal singularities, then so does Y, so this class of singularities allows us to run the Minimal Model Program by iterating the contraction of a K_X-negative extremal ray.

CASE III. $\dim Y = \dim X$, but the codimension of the exceptional variety of ϕ is at least two. In this case, ϕ is called a *small contraction*. Here we run into difficulties, because Y will turn out to be a pretty unpleasant variety. For one thing, K_Y will not be $\mathbb{Q}$-Cartier and this torpedos most of the steps required for running the Minimal Model Program. Therefore this case needs a new solution.

Because we need to allow singularities, we have to redefine the meaning of "minimal".

Definition 5.4. A variety X is said to have $\mathbb{Q}$-*factorial singularities* if every Weil divisor on X is $\mathbb{Q}$-Cartier.

A variety X is called a *minimal model* if it has terminal $\mathbb{Q}$-factorial singularities and K_X is nef.

If X is a smooth 3-fold, then it turns out that the morphism provided by the Contraction Theorem cannot be a small contraction. Still, small contractions might arise while running the Minimal Model Program on a smooth 3-fold. Namely, running the Minimal Model Program for the first time might produce a divisorial contraction, with the resulting variety Y singular. Then a second application of the Minimal Model Program might very well produce a small contraction, so we have to deal with this situation even if we are primarily interested in minimal models of smooth varieties.

Allowing mild singularities lets us incorporate divisorial contractions into the Minimal Model Program, but how are we to deal with small contractions? First, it is clear that if $\phi : X \to Y$ is a small contraction, then K_Y is not $\mathbb{Q}$-Cartier. (Indeed, if it were, then consider a curve C that is contracted by ϕ. On the one hand, we have $K_X \cdot C < 0$; on the other hand, since ϕ is small we must have $K_X \equiv \phi^* K_Y$, so $K_X \cdot C = K_Y \cdot \phi_*(C) = 0$.)

The solution to a small contraction is to replace X by some variety other than Y. The variety to use is the "flip" X^+ of X defined as follows.

Definition 5.5. [**KM98**, 2.8] Let $\phi : X \to Y$ be a small contraction such that $-K_X$ is $\mathbb{Q}$-Cartier and ϕ-ample. A variety X^+ together with a proper birational morphism $\phi^+ : X^+ \to Y$ is called a *flip* of ϕ if K_{X^+} is $\mathbb{Q}$-Cartier and ϕ^+-ample, and ϕ^+ is a small contraction. By slight abuse of terminology, the rational map $(\phi^+ \circ \phi^{-1}) : X \dashrightarrow X^+$ is also called a flip. A flip gives the following diagram:

$$\begin{tikzcd} X \arrow[rr, dashed] \arrow[dr, "\phi", "-K_X \text{ is } \phi\text{-ample}"'] & & X^+ \arrow[dl, "\phi^+"', "K_{X^+} \text{ is } \phi^+\text{-ample}"] \\ & Y & \end{tikzcd}$$

The variety X^+ in the definition of a flip is birational to the original variety X. While it is far from clear why a flip exists, it turns out that the flip of ϕ is unique if it exists. Indeed, if there is a flip, then

$$X^+ \simeq \operatorname{Proj} \bigoplus_{n \geq 0} \phi_* \mathscr{O}_X(nm_0 K_X)$$

where m_0 is a positive integer such that $m_0 K_X$ is Cartier.

Now the Minimal Model Program proceeds as follows:

5.6 **MMP.**

(1) Start with an arbitrary variety X.
(2) Replace X by a smooth projective model. (This is possible by Hironaka's resolution of singularities [**Hir64**]).
(3) If X is minimal, stop.
(4) If X is not minimal, use the Contraction Theorem (4.13) to produce a morphism $\phi : X \to Y$.
(5) If ϕ is a Fano fiber space, stop.
(6) If ϕ is a divisorial contraction, replace X by Y and return to (3).
(7) If ϕ is a small contraction, perform a flip, replace X by X^+ and return to (3).

In order for this to work one needs to ensure that the steps are indeed possible to take and that they produce varieties that are suitable for the next step. In particular, one needs to prove that flips exist and the flip of a variety with terminal $\mathbb{Q}$-factorial singularites also has terminal $\mathbb{Q}$-factorial singularites.

Once this is done, it follows that the above algorithm can always take the next step, but one also needs to make sure that the process terminates. Divisorial contractions lower the Picard number and flips do not change them. This means that there can be only finitely many divisorial contractions in any given run of the MMP. The only missing step at this point is to prove that that there cannot be an infinite sequence of flips.

All of these steps have been established for threefolds in the 1980s (for explicit references, see [**KM98**], or the bibliography at the end). The existence of flips for fourfolds has been recently proved by Shokurov [**Sho03**], but at the time of the writing of this article, termination of flips is only known in dimension three.

5.7. Flops. Recall that a minimal model is a projective variety X with terminal singularities such that K_X is nef. Although canonical models are unique, minimal models are not necessarily. It is, however, conjectured that any two minimal models in the same birational equivalence class are connected by a sequence of flops:

Definition 5.7.1. [**KM98**, 6.10] Let $\phi : X \to Y$ be a small contraction such that K_X is $\mathbb{Q}$-Cartier and numerically ϕ-trivial. Assume that D is a $\mathbb{Q}$-Cartier divisor on X such that $-D$ is ϕ-ample. A variety X° together with a proper birational morphism $\phi^\circ : X^\circ \to Y$ is called a *flop* of ϕ if K_{X° is $\mathbb{Q}$-Cartier and numerically ϕ°-trivial, D°, the proper transform of D, is ϕ°-ample and ϕ° is a small contraction. As in the case of flips, by slight abuse of terminology, the rational map $(\phi^\circ \circ \phi^{-1}) : X \dashrightarrow X^\circ$ is also called a flop. A flop gives the following diagram:

$$\begin{tikzcd} X \arrow[rr, dashed] \arrow[dr, "\phi"'] & & X^\circ \arrow[dl, "\phi^\circ"] \\ & Y & \end{tikzcd}$$

$-D$ is ϕ-ample, and
K_X is numerically ϕ-trivial

D° is ϕ°-ample, and
K_{X° is numerically ϕ°-trivial

REMARK 5.7.2. It is known in dimension 3, that a pair of birational minimal varieties are connected by a sequence of flops.

REMARK 5.7.3. Using the notation of (5.7.1), observe that K_X is nef if and only if K_{X° is.

