FAMILIES OVER SPECIAL BASE MANIFOLDS AND A CONJECTURE OF CAMPANA

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ABSTRACT. Consider a smooth, projective family of canonically polarized varieties over a smooth, quasi-projective base manifold Y, all defined over the complex numbers. It has been conjectured that the family is necessarily isotrivial if Y is special in the sense of Campana. We prove the conjecture when Y is a surface or threefold.

The proof uses sheaves of symmetric differentials associated to fractional boundary divisors on log canonical spaces, as introduced by Campana in his theory of *Orbifoldes Géométriques*. We discuss a weak variant of the Harder-Narasimhan Filtration and prove a version of the Bogomolov-Sommese Vanishing Theorem that take the additional fractional positivity along the boundary into account. A brief, but self-contained introduction to Campana's theory is included for the reader's convenience.

CONTENTS

1. INTRODUCTION AND STATEMENT OF MAIN RESULT

1.A. **Introduction.** Complex varieties are traditionally classified by their Kodaira-Iitaka dimension. A smooth, projective variety Y is said to be of "general type" if the Kodaira-Iitaka dimension of the canonical bundle is maximal, i.e. $\kappa(\Omega_Y^{\dim Y}) = \dim Y$. Refining the distinction between "general type" and "other," Campana suggested in a series of remarkable papers to consider the class of "special" varieties Y, characterized by the fact that the Kodaira-Iitaka dimension $\kappa(\mathscr{A})$ is small whenever $\mathscr{A}$ is an invertible subsheaf of Ω_Y^p, for some p. Replacing Ω_Y^p with the sheaf of logarithmic differentials, the notion also makes sense for quasi-projective varieties.

Conjecturally, special varieties have a number of good topological, geometrical and arithmetic properties. In particular, Campana conjectured that any map from a special quasi-projective variety to the moduli stack of canonically polarized manifolds is necessarily constant. Equivalently, it is conjectured that any smooth projective family of canonically polarized manifolds over a special quasi-projective base variety is necessarily isotrivial. This generalizes the classical Shafarevich Hyperbolicity Theorem and recent results obtained for families over base manifolds that are not of general type, cf. [KK08a, KK08b] and the references therein.

In this paper, we prove Campana's conjecture for quasi-projective base manifolds Y° of dimension $\dim Y^\circ \le 3$. Throughout the present paper we work over the field of complex numbers.

1.B. **Main result.** Before formulating the main result in Theorem 1.5 below, we briefly recall the precise definition of a special logarithmic pair. The classical Bogomolov-Sommese Vanishing Theorem is our starting point.

Theorem 1.1 (Bogomolov-Sommese Vanishing Theorem, [EV92, Sect. 6])**.** *Let Y be a smooth projective variety and $D \subset Y$ a reduced, possibly empty divisor with simple normal crossings. If $p \le \dim Y$ is any number and $\mathscr{A} \subseteq \Omega_Y^p(\log D)$ any invertible subsheaf, then the Kodaira-Iitaka dimension of $\mathscr{A}$ is at most p, i.e., $\kappa(\mathscr{A}) \le p$.* $\square$

In a nutshell, we say that a pair (Y, D) is special if the inequality in the Bogomolov-Sommese Vanishing Theorem is always strict.

Definition 1.2 (Special logarithmic pair)**.** *In the setup of Theorem 1.1, a pair (Y, D) is called* special *if the strict inequality $\kappa(\mathscr{A}) < p$ holds for all p and all invertible sheaves $\mathscr{A} \subseteq \Omega_Y^p(\log D)$. A smooth, quasi-projective variety Y° is called* special *if there exists a smooth compactification Y such that $D := Y \setminus Y^\circ$ is a divisor with simple normal crossings and such that the pair (Y, D) is special.*

Remark 1.3 (Special quasi-projective variety)*.* It is an elementary fact that if Y° is a smooth, quasi-projective variety and Y_1, Y_2 two smooth compactifications such that $D_i := Y_i \setminus Y^\circ$ are divisors with simple normal crossings, then (Y_1, D_1) is special if and only if (Y_2, D_2) is. The notion of special should thus be seen as a property of the quasi-projective variety Y°.

With this notation in place, Campana's conjecture is then formulated as follows.

Conjecture 1.4 (Generalization of Shafarevich Hyperbolicity, [Cam08, Conj. 12.19])**.** *Let $f : X^\circ \to Y^\circ$ be a smooth family of canonically polarized varieties over a smooth quasi-projective base. If Y° is special, then the family f is isotrivial.*

Theorem 1.5 (Campana's conjecture in dimension three)**.** *Conjecture 1.4 holds if $\dim Y^\circ \le 3$.*

Remark 1.5.1. In the case of $\dim Y^\circ = 2$, Conjecture 1.4 is claimed in [Cam08, Thm. 12.20]. However, we had difficulties following the proof, and offered a new proof of Campana's conjecture in dimension two, [JK09, Cor. 4.5].

Remark 1.6. In analogy to the maximally rationally connected fibration, Campana proves the existence of a quasi-holomorphic "core map", $c : Y^\circ \dashrightarrow C(Y^\circ)$, which is characterized by the fact that its fibers are special any by a certain maximality property. One equivalent reformulation of Conjecture 1.4 is that the core map always factors the moduli map $\mu : Y^\circ \to \mathfrak{M}$, i.e., that there exists a commutative diagram of rational maps

$$
\begin{array}{ccc}
 & C(Y^\circ) & \\
{}^{c}\nearrow & & \searrow \\
Y^\circ \xrightarrow{\qquad \mu \qquad} & & \mathfrak{M}.
\end{array}
$$

1.C. **Outline of the paper.** In Part I of this paper, we introduce the notion of $\mathcal{C}$-pairs, also called *Orbifoldes Géométriques* by Campana, and prove a number of basic results that will be important later. The notion of a $\mathcal{C}$-pair offers the formal framework suitable for the discussion of differentials on charts of moduli stacks and on the associated coarse moduli spaces. Section 2 contains a brief introduction to $\mathcal{C}$-pairs and their use for our purposes. Several sheaves of differentials and the associated version of Kodaira-Iitaka dimension for subsheaves of $\mathcal{C}$-differentials are also introduced.