Though minimal models are not unique in general, there is a case when they are. The reader should have no problem supplying a proof for this.

EXERCISE 5.7.4. If X has terminal singularities and K_X is ample, then X has a unique minimal model (itself).

6. Where next?

Once we reach a minimal model, we are happy, but we need more work before we can rest. In this article we will only briefly touch on this issue, and spend even less than a glimpse on it, but this should not give the impression that this part is not an equally important part of classification theory.

According to the Basepoint-free Theorem (4.6), if K_X is nef and big, some multiple of it is basepoint-free. This means that if X is of general type, then a minimal model provided by the MMP admits a canonical model and a birational morphism onto it. The canonical model admits pluricanonical embeddings giving rise to many ways of investigation.

One hopes that something similar can be done even if X is not of general type. This is summarized in the following.

Conjecture 6.1 (Abundance Conjecture). *Let X be a proper variety with log canonical singularities. Then*

(6.1.1) $\oplus_{m=0}^{\infty} H^0(X, \mathscr{O}_X(mK_X))$ *is a finitely generated ring.*
(6.1.2) *If K_X is nef, then $|mK_X|$ is basepoint-free for some $m > 0$.*

REMARK 6.2. It is relatively easy to see that if K_X is nef, then (6.1.2) implies (6.1.1). In general, the Minimal Model Program reduces (6.1.1) to (6.1.2), and as a consequence, frequently only (6.1.2) is called the Abundance Conjecture.

The conjecture is known to be true in dimensions two and three and as it was pointed out above the basepoint-free theorem (4.6) implies that it also holds if K_X is nef and big.

7. Minimal models of surfaces, revisited

In this section, we will examine how the Minimal Model Program handles the classification of surfaces. Eschewing logical difficulties, the goal is simply to see how the foundational theorems and conjectures of the MMP work in the case of surfaces; we make no claims that the "proofs" described here actually fit in a non-circular manner into algebraic geometry as it presently exists. As we have done all along, we restrict our attention to characteristic 0, although most of the results concerning the classification of surfaces remain valid (with a few more cases in characteristics 2 and 3).

Let X be a smooth projective surface. To run the MMP, we look at the canonical class K_X. If it is nef, then we stop and scratch our heads (the MMP cannot help us anymore). In a moment we will see that this case can be analyzed a bit for surfaces: in particular, one can classify surfaces of Kodaira dimension 0 and 1, while those of Kodaira dimension 2 (surfaces of general type) remain a mystery. (One can try to approach such surfaces using the canonical model, but we will not dwell on this here.) The goal of the MMP is to contract (numerical equivalence classes of) curves until we reach this case.

7.1. K_X **is not nef.** If the canonical class is not nef, then the cone of curves contains extremal rays. Let Γ be an integral rational generator for an extremal ray, and let the resulting extremal contraction be $\phi : X \to X'$. We find three cases:

(i) The map ϕ is birational. In this case, one can check that X' is a smooth blow-down of X along a (-1)-curve.
(ii) The map ϕ has 1-dimensional fibers. In this case, every fiber must be integral (since any curve in a fiber is numerically equivalent to a multiple of Γ) and rational, which implies (using Tsen's theorem) that ϕ exhibits X as a projective bundle over a smooth curve, i.e., X is a ruled surface.
(iii) The map ϕ is the constant map to a point. In this case, since every contracted curve is numerically equivalent to a multiple of Γ, we see that the Picard number of X is 1 and X is Fano. It follows that $X \cong \mathbb{P}^2$.

To summarize, when K_X is not nef, we can find a birational map $X \to \widetilde{X}$ which is a sequence of finitely many blow-downs along (-1)-curves such that either $K_{\widetilde{X}}$ is nef, or $\widetilde{X} \to B$ is a ruled surface, or $\widetilde{X} \cong \mathbb{P}^2$. This leaves us with the problem of studying what happens when the minimal model program cannot help us.

7.2. K_X **is nef.** We can split this up according to the Kodaira dimension. It is easy to see that the Kodaira dimension of cases (ii) and (iii) above is -1 (as no multiple of the canonical class ever acquires global sections). Conversely, we can invoke a classical theorem of Enriques.

Theorem 7.2.1. *A smooth complex surface X is birationally ruled if and only if $\mathrm{H}^0(X, \omega_X^{\otimes m}) = 0$ for all $m \leq 6$.*

Since any birationally ruled surface has Kodaira dimension -1, we see that the case of surfaces with $\kappa = -1$ is completely described by (ii) and (iii) above.

Corollary 7.2.2. *If K_X is nef then X is not birationally ruled.*

Thus, (i) above is disjoint from (ii) and (iii). Castelnuovo's criterion also tells us what happens when K_X is at the boundary of the nef cone:

Corollary 7.2.3. *If $K_X \equiv 0$, then $12K_X \sim 0$.*

(More precisely, one has $mK_X \sim 0$ for $m = 1, 2, 3, 4,$ or 6.) Thus, the rougher relation of being numerically equivalent to 0 implies that a small multiple of K_X is in fact linearly equivalent to 0. This tells us (among other things) that the Kodaira dimension of X is 0 if and only if $K_X \equiv 0$ if and only if K_X is torsion in $\mathrm{Pic}(X)$. The class of surfaces with $\kappa = 0$ includes the Abelian surfaces, $K3$ surfaces, certain $\mathbb{Z}_2$-quotients of these, the Enriques surfaces, and hyperelliptic surfaces.

If $K_X^2 > 0$, then one sees easily by a simple computation using the Riemann-Roch theorem that $\kappa(X) = 2$. Then by the Basepoint-free Theorem (4.6) it follows that X admits a birational morphism onto its canonical model. The investigation of canonical models will not be addressed in these notes.

The remaining case is when K_X is nef, $K_X^2 = 0$, but $K_X \not\equiv 0$. We claim that in this case $\kappa(X) = 1$ and X is an elliptic surface, i.e., it admits a fibration $\pi : X \to B$ such that almost all fibers of π are non-singular elliptic curves. Conversely, if $\kappa(X) = 1$, it is easy to verify that $K_X^2 = 0$. Since birationally ruled surfaces have Kodaira dimension 0, we see that any surface of Kodaira dimension 1 must be elliptically fibered. This will "complete" the classification of surfaces.

The Abundance Conjecture holds for surfaces (6.2) and it implies the existence of an elliptic fibration structure immediately: for some $m > 0$, the m-pluricanonical map provides a morphism $\chi : X \to B$.