Even though our presentation differs from that of Campana's papers, most of the material covered in Part I is not new and appears, e.g., in [Cam08]. We have chosen to include a complete and entirely self-contained introduction because we found some parts of [Cam08] hard to read, and because some of the basic notions have still not found their final form in the literature.

In contrast, the results of Part II are new to the best of our knowledge. In Section 6, we discuss a weak variant of the Harder-Narasimhan Filtration that works for sheaves of $\mathcal{C}$-differentials and takes the extra fractional positivity of these sheaves into account. Even though we believe that a refinement of the Harder-Narasimhan Filtration works in the more general context of vector bundles with fractional elementary transformations, and might be of independent interest, we develop the theory only to the absolute minimum required to prove Theorem 1.5.

In Section 7, we generalize the classical Bogomolov-Sommese Vanishing Theorem 1.1 to sheaves of $\mathcal{C}$-differentials on $\mathcal{C}$-pairs with log canonical singularities. Again, this is a generalization of the results obtained in [GKK08] that respects the fractional positivity along the boundary.

In Part III we prove Theorem 1.5. In Section 9, we recall the notion of a Viehweg-Zuo sheaf associated to a smooth projective family of canonically polarized manifolds over a smooth quasi-projective base, and state a recent refinement of their result. In Section 10, we perform a series of reductions to simplify the problem. We then employ the results of the previous parts to end the proof in Section 11.

PART I. $\mathcal{C}$-**PAIRS AND THEIR DIFFERENTIALS**

2. $\mathcal{C}$-PAIRS, ADAPTED MORPHISMS AND COVERS

2.A. $\mathcal{C}$-**pairs, introduction and definitions.** Let $\gamma : Y \to X$ be a finite morphism of degree N between n-dimensional smooth varieties and assume that γ is totally branched over a smooth divisor $D \subset X$. In this setting, if $\sigma \in \Gamma\big(X,\, \Omega_X^p(*D)\big)$ is a p-form, possibly with poles of arbitrary order along D, its pull-back $\gamma^*(\sigma)$ is again a p-form, now with poles along $D_\gamma := \operatorname{supp} \gamma^*(D)$. It is an elementary fact that to check whether σ does indeed have poles, it suffices to look at its pull-back $\gamma^*(\sigma)$. More precisely, it is true that σ has poles of positive order if and only if $\gamma^*(\sigma)$ does. A similar statement holds for forms with logarithmic poles along D. This is, however, no longer true if we look at symmetric products of Ω_X^p.

For an example that will be important later, choose local coordinates $z_1, \ldots, z_n$ on X such that $D = \{z_1 = 0\}$. The symmetric form

$$(2.0.1) \quad \sigma := \frac{1}{z_1^a}(dz_1)^{\otimes b_1} \otimes (dz_2)^{\otimes b_2} \otimes \cdots \otimes (dz_n)^{\otimes b_n} \in \Gamma\big(X,\, \operatorname{Sym}^{b_1 + \cdots + b_n} \Omega_X^1(*D)\big)$$

has a pole of order a along D. However, an elementary computation shows that $\gamma^*(\sigma)$ does not have any pole if the pole order of σ is sufficiently small with respect to b_1, that is $a \le b_1 \cdot \frac{N-1}{N}$.

In our proof of Theorem 1.5, we consider morphisms $\gamma : Y \to X$, where X is a suitable subvariety of the coarse moduli space and Y is a chart for the moduli stack, or simply has a morphism to the moduli stack. Tensor products of Ω_Y^p and Ω_X^p and the pull-back map appear naturally in this context when one discusses positivity and the Kodaira-Iitaka dimension of invertible subsheaves of Ω_Y^p, and tries to relate that to objects living on the coarse moduli space. The formal set-up for this discussion has been given by Campana in his theory of *Orbifoldes Géométriques*. Since the word *orbifold* is already used in a different context, and since the notion of a *geometric orbifold* is not widely accepted, we have chosen to use the words $\mathcal{C}$-*pair*, $\mathcal{C}$-*form* and $\mathcal{C}$-*differential* in this paper, where "$\mathcal{C}$" stands for Campana. In this language, we will say that the form σ defined in (2.0.1) is a $\mathcal{C}$-form on the $\mathcal{C}$-pair $(X, \frac{N-1}{N} \cdot D)$ if and only if $a \le b_1 \cdot \frac{N-1}{N}$ holds.

Notation 2.1. We will often need to consider numbers $\frac{N-1}{N}$, where N is either a positive integer or $N = \infty$. Throughout the paper we follow the convention that $\frac{\infty-1}{\infty} := 1$.