Since by definition mK_X is a pullback along χ of a Cartier divisor on B, it follows (upon replacing B by its normalization if necessary) that in fact $mK_X = \chi^*(mK_B+L_m)$. Choosing m large enough guarantees that χ has connected fibers.

Since $K_X^2 = 0$, it is not big, χ cannot be generically finite. (Indeed, if it were, one could easily find two hyperplane sections of the image which intersected transversely and did not pass through the locus of positive-dimensional fibers, and this would cause a contradiction.) Since K_X is nef and $K_X \not\equiv 0$, the image of the canonical map cannot be a point. Hence B must be a curve.

That the generic fiber is smooth now follows from the fact that we are in characteristic 0, so the generic fiber of any dominant map from a smooth variety to any variety must be smooth. Furthermore, since mK_X is the pullback of a Cartoer divisor from B, the smooth fibers must be elliptic curves.

It is important to note that one usually proves the abundance conjecture for surfaces by showing that a surface of Kodaira dimension 1 admits an elliptic fibration structure. The point is to show that for some $m > 0$, there are sufficiently many members of the linear system mK_X (i.e., at least two). It immediately follows that any member of such a system has arithmetic genus one. Showing the existence of enough sections of $\omega_X^{\otimes m}$ directly is fairly complicated, and may be found in any of the standard references on algebraic surfaces.

8. Examples

8.1. Mumford's example of a non-ample divisor that's positive on every curve. In this section we give an example, due to Mumford, of a complete non-singular surface X, and a divisor D, such that $D \cdot Y > 0$ for all effective curves Y, but D is not ample. We'll follow the argument given by Hartshorne [**Har70**].

Let C be a non-singular complete curve, and $\mathscr{E}$ a bundle of rank 2 on C. Let X be the ruled surface $\mathbb{P}(\mathscr{E})$, $\pi : X \to C$ the associated projection, and D the divisor corresponding to the line bundle $\mathscr{O}_X(1)$. Finally let $\mathscr{S} = \oplus_{m\geq 0} \operatorname{Sym}^m(\mathscr{E})$. Note that then $\mathbb{P}(\mathscr{E}) \simeq \mathbf{Proj}\mathscr{S}$.

Recall the following correspondence between effective curves and subsheaves of the symmetric powers of $\mathscr{E}$.

Theorem 8.1.1. [**Har70**, Chapter I, Proposition 10.2] *For any positive integer m there exists a one-to-one correspondence between*

- *effective curves Y on X (possibly reducible with multiple components), having no fibers as components, of degree m over C, and*
- *sub-line bundles $\mathscr{M}$ of $\operatorname{Sym}^m(\mathscr{E})$.*

The correspondence is given by

$$Y \mapsto \pi_*(\mathscr{O}_X(m) \otimes \mathscr{O}_X(-Y))$$

and

$$\mathscr{M} \mapsto \textit{subschemes of } X \textit{ defined by the homogenous ideal } \mathscr{M}\cdot\mathscr{S}$$

Furthermore, under this correspondence

$$D \cdot Y = m(\deg \mathscr{E}) - \deg \mathscr{M}.$$

Recall that a vector bundle $\mathscr{E}$ on a curve is stable if for every sub-vector bundle

$$(\deg \mathscr{F})/(\operatorname{rank} \mathscr{F}) < (\deg \mathscr{E})/(\operatorname{rank} \mathscr{E}).$$

We then have the following:

Theorem 8.1.2. [**Har70**, Chapter I, Theorem 10.5] *Let C be a curve of genus $g \geq 2$ over the complex numbers. Then, for any $r > 0$, $d \in \mathbb{Z}$, there exists a stable bundle $\mathscr{E}$ of rank r and degree d, such that $\operatorname{Sym}^m(\mathscr{E})$ is stable for all $m > 0$.*

Now, let C be a curve of genus $g \geq 2$ over the complex numbers. Then there exists a stable bundle $\mathscr{E}$ of rank 2 and degree 0 such that $\operatorname{Sym}^m(\mathscr{E})$ is stable for all $m > 0$. As above, let $X = \mathbb{P}(\mathscr{E})$, D the divisor corresponding to $\mathscr{O}_X(1)$, and Y an effective curve on X. If Y is a fiber of π, then $D \cdot Y = 1$. If Y is an irreducible curve of degree m over C, then Y corresponds to a sub-line bundle $\mathscr{M} \subseteq \operatorname{Sym}^m(\mathscr{E})$. Now, $\operatorname{Sym}^m(\mathscr{E})$ is stable and of degree 0 (since $\deg \mathscr{E} = 0$), thus $\deg \mathscr{M} < 0$. Therefore,

$$D \cdot Y = m(\deg \mathscr{E}) - \deg \mathscr{M} = -\deg \mathscr{M} > 0.$$

Hence $D \cdot Y > 0$ for every effective curve Y on X.

However, D is not ample because $D^2 = 0$.

ACKNOWLEDGEMENT. The following examples are based on examples in [**Deb01**, §6.6]. We thank Olivier Debarre for allowing us to include them here.

8.2. The Contraction Theorem in action. (cf. [**Deb01**, 6.14]). The Contraction Theorem (4.13) guarantees the existence of an extremal contraction $\pi : X \to Y$, under which the entire numerical equivalence class of the generator of the extremal ray is contracted. We know that if the dimension of Y is lower than the dimension of X, then π is a Fano fibration over Y. This example deals with the simple case where X is a projective bundle over Y.

Let $\mathscr{E}$ be a rank $r + 1$ vector bundle over a smooth projective variety Y, and let $X = \mathbb{P}(\mathscr{E})$. Let ℓ be the class of a line in one of the fibers of π. It is clear that all such lines are numerically equivalent, and that a curve is contracted if and only if it is numerically equivalent to a multiple of ℓ. It follows that the ray generated by ℓ is extremal. We now verifiy that it is K_X–negative.