Definition 2.2 ($\mathcal{C}$-pair and $\mathcal{C}$-multiplicities, cf. [Cam08, Def. 2.1]). *A $\mathcal{C}$-pair is a pair (X, D) where X is a normal variety or complex space and D is a $\mathbb{Q}$-divisor of the form*

$$D = \sum_i \tfrac{n_i - 1}{n_i} \cdot D_i$$

where the D_i are irreducible and reduced distinct Weil divisors on X and $n_i \in \mathbb{N}^+ \cup \{\infty\}$. The numbers n_i are called $\mathcal{C}$-multiplicities of the components D_i, denoted $m_{(X,D)}(D_i)$. More generally, if $E \subset X$ is any irreducible, reduced Weil divisor, set

$$m_{(X,D)}(E) := \begin{cases} n_i & \text{if } \exists i \text{ such that } E = D_i \\ 1 & \text{otherwise} \end{cases}$$

2.B. **Adapted coordinates.** In Section 3, we compute sheaves of $\mathcal{C}$-differentials in local coordinates. For this, we consider "adapted" systems of coordinates, defined as follows.

Definition 2.3 (Adapted coordinates). *Let (X, D) be a $\mathcal{C}$-pair, and let $x \in \operatorname{supp}(D)$ be a point which is smooth both in X and in $\operatorname{supp}(D)$. If U is a neighborhood of x, open in the*

analytic topology, and if $z_1, \ldots, z_n \in \mathscr{O}_{\mathrm{hol}}(U)$ are local analytic coordinates about x, we say that the z_i form an adapted system of coordinates *if the set-theoretic equation*

$$\mathrm{supp}(D) \cap U = \{z_1 = 0\}$$

holds.

Remark 2.4. If (X, D) is a $\mathcal{C}$-pair, and $x \in \mathrm{supp}(D)$ is a point which is smooth both in X and in $\mathrm{supp}(D)$, then there always exists an open neighborhood of x with an adapted system of coordinates. The set of points for which there is no system of adapted coordinates is therefore contained in a closed subset of codimension ≥ 2.

The last remark shows that the set of points for which there is no system of adapted coordinates will not play any role when we use adapted coordinates in the discussion of *reflexive* sheaves of differentials. For a more general setup on smooth spaces, see [Cam08, Sect. 2.5].

2.C. **Adapted morphisms.** In Section 2.A, we attached a $\mathcal{C}$-pair to the base of a finite morphism. Conversely, in the discussion of a given $\mathcal{C}$-pair (X, D), we will often use morphisms $Y \to X$ which induce the $\mathcal{C}$-pair structure on X, at least to some extent. In this section, we introduce the necessary notation and prove the existence of these "adapted" morphisms.

Notation 2.5 (Multiplicity of a Weil divisor in a pull-back divisor). Let $\gamma : Y \to X$ be a surjective morphism of normal varieties of constant fiber dimension. If D is any divisor on X, its restriction $D|_{X_{\mathrm{reg}}}$ to the smooth locus of X is Cartier. In particular, there exists a pull-back $\gamma^*(D|_{X_{\mathrm{reg}}})$, which we can interpret as a Weil divisor on the normal space $\gamma^{-1}(X_{\mathrm{reg}})$. If $E \subset Y$ is any irreducible divisor, then E necessarily intersects $\gamma^{-1}(X_{\mathrm{reg}})$, and it makes sense to consider the coefficient m of the pull-back divisor $\gamma^*(D|_{X_{\mathrm{reg}}})$ along $E|_{\gamma^{-1}(X_{\mathrm{reg}})}$. Abusing notation, we say that E *appears in* $\gamma^*(D)$ *with multiplicity* m.

Convention 2.6 (Pull-back of Weil divisors). In the setup of Notation 2.5, the pull-back morphism for Cartier divisors defined on X_{reg} extends to a well-defined pull-back morphism

$$\gamma^* : \{\text{Weil divisors on } X\} \to \{\text{Weil divisors on } Y\}$$

that respects linear equivalence. Throughout this article, whenever a surjective morphism of constant fiber dimension is given, we will use the pull-back morphism for Weil divisors and their linear equivalence classes without extra mention.

Definition 2.7 (Adapted morphism)**.** *Let (X, D) be a $\mathcal{C}$-pair, with $D = \sum_i \frac{n_i - 1}{n_i} D_i$. A surjective morphism $\gamma : Y \to X$ from an irreducible and normal space is called* adapted *if the following holds:*

    (2.7.1) *for any number i with $n_i < \infty$ and any irreducible divisor $E \subset Y$ that surjects onto D_i, the divisor E appears in $\gamma^*(D_i)$ with multiplicity precisely n_i.*
    (2.7.2) *the fiber dimension is constant on X.*

The morphism γ is called subadapted *if in (2.7.1) we require only that E appears in $\gamma^*(D_i)$ with multiplicity at least n_i.*

The preimage of the logarithmic part of D will appear again and again when we use adapted covers to discuss the differentials associated with a $\mathcal{C}$-pair. We will thus introduce a specific notation for this divisor.

Notation 2.8 (Adapted logarithmic divisor). Given a $\mathcal{C}$-pair (X, D) and an adapted or subadapted morphism $\gamma : Y \to X$ as in Definition 2.7, we set

$$D_\gamma := \operatorname{supp} \gamma^*(\lfloor D \rfloor) \subset Y.$$

We call D_γ the *adapted logarithmic divisor* associated with γ.

Given a $\mathcal{C}$-pair (X, D) as in Definition 2.2 and general hyperplane H, we construct an adapted morphism $\gamma : Y \to X$ which is also a finite cyclic cover totally branched over H. The proof is fairly standard and is included only for completeness.

Proposition 2.9 (Existence of an adapted morphism)**.** *Let (X, D) be a $\mathcal{C}$-pair as in Definition 2.2. If X is projective and if the components $D_i \subseteq D$ are Cartier, then there exists a very ample line bundle $L \in \operatorname{Pic}(X)$ such that for general $H \in |L|$, there exists a finite cover $\gamma : Y \to X$ with the following properties.*

- (2.9.1) *The domain Y is normal.*
- (2.9.2) *The morphism γ is adapted in the sense of Definition 2.7. It is cyclic, in particular Galois.*
- (2.9.3) *The branch locus of γ is the union of H and those components D_i with $\mathcal{C}$-multiplicities $n_i \neq \infty$.*
- (2.9.4) *The morphism γ is totally branched over H.*
- (2.9.5) *If $D_i \subseteq \lfloor D \rfloor$ is any component, then γ in unbranched over the general point of D_i.*

Proof. For convenience of notation, we sort the indices n_i so that the first $\mathcal{C}$-multiplicities $n_1, n_2, \ldots, n_k$ are those that are finite. Let N be the least common multiple of the $\mathcal{C}$-multiplicities $(n_i)_{i \leq k}$ that are not ∞, consider a very ample Cartier divisor A such that

$$L := A^{\otimes N} - \sum_{i \leq k} \frac{N}{n_i} \cdot D_i$$

is still very ample, and consider a general hyperplane $H \in |L|$. Let $\sigma \in H^0\left(X, A^{\otimes N}\right) \backslash \{0\}$ be a non-vanishing section associated to the divisor $H + \sum_{i \leq k} \frac{N}{n_i} \cdot D_i \in |A^{\otimes N}|$. Abusing notation, let A and $A^{\otimes N}$ also denote the total spaces of the associated bundles. Consider the multiplication map $m : A \to A^{\otimes N}$, identify the section σ with a subvariety of the space $A^{\otimes N}$, and let $\tilde{\sigma} \subset A$ be the preimage $\tilde{\sigma} = m^{-1}(\sigma)$. The map $m|_{\tilde{\sigma}} : \tilde{\sigma} \to \sigma$ is clearly a cyclic cover, with an associated action of $\mathbb{Z}/N\mathbb{Z}$, acting via multiplication with N^{th} roots of unity.

The restricted morphism $m|_{\tilde{\sigma}} : \tilde{\sigma} \to \sigma$ is obviously unbranched away from $H \cup \bigcup_{i \leq k} D_i$. Over the general point of H, the variety $\tilde{\sigma}$ is smooth and the morphism $m|_{\tilde{\sigma}}$ is totally branched to order N.

Now let x be a general point of one of the D_i with $i \leq k$. Choose an open neighborhood of x with a system of adapted coordinates, $z_1, \ldots, z_n$, and choose bundle coordinates y and y' on A and $A^{\otimes N}$, respectively, such that the multiplication map m is given as $y \mapsto y^N = y'$. In these coordinates, we have $D_i = \{z_1 = 0\}$, and the subvarieties σ and $\tilde{\sigma}$ are given as

$$\sigma = \left\{ y' - z_1^{\frac{N}{n_i}} = 0 \right\} \quad \text{and} \quad \tilde{\sigma} = \left\{ y^N - z_1^{\frac{N}{n_i}} = 0 \right\}.$$

Recalling that

$$y^N - z_1^{\frac{N}{n_i}} = (y^{n_i})^{\frac{N}{n_i}} - z_1^{\frac{N}{n_i}} = \prod_{k=0}^{\frac{N}{n_i}-1} \left(y^{n_i} - \varepsilon^k \cdot z_1 \right)$$

for $\varepsilon = \exp\left(\frac{n_i}{N} \cdot \sqrt{-1}\right)$, similar to [BHPVdV04, Sect. III.9], we obtain that

$$\tilde{\sigma} = \bigcup_{k=0}^{\frac{N}{n_i}-1} \left\{ y^{n_i} = \varepsilon^k \cdot z_1 \right\}$$

is the union of $\frac{N}{n_i}$ distinct smooth components, each totally branched to order n_i over D_i. Defining Y as the normalization of $\tilde{\sigma}$, we obtain the claim. □

Notation 2.10 (Cyclic adapted cover with extra branching). Given a $\mathcal{C}$-pair (X, D) and a general hyperplane H as in Proposition 2.9, we call the associated morphism γ a *cyclic adapted cover with extra branching along H* and set $H_\gamma := \operatorname{supp} \gamma^*(H)$.

The standard adjunction formula immediately gives the following useful relation between the log canonical divisor $K_Y + D_\gamma$ and the pull-back of $K_X + D$.

Lemma 2.11. *If $\gamma : Y \to X$ is a cyclic adapted cover with extra branching along H, the following equivalence of Weil divisor classes holds,*

$$K_Y + D_\gamma = \gamma^*(K_X + D) + (N - 1) \cdot H_\gamma,$$

where N is the degree of the finite morphism γ.

Proof. Again, we sort the indices n_i so that the first $\mathcal{C}$-multiplicities $n_1, n_2, \ldots, n_k$ are those that are finite. By definition of adapted cover, the cycle-theoretic preimage $\gamma^*(D_i)$ is a sum of divisors $D_{i,j}$ that appear with multiplicity precisely n_i if $i \leq k$, and with multiplicity one if $i > k$

$$\gamma^*(D_i) = \begin{cases} \sum_j n_i \cdot D_{i,j} & \text{if } i \leq k \\ \sum_j D_{i,j} & \text{if } i > k \end{cases}$$

In particular,

$$\gamma^*(D) = \sum_{i \leq k} \sum_j n_i \tfrac{n_i-1}{n_i} \cdot D_{i,j} + \sum_{i > k} \sum_j D_{i,j} = \sum_{i \leq k} \sum_j (n_i - 1) \cdot D_{i,j} + D_\gamma.$$