X comes with a line bundle, denoted by $\mathscr{O}_X(1)$ whose restriction to each projective fiber is $\mathscr{O}(1)$, and defined by the property that $\pi_*\mathscr{O}_X(1) = \mathscr{E}$. We have the following exact sequence for the relative cotangent bundle, $\Omega_{X/Y}$.

$$0 \to \Omega_{X/Y} \to \pi^*\mathscr{E} \otimes \mathscr{O}_X(-1) \to \mathscr{O}_X \to 0$$

from which we deduce that

$$\mathscr{O}_X(K_{X/Y}) \simeq \det(\Omega_{X/Y}) \simeq \pi^* \det(\mathscr{E}) \otimes \mathscr{O}_X(-r-1)$$

Combining this with $K_X = K_{X/Y} + \pi^* K_Y$ we obtain that

$$\mathscr{O}_X(K_X) \simeq \pi^*(\det(\mathscr{E}) \otimes \mathscr{O}_Y(K_Y)) \otimes \mathscr{O}_X(-r-1).$$

The intersection of ℓ with any class pulled back from Y is trivial, and $\deg\left(\mathscr{O}_X(1)|_\ell\right) = 1$. It follows that

$$K_X \cdot \ell = -(r+1) \tag{8.2.1}$$

8.3. A Fano fibration that is not a projective bundle, and a divisorial contraction that is not a smooth blow-up. (cf. [**Deb01**, 6.16]). In this subsection we follow [**Deb01**, 6.16] almost verbatim. Let C be a smooth projective curve of genus g. Let d be a positive integer, and $J^d(C)$ the Jacobian of C which parameterizes isomorphism classes of invertible sheaves of degree d on C. Further let $C^{(d)}$ be the symmetric product of d copies of C, that is, C^d/S_d. Then there is a map

$$\pi_d : C^{(d)} \to J^d(C),$$

called the *Abel-Jacobi map*, defined by

$$(p_1, \cdots, p_d) \mapsto \mathscr{O}_C(p_1 + \cdots + p_d).$$

For $d > 2g-1$, $h^1(C, \mathscr{O}_C(p_1 + \cdots + p_d)) = 0$, so

$$h^0(C, \mathscr{O}_C(p_1 + \cdots + p_d)) = d + (1-g)$$

by Riemann-Roch. Thus $\pi : C^{(d)} \to J^d(C)$ is a $\mathbb{P}^{d-g}$-bundle, hence it is the contraction of a $K_{C^{(d)}}$-negative extremal ray by the previous example (8.2).

Note that for any positive d, all fibers of π_d are projective spaces, although π_d need not be flat. Let ℓ_d be the class of a line in a fiber. Observe that a curve is contracted by π_d if and only if it is contained in one of the fibers, which holds if and only if the curve is numerically equivalent to a (rational) multiple of ℓ_d. We conclude that π_d is the contraction of the extremal ray $\mathbb{R}_{\geq 0}[\ell_d]$, but at this point we do not yet know whether it is a $K_{C^{(d)}}$-negative extremal ray.

CLAIM 8.3.1. $K_{C^{(d)}} \cdot \ell_d = g - d - 1$.

PROOF. The formula holds for $d > 2g-1$ by (8.2.1). We will prove the formula by descending induction on d. Assume it holds for d. Pick a point of C to get an embedding $\iota : C^{(d-1)} \hookrightarrow C^{(d)}$. Then $C^{(d-1)} \cdot \ell_d = 1$. The adjunction formula and the projection formula yield, that

$$\begin{aligned} K_{C^{(d-1)}} \cdot \ell_{d-1} &= \iota^*(K_{C^{(d)}} + C^{(d-1)}) \cdot \ell_{d-1} \\ &= (K_{C^{(d)}} + C^{(d-1)}) \cdot \iota_*\ell_{d-1} \\ &= (K_{C^{(d)}} + C^{(d-1)}) \cdot \ell_d \\ &= (g-d-1) + 1 = g - (d-1) - 1. \end{aligned} \tag{8.3.2}$$

□

It follows that for $d \geq g$, the map π_d is the contraction of the $K_{C^{(d)}}$-negative extremal ray $\mathbb{R}_{\geq 0}[\ell_d]$.

(8.3.3) If $d = g+1$, the general fiber is $\mathbb{P}^1$, but some fibers are larger when $g \geq 3$, so the contraction is not a projective bundle.

(8.3.4) If $d = g$, the general fiber of π_d is a point. The contraction of the locus of $\mathbb{R}^+[\ell_d]$ is

$$\{L \in J^g(C) : h^1(C, L) > 0\}$$

and the general fiber over the image of the exceptional locus is $\mathbb{P}^1$. But some fibers are larger when $g \geq 6$, because the curve C has a g^1_{g-2}, and so the contraction is not a smooth blow-up.

8.4. A flip. (cf. [**Deb01**, 6.17]). Recall that if $\mathscr{E}$ is a locally free sheaf on Z, then a morphism $X \to \mathbb{P}(\mathscr{E})$ is equivalent to a morphism $\phi : X \to Z$, an invertible sheaf $\mathscr{L}$ on X, and a surjection $\phi^*\mathscr{E} \to \mathscr{L}$ (cf. [**Har77**, II.7.12]). The identity map $\mathbb{P}(\mathscr{E}) \to \mathbb{P}(\mathscr{E})$ corresponds to the projection $\pi : \mathbb{P}(\mathscr{E}) \to Z$ and the natural surjection $\pi^*\mathscr{E} \to \mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$.

Let r, s be positive integers. Let

$$\pi_{Y_{r,s}} : Y_{r,s} = \mathbb{P}(\mathscr{O}_{\mathbb{P}^s} \oplus \mathscr{O}_{\mathbb{P}^s}(1)^{\oplus r+1}) \to \mathbb{P}^s$$

and

$$\pi_{X_{r,s}} : X_{r,s} = \mathbb{P}(\mathscr{O}_{\mathbb{P}^s\times\mathbb{P}^r} \oplus \mathscr{O}_{\mathbb{P}^s\times\mathbb{P}^r}(1,1)) \to \mathbb{P}^s \times \mathbb{P}^r$$

be projective space bundles. Let $\varepsilon_{r,s} : X_{r,s} \to Y_{r,s}$ be the morphism determined by the composition

$$\phi : X_{r,s} \xrightarrow{\pi_{X_{r,s}}} \mathbb{P}^s \times \mathbb{P}^r \xrightarrow{p_1} \mathbb{P}^s,$$

and the surjection

$$\begin{aligned}\phi^*(\mathscr{O}_{\mathbb{P}^s} \oplus \mathscr{O}_{\mathbb{P}^s}(1)^{\oplus r+1}) \simeq \pi^*(\mathscr{O}_{\mathbb{P}^s\times\mathbb{P}^r} \oplus \mathscr{O}_{\mathbb{P}^s\times\mathbb{P}^r}(1,0)^{\oplus r+1}) \twoheadrightarrow \\ \twoheadrightarrow \pi^*(\mathscr{O}_{\mathbb{P}^s\times\mathbb{P}^r} \oplus \mathscr{O}_{\mathbb{P}^s\times\mathbb{P}^r}(1,1)) \twoheadrightarrow \mathscr{O}_{X_{r,s}}(1).\end{aligned}$$

Note that $\varepsilon_{r,s}^*\mathscr{O}_{Y_{r,s}}(1) \simeq \mathscr{O}_{X_{r,s}}(1)$ by construction. Since $X_{r,s}$ is symmetric in r and s, we have the following diagram.