Together with the standard adjunction formula for a finite morphism,

$$K_Y = \gamma^*(K_X) + \sum_{i \leq k} \sum_j (n_i - 1) \cdot D_{i,j} + (N - 1) \cdot H_\gamma,$$

this gives the claim. □

2.D. **Adapted differentials.** If $\gamma : Y \to X$ is a cyclic adapted cover with extra branching along H and if X and Y are smooth, it will be useful later to slightly enlarge the sheaf $\gamma^*\Omega_X^1(\log \lfloor D \rfloor)$ and consider a sheaf $\Omega_Y^1(\log D_\gamma)_{\mathrm{adpt}}$ of "adapted differentials" with

$$\det \Omega_Y^1(\log D_\gamma)_{\mathrm{adpt}} \cong \mathscr{O}_Y\left(\gamma^*(K_X + D) \right).$$

If X and Y are singular, we do a similar construction, using the reflexive hull of $\gamma^*\Omega_X^1(\log \lfloor D \rfloor)$. The following notation is useful in this context and is used throughout the present paper.

Notation 2.12 (Reflexive sheaves and operations). Let Z be a normal variety and $\mathscr{A}$ a coherent sheaf of $\mathscr{O}_Z$-modules. For $n \in \mathbb{N}$, set $\mathscr{A}^{[n]} := \otimes^{[n]}\mathscr{A} := (\mathscr{A}^{\otimes n})^{**}$, $\mathrm{Sym}^{[n]}\mathscr{A} := (\mathrm{Sym}^n \mathscr{A})^{**}$, etc. Likewise, for a morphism $\gamma : X \to Z$ of normal varieties, set $\gamma^{[*]}\mathscr{A} := (\gamma^*\mathscr{A})^{**}$. If $\mathscr{A}$ is reflexive of rank one, we say that $\mathscr{A}$ is $\mathbb{Q}$-Cartier if there exists a number $n \in \mathbb{N}$ such that $\mathscr{A}^{[n]}$ is invertible.

Adapted differentials are now defined as follows.

Definition 2.13 (Adapted differentials). *If $\gamma : Y \to X$ is a cyclic adapted cover with extra branching along H and $1 \le p \le \dim X$, we define a sheaves*

$$\Omega_Y^{[p]}(\log D_\gamma)_{\mathrm{adpt}} \subseteq \Omega_Y^{[p]}(\log D_\gamma),$$

called sheaves of adapted differentials associated with the adapted cover γ, on the level of presheaves as follows. If $U \subseteq Y$ is any open set and $\sigma \in \Gamma\big(U, \Omega_Y^{[p]}(\log D_\gamma)\big)$ any section, then σ is in $\Gamma\big(U, \Omega_Y^{[p]}(\log D_\gamma)_{\mathrm{adpt}}\big)$ if and only if the restriction of σ to the open set $V := U \setminus \gamma^{-1}(\lceil D \rceil)$ satisfies $\sigma|_V \in \Gamma\big(V, \gamma^{[*]}\Omega_X^{[p]}\big)$.

We end this section by noting a few properties of the sheaf of adapted differentials for later use.

Remark 2.14 (Reflexivity, inclusions of adapted differentials). It is immediate from the definition that the sheaf $\Omega_Y^{[p]}(\log D_\gamma)_{\mathrm{adpt}}$ of adapted differentials is reflexive. Since $\gamma^{[*]}\big(\Omega_X^{[p]}(\log\lfloor D \rfloor)\big) \subseteq \Omega_Y^{[p]}(\log D_\gamma)$, it is also clear that there exist inclusions

$$\gamma^{[*]}\big(\Omega_X^{[p]}(\log\lfloor D \rfloor)\big) \subseteq \Omega_Y^{[p]}(\log D_\gamma)_{\mathrm{adpt}} \subseteq \Omega_Y^{[p]}(\log D_\gamma).$$

Remark 2.15 (Determinant of adapted differentials). There exist isomorphisms of sheaves

$$\det\left(\Omega_Y^{[1]}(\log D_\gamma)_{\mathrm{adpt}}\right) \cong \mathscr{O}_Y\big(K_Y + D_\gamma - (N-1)\cdot H_\gamma\big) \qquad \text{by Construction}$$
$$\cong \mathscr{O}_Y\big(\gamma^*(K_X + D)\big). \qquad\qquad \text{by Lemma 2.11}$$

Remark 2.16 (Normal bundle sequence for adapted differentials). Let $F \subset X$ be a smooth curve. Assume that the pair $(X, \lceil D \rceil \cup H)$ is snc along F, and that F intersects the support $\mathrm{supp}(D + H)$ transversely. The preimage $\tilde{F} := \gamma^{-1}(F) \subset Y$ is then smooth, intersects $D_\gamma \cup H_\gamma$ transversely, and the standard conormal sequence of logarithmic differentials,

$$0 \to N_{\tilde{F}/Y}^* \to \Omega_Y^1(\log D_\gamma)|_{\tilde{F}} \to \Omega_{\tilde{F}}^1(\log D_\gamma|_{\tilde{F}}) \to 0,$$

restricts to an exact sequence

$$0 \to \underbrace{N_{\tilde{F}/Y}^*}_{\cong\, \gamma^*(N_{F/X}^*)} \to \Omega_Y^1(\log D_\gamma)_{\mathrm{adpt}}|_{\tilde{F}} \to \underbrace{\Omega_{\tilde{F}}^1(\log D_\gamma|_{\tilde{F}}) \otimes \mathscr{O}_{\tilde{F}}\big(-(N-1)H_\gamma|_{\tilde{F}}\big)}_{\cong\, \gamma^*(\Omega_F^1)\otimes \mathscr{O}_{\tilde{F}}(\gamma^* D|_F)} \to 0.$$

3. $\mathcal{C}$-DIFFERENTIALS

Given a $\mathcal{C}$-pair (X, D) and numbers p and d, we next define the sheaf of $\mathcal{C}$-differentials, written as $\mathrm{Sym}_{\mathcal{C}}^{[d]}\Omega_X^p(\log D)$. A section $\sigma \in \Gamma\big(X, \mathrm{Sym}_{\mathcal{C}}^{[d]}\Omega_X^p(\log D)\big)$ is a symmetric form on X, possibly with logarithmic poles along the support of D, which satisfies extra conditions. There are two essentially equivalent ways to specify what these conditions are.

   (3.0.1) The pole order of σ along a component of D is small compared to the multiplicity of the component in D, and to the pole order of forms $f \cdot \sigma \in \Gamma\big(X, \mathrm{Sym}_{\mathcal{C}}^{[d]}\Omega_X^p(\log D)\big)$, where f is a rational or meromorphic function.

(3.0.2) The pull-back of σ to any adapted covering γ has only logarithmic poles along D_γ, and no other poles elsewhere.

The sheaf of $\mathcal{C}$-differentials has been defined in [Cam08] writing down Condition (3.0.1) in adapted coordinates on smooth spaces. For our purposes, however, Condition (3.0.2) is more convenient. The relation between the definitions is perhaps most clearly seen when the $\mathcal{C}$-differentials are computed explicitly in local coordinates. This is done in Computation 3.8 below.

3.A. **Useful results of sheaf theory.** Before defining the sheaf of $\mathcal{C}$-differentials in Definition 3.5 below, we recall a few facts and definitions concerning saturated and reflexive sheaves.

Definition 3.1 (Saturation of a subsheaf). *Let X be a normal variety, $\mathscr{B}$ a coherent, reflexive sheaf of $\mathscr{O}_X$-modules and $\mathscr{A}$ a subsheaf, with inclusion $\iota : \mathscr{A} \to \mathscr{B}$. The saturation of $\mathscr{A}$ in $\mathscr{B}$ is the kernel of the natural map*

$$\mathscr{B} \to \operatorname{coker}(\iota)\Big/\operatorname{tor}.$$

If the ambient sheaf $\mathscr{B}$ is understood from the context, the saturation of $\mathscr{A}$ is often denoted as $\overline{\mathscr{A}}$. If $\operatorname{coker}(\iota)$ is torsion free, we say that $\mathscr{A}$ is saturated in $\mathscr{B}$.

Proposition 3.2 (Reflexivity of the saturation, cf. [OSS80, Claim on p. 158]). *In the setup of Definition 3.1, the saturation $\overline{\mathscr{A}}$ is reflexive.* $\square$

The next proposition shows that the reflexive symmetric product of a saturated sheaf remains saturated.

Proposition 3.3 (Saturation and symmetric products). *Let X be a normal variety, $\mathscr{B}$ a coherent, reflexive sheaf of $\mathscr{O}_X$-modules and $\mathscr{A}$ a saturated subsheaf, with inclusion $\iota : \mathscr{A} \to \mathscr{B}$. If m is any number, then the natural inclusion of reflexive symmetric products,*

$$\operatorname{Sym}^{[m]} \iota : \operatorname{Sym}^{[m]} \mathscr{A} \to \operatorname{Sym}^{[m]} \mathscr{B}$$

represents $\operatorname{Sym}^{[m]} \mathscr{A}$ as a saturated subsheaf of $\operatorname{Sym}^{[m]} \mathscr{B}$.

Proof. There exists a closed subset $Z \subset X$ of $\operatorname{codim}_X Z \geq 2$ such that $\mathscr{A}$, $\mathscr{B}$ and $\operatorname{coker}(\iota)$ are locally free on $X^\circ := X \setminus Z$. It follows from standard sequences [Har77, II, Ex. 5.16] that the cokernel of $\operatorname{Sym}^{[m]} \iota$ is torsion-free on X°. In particular, the natural inclusion

$$(3.3.1) \qquad\qquad \operatorname{Sym}^{[m]} \mathscr{A} \to \overline{\operatorname{Sym}^{[m]} \mathscr{A}}$$

is isomorphic away from Z. By definition and by Proposition 3.2, respectively, both sides of (3.3.1) are reflexive. The inclusion (3.3.1) must thus be isomorphic. $\square$

Definition 3.4 (Sheaf of sections with arbitrary pole order). *Let X be a variety, let $D \subset X$ be a reduced Weil divisor and $\mathscr{F}$ a reflexive coherent sheaf of $\mathscr{O}_X$-modules. We will often consider sections of $\mathscr{F}$ with poles of arbitrary order along D, and let $\mathscr{F}(*D)$ be the associated sheaf of these sections. More precisely, we define*

$$\mathscr{F}(*D) := \varinjlim_m \big(\mathscr{F} \otimes \mathscr{O}_X(m \cdot D)\big)^{**}.$$

3.B. **The definition of $\mathcal{C}$-differentials.** We next define a $\mathcal{C}$-differential. Our approach is slightly different than Campana's approach in [Cam08], as Campana defines $\mathcal{C}$-differentials in local adapted coordinates. However, we will recover his definition in Section 3.C.