(8.4.1)
$$\begin{tikzcd} & X_{r,s} \arrow[dl, "\varepsilon_{r,s}"'] \arrow[dr, "\varepsilon_{s,r}"] & \\ Y_{r,s} \arrow[rr, dashed, "f_{r,s}"] & & Y_{s,r} \end{tikzcd}$$

Proposition 8.4.2. *If $r < s$, then $f_{r,s}$ is a flip (see Definition 5.5).*

PROOF. We will prove this by finding small contractions $Y_{r,s} \to Z_{r,s}$ and $Y_{s,r} \to Z_{r,s}$ and then verifying that $K_{Y_{r,s}}$ and $K_{Y_{s,r}}$ have the right numerical properties on contracted curves. First we have to examine things in more detail.

We claim that $\varepsilon_{r,s}$ is a blowup along a smooth subvariety. There is a section $\mathbb{P}^s \to Y_{r,s}$ determined by the projection $\mathscr{O}_{\mathbb{P}^s} \oplus \mathscr{O}_{\mathbb{P}^s}(1)^{\oplus r+1} \to \mathscr{O}_{\mathbb{P}_s}$. Let $P_{r,s}$ be the image of this section. The ideal sheaf of $P_{r,s}$ is the image of the natural map $\pi^*\mathscr{O}_{\mathbb{P}^s}(1)^{\oplus r+1} \otimes \mathscr{O}_{Y_{r,s}}(-1) \to \mathscr{O}_{Y_{r,s}}$. The preimage of this under $\varepsilon_{r,s}$ is the map

$$\pi^*\mathscr{O}_{\mathbb{P}^s\times\mathbb{P}^r}(1,0)^{\oplus r+1} \otimes \mathscr{O}_{X_{r,s}}(-1) \twoheadrightarrow \pi^*\mathscr{O}_{\mathbb{P}^s\times\mathbb{P}^r}(1,1) \otimes \mathscr{O}_{X_{r,s}}(-1) \to \mathscr{O}_{X_{r,s}}.$$

Thus the ideal sheaf of $\varepsilon^{-1}(P_{r,s})$ is invertible.

It follows by the universal property of blowups that there is a unique morphism from $X_{r,s}$ to the blowup of $Y_{r,s}$ along $P_{r,s}$ (cf. [**Har77**, II.7.14]). We leave to the reader to check that this is an isomorphism, which proves the claim.

Let $E_{r,s}$ be the exceptional divisor of $\varepsilon_{r,s}$. We have seen that $E_{r,s}$ is defined by the vanishing of the map $\pi^*\mathscr{O}_{\mathbb{P}^s\times\mathbb{P}^r}(1,1) \to \mathscr{O}_{X_{r,s}}(1)$. As this is symmetric in r and s, it follows that $E_{r,s} = E_{s,r}$.

We pause for an illustrative remark. Since $E_{r,s}$ is a section of $\pi_{X_{r,s}}$, it follows that $E_{r,s} \cong \mathbb{P}^s \times \mathbb{P}^r$. Moreover, $P_{r,s} \cong \mathbb{P}^s$, and the morphism $\varepsilon_{r,s}$ restricted to $E_{r,s}$ is the projection onto the first factor. This means that in diagram (8.4.1), $\varepsilon_{r,s}$ contracts the $\mathbb{P}^r$ factor, while $\varepsilon_{s,r}$ contracts the $\mathbb{P}^s$ factor. We might speculate that there is a morphism $Y_{r,s} \to Z_{r,s}$ contracting $P_{r,s}$ such that $Z_{r,s} = Z_{s,r}$. Then $X_{r,s} \to Z_{r,s}$ would contract the whole divisor $E_{r,s}$. Now we construct $Z_{r,s}$.

We claim that the linear system corresponding to $\mathscr{O}_{Y_{r,s}}(1)$ is base point free. Since we have a surjection

$$\pi^*(\mathscr{O}_{\mathbb{P}^s} \oplus \mathscr{O}_{\mathbb{P}^s}(1)^{\oplus r+1}) \twoheadrightarrow \mathscr{O}_{Y_{r,s}}(1), \tag{8.4.3}$$

at every point of $Y_{r,s}$ at least one of the summands maps into $\mathscr{O}_{Y_{r,s}}(1)$. At every point, each of the summands has a nonvanishing global section, and hence so does $\mathscr{O}_{Y_{r,s}}(1)$, which proves the claim.

Let $c_{r,s} : Y_{r,s} \to \mathbb{P}^N$ be the morphism defined by $H^0(Y_{r,s}, \mathscr{O}_{Y_{r,s}}(1))$. We claim that this contracts $P_{r,s}$ to a point and embeds the complement of $P_{r,s}$. For the former, note first that every section $\mathscr{O}_{Y_{r,s}} \to \mathscr{O}_{Y_{r,s}}(1)$ factors through the surjection (8.4.3), which can be deduced from

$$\pi_*\mathscr{O}_{Y_{r,s}}(1) \simeq \mathscr{O}_{\mathbb{P}^s} \oplus \mathscr{O}_{\mathbb{P}^s}(1)^{\oplus r+1}.$$

By projecting onto the first summand, we obtain from each section an endomorphism of $\mathscr{O}_{Y_{r,s}}$ (we don't claim this is unique). Note further that the section vanishes on $P_{r,s}$ if and only if this endomorphism is zero. Conversely, if it is nonzero, then the section does not vanish on any point of $P_{r,s}$. This implies that $c_{r,s}(P_{r,s})$ is a point. The second part of the claim is left to the reader.

Let $Z_{r,s}$ be the image of $c_{r,s}$. Since $Z_{r,s}$ is also the image of $X_{r,s}$ by the morphism given by $H^0(X, \mathscr{O}_{X_{r,s}}(1))$, it follows that $Z_{r,s} = Z_{s,r}$, so we have a commutative diagram.

$$\begin{tikzcd}
Y_{r,s} \arrow[rr, dashed, "f_{r,s}"] \arrow[dr] & & Y_{s,r} \arrow[dl] \\
& Z_{r,s} &
\end{tikzcd}$$

It remains to be shown that $K_{Y_{r,s}}$ is negative on contracted curves and $K_{Y_{s,r}}$ is positive on contracted curves. By the technique we used in (8.2), we compute

$$\mathscr{O}_{Y_{r,s}}(K_{Y_{r,s}}) \simeq \pi^*\mathscr{O}_{\mathbb{P}^s}(r-s) \otimes \mathscr{O}_{Y_{r,s}}(-r-2).$$

Let ℓ be a line in $P_{r,s}$. Then $\mathscr{O}_{Y_{r,s}}(-r-2)|_\ell$ is trivial since ℓ is contracted by $c_{r,s}$ and $\mathscr{O}_{Y_{r,s}}(1) \simeq c_{r,s}^*\mathscr{O}_{\mathbb{P}^N}(1)$. Moreover, $\pi^*\mathscr{O}_{\mathbb{P}^s}(r-s)|_\ell$ has degree $r-s$ since $P_{r,s}$ is a section of π. It follows that $K_{Y_{r,s}} \cdot \ell = r-s$. Likewise $K_{Y_{s,r}} \cdot \ell' = s-r$ where ℓ' is a line in $P_{s,r}$. This finishes the proof since we assumed $r < s$. $\square$

8.5. A small contraction whose exceptional locus is disconnected. (cf. [**Deb01**, 6.19], [**Kaw89**, p. 599]). On a smooth 4-fold X'', let C'' be a smooth curve and S'' be a smooth surface that meet transversely in a finite number of points $x_1, \dots, x_r$. First, blow-up X'' along the curve C'' to get a new 4-fold X'. Let C' be the exceptional divisor of this blow-up and let S' be the strict transform of S'', which is isomorphic to S'' blown-up at the points $x_1, \dots, x_r$ (because the intersection of C'' and S'' is transverse). Let E_i' be the exceptional curve in S' lying above x_i, and let P_i' be the complete inverse image of x_i in X'. Now blow up X' along the S' to get a 4-fold X. Let S be the exceptional divisor, and let C be the strict transform of C'. Let P_i be the strict transform of P_i' and let $E_i = P_i \cap S$. Let Γ_i be a fiber over one of the points E_i' and let L be a line in one of the planes in C that gets contracted to a point in C''. Let $\alpha : X \to X''$ denote the composition of the two blow-ups (cf. [**Deb01**, Figure on p. 161]).

The key to the analysis is to consider the relative effective cone of curves $NE(\alpha)$ generated by the effective curves in X which are contracted to a point in X''. Since this cone is dual to the cone of the relative Neron-Severi group, it has dimension 2. Note also that each Γ_i has the same class in $NE(\alpha)$ because they are fibers in the family $S \to S'$; call that class $[\Gamma]$. Using basic facts in intersection theory, one can show that $[L]$ and $[\Gamma]$ form a basis for the linear subspace generated by $NE(\alpha)$ (using the fact that $[C]$ and $[S]$ form a dual basis). One further shows that $E_i \equiv L - \Gamma$. Since $NE(\alpha)$ is generated by $[L]$, $[\Gamma]$, and $[E_i]$, we find that as a cone $NE(\alpha)$ is generated by $[\Gamma]$ and $[E_i] = [L] - [\Gamma]$. Because $NE(\alpha)$ is an extremal subcone of $NE(X)$, we conclude that the two extremal rays of $NE(\alpha)$ must be extremal rays of $NE(X)$. The contraction associated to $[\Gamma]$ is simply the blow-up $X \to X'$, but the contraction $c : X \to Y$ associated to $[L] - [\Gamma]$ is new. It must contract each of the E_i's and because $P_i \cong \mathbb{P}^2$, each P_i must be contracted as well. It is not too hard to check that no other curves are numerically equivalent to the E_i, which by the Contraction Theorem (4.13) means that the exceptional locus of c is the disjoint union of $P_1, \ldots, P_r$. Since this is a small contraction, it follows that one must find a flip for $c : X \to Y$ in order to continue with the Minimal Model Program.

8.6. A flip in dimension 3. (cf. [**Deb01**, 6.20]). In this example we construct a flip in dimension three. We work over an algebraically closed field of characteristic zero. We will begin with a map that contracts a curve with positive intersection with the canonical bundle. This is the end result of a flip. It will then be easier to construct the map which is a small contraction of a K-negative extremal curve.

We begin with the projective bundle $X^+ = \mathbb{P}(\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(1) \oplus \mathscr{O}_{\mathbb{P}^1}(2))$ over $\mathbb{P}^1$. The line bundle $\mathscr{O}_{X^+}(1)$ gives a map $f : X^+ \to Y$ to $\mathbb{P}^5$ where the image Y is a cone over a cubic scroll. (We remind the reader that a cubic scroll is the embedding of the Hirzebruch surface $F_1 \cong \mathbb{P}(\mathscr{O}_{\mathbb{P}^1}(1) \oplus \mathscr{O}_{\mathbb{P}^1}(2))$ in $\mathbb{P}^4$ by the line bundle $\mathscr{O}_{F_1}(1)$. Both the exceptional curve and the fibers map to lines under this map.)

Let H denote the class of a divisor in the linear system associated to $\mathscr{O}_{X^+}(1)$ and let F denote the class of a fiber. Then the map f contracts the unique curve C in the class $H^2 - 3HF$ to the cone point of Y. Figure 8.6.1 depicts this map. We would like f to be the flip of another map $g : X \to Y$.

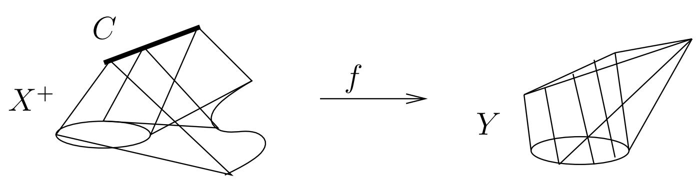


FIGURE 8.6.1. The map f contracts the curve C.