Definition 3.5 ($\mathcal{C}$-differentials, cf. [Cam08, Sect. 2.6-7])**.** *If (X, D) is a $\mathcal{C}$-pair we define a sheaf*

$$\underbrace{\operatorname{Sym}_{\mathcal{C}}^{[d]} \Omega_X^p(\log D)}_{=:\mathscr{A}} \subseteq \underbrace{\left(\operatorname{Sym}^{[d]} \Omega_X^p\right)(*\lceil D\rceil)}_{=:\mathscr{B}}$$

on the level of presheaves as follows: if $U \subseteq X$ is open and $\sigma \in \Gamma\left(U, \mathscr{B}\right)$ any form, possibly with poles along D, then σ is a section of $\mathscr{A}$ if and only if for any open subset $U' \subseteq U$ and any adapted morphism $\gamma : V \to U'$, the reflexive pull-back has at most logarithmic poles along D_γ, and no other poles elsewhere, i.e.

$$(3.5.1) \qquad\qquad \gamma^{[*]}(\sigma) \in \Gamma\left(V, \operatorname{Sym}^{[d]} \Omega_V^p(\log D_\gamma)\right).$$

Explanation 3.6. Inclusion (3.5.1) of Definition 3.5 can also be expressed as follows. If $E \subset V$ is any irreducible Weil divisor which dominates a component of $\lfloor D \rfloor$, then $\gamma^*(\sigma)$ may have at most logarithmic poles along E. If E does not dominate a component of $\lfloor D \rfloor$, then $\gamma^*(\sigma)$ may not have any poles along E.

Remark 3.7. Definition 3.5 remains invariant if we remove arbitrary small sets from U'. It is therefore immediate that the sheaf $\operatorname{Sym}_{\mathcal{C}}^{[d]} \Omega_X^p(\log D)$ is torsion free and normal as a sheaf of $\mathscr{O}_X$-modules, cf. [OSS80, Def. 1.1.11 on p. 150]. Once we have seen in Corollary 3.14 that $\operatorname{Sym}_{\mathcal{C}}^{[d]} \Omega_X^p(\log D)$ is also coherent, this will imply that it is in fact reflexive.

3.C. **$\mathcal{C}$-differentials in local coordinates.** It is sometimes useful to represent $\mathcal{C}$-differentials explicitly in local coordinates. The following computations yields several results which will be needed later on.

Computation 3.8. Let (X, D) be a $\mathcal{C}$-pair as in Definition 2.2. Let $D_i \subseteq D$ be a component, let $x \in D_i$ be a smooth point, and let $U \subseteq X$ be an open neighborhood of x with an adapted system of coordinates as in Definition 2.3. Finally, consider a section

$$\sigma := \frac{f(z_1, \ldots, z_n)}{z_1^a} \cdot (dz_1)^{m_1} \cdot (dz_2)^{m_2} \cdots (dz_n)^{m_n} \in \Gamma\left(U, \operatorname{Sym}^{[d]} \Omega_X^1\left(*\lceil D\rceil\right)\right),$$

where $d = \sum m_i$ and $f \in \mathscr{O}_U$ is a holomorphic function that does not vanish along $D_i \cap U = \{z_1 = 0\}$. We aim to express Condition (3.5.1) in this context. To this end, after possibly replacing U by one of its open subsets, let $\gamma : V \to U$ be any adapted morphism, and $E \subset V$ any divisor that dominates $D_i \cap U$.

If D_i appears in D with $\mathcal{C}$-multiplicity $n_i = \infty$, it is a standard fact that $\gamma^{[*]}(\sigma)$ has logarithmic poles along E if and only if σ has logarithmic poles along D_i, see e.g. [GKK08, Cor. 2.12.1]. Condition (3.5.1) therefore says that σ is a section of $\operatorname{Sym}_{\mathcal{C}}^{[d]} \Omega_X^1(\log D)$ if and only if $a \leq m_1$.

If D_i appears in D with $\mathcal{C}$-multiplicity $n_i < \infty$, then E appears in $\gamma^*(D_i)$ with multiplicity n_i. The reflexive pull-back $\gamma^{[*]}(\sigma)$ is thus a rational section of the sheaf $\operatorname{Sym}^{[d]} \Omega_V^1(\log D_\gamma)$ whose pole order along E is precisely

$$(3.8.1) \qquad\qquad P(\sigma, D_i) := n_i \cdot a - (n_i - 1) \cdot m_1.$$

We obtain from Condition (3.5.1) that σ is a section of $\operatorname{Sym}_{\mathcal{C}}^{[d]} \Omega_X^1(\log D)$ if and only if $P(\sigma, D_i) \leq 0$.

Computation 3.9. In the setup of Computation 3.8, if τ is an arbitrary section of $\left(\operatorname{Sym}^{[d]} \Omega_X^1\right)(* \lceil D \rceil)$, write τ locally as

$$\tau := \sum_{m_1 \ldots m_n} \underbrace{\frac{f_{m_1 \ldots m_n}(z_1, \ldots, z_n)}{z_1^{a_{m_1 \ldots m_n}}} \cdot (dz_1)^{m_1} \cdot (dz_2)^{m_2} \cdots (dz_n)^{m_n}}_{=: \sigma_{m_1 \ldots m_n}},$$

where the functions $f_{m_1 \ldots m_n}$ are either constantly zero, or do not vanish along $D_i \cap U$. Again, we aim to formulate Condition (3.5.1) for the section τ. Choose an adapted covering γ and a divisor E as in Computation 3.8.

If D_i appears in D with $\mathcal{C}$-multiplicity $n_i = \infty$, it is again clear that $\gamma^{[*]}(\tau)$ has logarithmic poles along E if and only if τ has logarithmic poles along D_i. Condition (3.5.1) therefore says that τ is a section of $\operatorname{Sym}_{\mathcal{C}}^{[d]} \Omega_X^1(\log D)$ if and only if $a_{m_1 \ldots m_n} \leq m_1$ for all multi-indices $m_1, \ldots, m_n$ with $f_{m_1 \ldots m_n} \not\equiv 0$.