We now describe the construction of g and X. On X^+ there is a minimal subscroll isomorphic to F_1 containing C. The standard normal bundle sequence,

$$0 \to N_{C/F_1} \to N_{C/X} \to (N_{F_1/X})|_C \to 0$$

allows us to compute the normal bundle of C in X:

$$N_{C/X} \cong \mathscr{O}_{\mathbb{P}^1}(-1) \oplus \mathscr{O}_{\mathbb{P}^1}(-2).$$

We blow up X^+ along C to obtain X_1^+. The exceptional divisor S_1 is isomorphic to the Hirzebruch surface F_1. Let E be the exceptional curve on S_1. A similar calculation shows that the normal bundle of E in X_1^+ is isomorphic to $\mathscr{O}_{\mathbb{P}^1}(-1) \oplus \mathscr{O}_{\mathbb{P}^1}(-1)$. We blow up X_1^+ along E. Let X_2 denote the resulting threefold.

The exceptional divisor S_2 of the new blow-up is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. The new exceptional divisor S_2 meets the old exceptional divisor S_1 along the proper transform of E, which by abuse of notation we will continue to call E. Figure 8.6.2 depicts the exceptional divisors of these blow-ups.

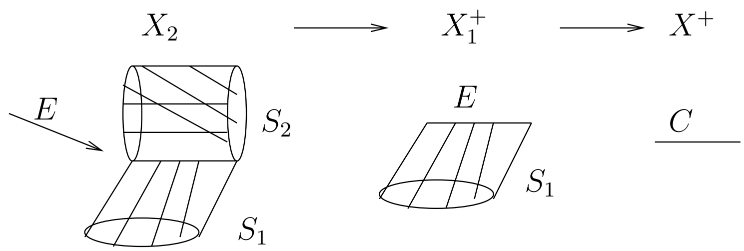


FIGURE 8.6.2. The exceptional divisors of the blow-up.

We now observe that E is a K_{X_2}-negative extremal curve. (If not, it could be expressed as a linear combination of the proper transform of a fiber of S_1 and a fiber of S_2. Checking the intersection product of E with S_1 and S_2 leads to a contradiction.) By the Contraction Theorem (4.13) we can contract E to obtain the threefold X_1. When we contract E, the image of the old exceptional divisor is isomorphic to the projective plane $\mathbb{P}^2$. The image of the second exceptional divisor is $\mathbb{P}^1$. Note that E is a fiber of $\mathbb{P}^1 \times \mathbb{P}^1$ for one of the projections. Once we contract it, all the fibers linearly equivalent to it on the surface get contracted. The image of the two exceptional divisors now looks like $\mathbb{P}^2$ with a rational curve C' meeting it transversely.

Next observe that the lines on the $\mathbb{P}^2$ are K_{X_1}-negative extremal curves. Again by the Contraction Theorem (4.13) we can contract them. We get a threefold X with a rational double point. The image of C', which we will continue to call C', is a K_X-negative extremal curve. We can contract this curve to obtain the cone over a cubic scroll. Let the resulting map be denoted by $g : X \to Y$. Our original map $f : X^+ \to Y$ is the flip of g. Note that the K_{X^+} positive curve C replaces the K_X-negative curve C'. We thus obtain our example of a flip in dimension three. Figure 8.6.3 depicts these contractions.

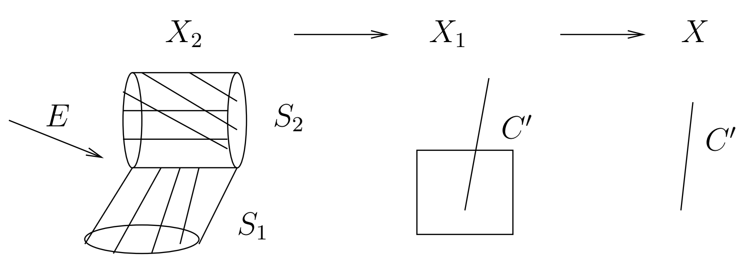


FIGURE 8.6.3. The contractions to get Y from X_2

References

[Ben83] X. BENVENISTE: *Sur l'anneau canonique de certaines variétés de dimension* 3, Invent. Math. **73** (1983), no. 1, 157–164. MR707354 (85g:14020)

[DB81] P. DU BOIS: *Complexe de de Rham filtré d'une variété singulière*, Bull. Soc. Math. France **109** (1981), no. 1, 41–81. MR613848 (82j:14006)

[Deb01] O. DEBARRE: *Higher-dimensional algebraic geometry*, Universitext, Springer-Verlag, New York, 2001. MR1841091 (2002g:14001)

[Elk81] R. ELKIK: *Rationalité des singularités canoniques*, Invent. Math. **64** (1981), no. 1, 1–6. MR621766 (83a:14003)

[Har70] R. HARTSHORNE: *Ample subvarieties of algebraic varieties*, Notes written in collaboration with C. Musili. Lecture Notes in Mathematics, Vol. 156, Springer-Verlag, Berlin, 1970. MR0282977 (44 #211)

[Har77] R. HARTSHORNE: *Algebraic geometry*, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52. MR0463157 (57 #3116)

[Hir64] H. HIRONAKA: *Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II*, Ann. of Math. (2) 79 (1964), 109–203; ibid. (2) **79** (1964), 205–326. MR0199184 (33 #7333)

[Ish85] S. ISHII: *On isolated Gorenstein singularities*, Math. Ann. **270** (1985), no. 4, 541–554. MR776171 (86j:32024)

[Kaw82] Y. KAWAMATA: *A generalization of Kodaira-Ramanujam's vanishing theorem*, Math. Ann. **261** (1982), no. 1, 43–46. MR675204 (84i:14022)

[Kaw84a] Y. KAWAMATA: *The cone of curves of algebraic varieties*, Ann. of Math. (2) **119** (1984), no. 3, 603–633. MR744865 (86c:14013b)

[Kaw84b] Y. KAWAMATA: *On the finiteness of generators of a pluricanonical ring for a* 3*-fold of general type*, Amer. J. Math. **106** (1984), no. 6, 1503–1512. MR765589 (86j:14032)

[Kaw88] Y. KAWAMATA: *Crepant blowing-up of* 3*-dimensional canonical singularities and its application to degenerations of surfaces*, Ann. of Math. (2) **127** (1988), no. 1, 93–163. MR924674 (89d:14023)

[Kaw89] Y. KAWAMATA: *Small contractions of four-dimensional algebraic manifolds*, Math. Ann. **284** (1989), no. 4, 595–600. MR1006374 (91e:14039)

[Kaw92] Y. KAWAMATA: *Termination of log flips for algebraic* 3*-folds*, Internat. J. Math. **3** (1992), no. 5, 653–659. MR1189678 (93j:14019)

[KM98] J. KOLLÁR AND S. MORI: *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, Cambridge, 1998, With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original. MR1658959 (2000b:14018)

[KMM87] Y. KAWAMATA, K. MATSUDA, AND K. MATSUKI: *Introduction to the minimal model problem*, Algebraic geometry, Sendai, 1985, Adv. Stud. Pure Math., vol. 10, North-Holland, Amsterdam, 1987, pp. 283–360. MR946243 (89e:14015)

[Kol84] J. KOLLÁR: *The cone theorem. Note to a paper: "The cone of curves of algebraic varieties"* [Ann. of Math. (2) **119** (1984), no. 3, 603–633; MR0744865 (86c:14013b)] *by Y. Kawamata*, Ann. of Math. (2) **120** (1984), no. 1, 1–5. MR750714 (86c:14013c)

[Kol87] J. KOLLÁR: *The structure of algebraic threefolds: an introduction to Mori's program*, Bull. Amer. Math. Soc. (N.S.) **17** (1987), no. 2, 211–273. MR903730 (88i:14030)

[Kol89] J. KOLLÁR: *Flops*, Nagoya Math. J. **113** (1989), 15–36. MR986434 (90e:14011)

[Kol91] J. KOLLÁR: *Flips, flops, minimal models, etc*, Surveys in differential geometry (Cambridge, MA, 1990), Lehigh Univ., Bethlehem, PA, 1991, pp. 113–199. MR1144527 (93b:14059)

[Kol92] J. KOLLÁR ET. AL: *Flips and abundance for algebraic threefolds*, Société Mathématique de France, Paris, 1992, Papers from the Second Summer Seminar on Algebraic Geometry held at the University of Utah, Salt Lake City, Utah, August 1991, Astérisque No. 211 (1992). MR1225842 (94f:14013)

[Kol96] J. KOLLÁR: *Rational curves on algebraic varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, vol. 32, Springer-Verlag, Berlin, 1996. MR1440180 (98c:14001)

[Kol97] J. KOLLÁR: *Singularities of pairs*, Algebraic geometry—Santa Cruz 1995, Proc. Sympos. Pure Math., vol. 62, Amer. Math. Soc., Providence, RI, 1997, pp. 221–287. MR1492525 (99m:14033)

[Kov94] S. J. KOVÁCS: *The cone of curves of a* $K3$ *surface*, Math. Ann. **300** (1994), no. 4, 681–691. MR1314742 (96a:14044)

[Kov99] S. J. KOVÁCS: *Rational, log canonical, Du Bois singularities: on the conjectures of Kollár and Steenbrink*, Compositio Math. **118** (1999), no. 2, 123–133. MR1713307 (2001g:14022)

[Kov00a] S. J. KOVÁCS: *A characterization of rational singularities*, Duke Math. J. **102** (2000), no. 2, 187–191. MR1749436 (2002b:14005)

[Kov00b] S. J. KOVÁCS: *Rational, log canonical, Du Bois singularities. II. Kodaira vanishing and small deformations*, Compositio Math. **121** (2000), no. 3, 297–304. MR1761628 (2001m:14028)

[Mat02] K. MATSUKI: *Introduction to the Mori program*, Universitext, Springer-Verlag, New York, 2002. MR1875410 (2002m:14011)

[MM86] Y. MIYAOKA AND S. MORI: *A numerical criterion for uniruledness*, Ann. of Math. (2) **124** (1986), no. 1, 65–69. MR847952 (87k:14046)

[Mor82] S. MORI: *Threefolds whose canonical bundles are not numerically effective*, Ann. of Math. (2) **116** (1982), no. 1, 133–176. MR662120 (84e:14032)

[Mor87] S. MORI: *Classification of higher-dimensional varieties*, Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), Proc. Sympos. Pure Math., vol. 46, Amer. Math. Soc., Providence, RI, 1987, pp. 269–331. MR927961 (89a:14040)

[Mor88] S. MORI: *Flip theorem and the existence of minimal models for* 3*-folds*, J. Amer. Math. Soc. **1** (1988), no. 1, 117–253. MR924704 (89a:14048)

[Rei80] M. REID: *Canonical* 3*-folds*, Journées de Géometrie Algébrique d'Angers, Juillet 1979/Algebraic Geometry, Angers, 1979, Sijthoff & Noordhoff, Alphen aan den Rijn, 1980, pp. 273–310. MR605348 (82i:14025)

[Rei83a] M. REID: *Minimal models of canonical* 3*-folds*, Algebraic varieties and analytic varieties (Tokyo, 1981), Adv. Stud. Pure Math., vol. 1, North-Holland, Amsterdam, 1983, pp. 131–180. MR715649 (86a:14010)

[Rei83b] M. REID: *Projective morphisms according to Kawamata*, preprint, 1983.

[Rei87] M. REID: *Young person's guide to canonical singularities*, Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), Proc. Sympos. Pure Math., vol. 46, Amer. Math. Soc., Providence, RI, 1987, pp. 345–414. MR927963 (89b:14016)

[Sho85] V. V. SHOKUROV: *A nonvanishing theorem*, Izv. Akad. Nauk SSSR Ser. Mat. **49** (1985), no. 3, 635–651. MR794958 (87j:14016)

[Sho92] V. V. SHOKUROV: *3-fold log flips.* Appendix by Yujiro Kawamata: *The minimal discrepancy coefficients of terminal singularities in dimension three.*, Russ. Acad. Sci., Izv., Math. **40** (1992), no. 1, 95–202. MR1162635 (93j:14012)

[Sho03] V. V. SHOKUROV: *Prelimiting flips*, Tr. Mat. Inst. Steklova **240** (2003), no. Biratsion. Geom. Linein. Sist. Konechno Porozhdennye Algebry, 82–219. MR1993750 (2004k:14024)

[Ste83] J. H. M. STEENBRINK: *Mixed Hodge structures associated with isolated singularities*, Singularities, Part 2 (Arcata, Calif., 1981), Proc. Sympos. Pure Math., vol. 40, Amer. Math. Soc., Providence, RI, 1983, pp. 513–536. MR713277 (85d:32044)

[Uen83] K. UENO (ed.): *Classification of algebraic and analytic manifolds*, Progress in Mathematics, vol. 39, Mass., Birkhäuser Boston, 1983. MR728604 (84m:14003)

[Vie82] E. VIEHWEG: *Vanishing theorems*, J. Reine Angew. Math. **335** (1982), 1–8. MR667459 (83m:14011)

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