If D_i appears in D with $\mathcal{C}$-multiplicity $n_i < \infty$, set

$$P(\tau, D_i) = \max \left\{ P(\sigma_{m_1 \ldots m_n}, D_i) \,|\, f_{m_1 \ldots m_n} \not\equiv 0 \right\},$$

where the $P(\sigma_{m_1 \ldots m_n}, D_i)$ are the numbers defined in Equation (3.8.1) above. It is then clear that the reflexive pull-back $\gamma^{[*]}(\tau)$ is a rational section of the sheaf $\operatorname{Sym}^{[d]} \Omega_V^1(\log D_\gamma)$ whose pole order along E is precisely $P(\tau, D_i)$. Again, we obtain from Condition (3.5.1) that τ is a section of $\operatorname{Sym}_{\mathcal{C}}^{[d]} \Omega_X^1(\log D)$ if and only if $P(\tau, D_i) \leq 0$.

Observation 3.10. Using the convention that $\frac{n_i - 1}{n_i} = 1$ if $n_i = \infty$, Computation 3.9 gives the following set of generators for $\operatorname{Sym}_{\mathcal{C}}^{[d]} \Omega_U^1(\log D)$ near the point x,

$$\left(\frac{1}{z_1^{\lfloor m_1 \cdot \frac{n_i - 1}{n_i} \rfloor}} \cdot (dz_1)^{m_1} \cdot (dz_2)^{m_2} \cdots (dz_n)^{m_n} \right)_{\sum m_j = d}.$$

Thus, it follows from Definition 2.3 that the sheaf $\operatorname{Sym}_{\mathcal{C}}^{[d]} \Omega_X^1(\log D)$ is locally free wherever the pair $(X, \lceil D \rceil)$ is snc. In particular, it is locally free in codimension one. Since it is normal, we also see that

$$(3.10.1) \qquad \operatorname{Sym}_{\mathcal{C}}^{[d]} \Omega_X^1(\log D) \subseteq \operatorname{Sym}^{[d]} \Omega_X^1(\log \lceil D \rceil).$$

In the case $d = 1$, we obtain additionally that $\operatorname{Sym}_{\mathcal{C}}^{[1]}\left(\Omega_X^1(\log D)\right) = \Omega_X^{[1]}(\log \lfloor D \rfloor)$.

Observation 3.11. In Computation 3.9, if $n_i < \infty$, the number $P(\tau, D_i)$ depends only on the section τ and on the component D_i, but not on the choice of adapted coordinates, or on the choice of the adapted morphism γ.

Observation 3.12. In Computation 3.9, if $n_i < \infty$ and if the number $P(\tau, D_i)$ is nonpositive, then $\gamma^{[*]}(\tau)$ is a section of the sheaf $\operatorname{Sym}^{[d]} \Omega_V^1(\log D_\gamma)$ that vanishes along E precisely with multiplicity $-P(\tau, D_i)$.

3.D. **Consequences of the local computation.** Computations 3.8 and 3.9 have several immediate consequences which we note for future reference. It is not very hard to see that the computations and observations of Section 3.C also hold for sections in $\operatorname{Sym}_{\mathcal{C}}^{[d]} \Omega_X^p(\log D)$, for all numbers p. The consequences of Computation 3.8 which we draw in this section also hold for all p, and are stated in that generality. To keep the paper reasonably sized, we leave it to the reader to make the analogous computations in case $p \neq 1$.

3.D.1. *Inclusions, reflexivity.* In complete analogy to Inclusion (3.10.1) above, we can view the sheaf of $\mathcal{C}$-differentials as a subsheaf of the logarithmic differentials, for any p. In Corollary 3.14, we apply this inclusion to prove reflexivity of the sheaf of $\mathcal{C}$-differentials.

Corollary 3.13 (Inclusion of $\mathcal{C}$-differentials into logarithmic differentials)**.** *There exists an inclusion* $\mathrm{Sym}_{\mathcal{C}}^{[d]} \Omega_X^p(\log D) \subseteq \mathrm{Sym}^{[d]} \Omega_X^p(\log \lceil D \rceil)$. $\qquad\square$

Corollary 3.14 (Reflexivity of $\mathcal{C}$-differentials)**.** *The sheaf* $\mathrm{Sym}_{\mathcal{C}}^{[d]} \Omega_X^p(\log D)$ *is a coherent, reflexive sheaf of* $\mathcal{O}_X$*-modules, locally free wherever the pair* $(X, \lceil D \rceil)$ *is snc.*

Proof. Corollary 3.13 represents $\mathscr{F} := \mathrm{Sym}_{\mathcal{C}}^{[d]} \Omega_X^p(\log D)$ as a subsheaf of the coherent sheaf $\mathscr{G} := \mathrm{Sym}^{[d]} \Omega_X^p(\log \lceil D \rceil)$. We have also seen in Observation 3.10 that $\mathscr{F}$ is locally free wherever that pair $(X, \lceil D \rceil)$ is snc. In particular it is locally free on an open subset $U \subseteq X$ whose complement has codimension ≥ 2. In this setting, it follows from the classical extension theorem of coherent sheaves, [Gro60, I. Thm. 9.4.7], that there exists a coherent subsheaf $\mathscr{F}' \subseteq \mathscr{G}$ whose restriction to U agrees with $\mathscr{F}$. Since $\mathscr{F}$ is normal, and since the complement of U is small, we have $\mathscr{F} = (\mathscr{F}')^{**}$. $\qquad\square$

3.D.2. *Independence of $P(\tau, D_i)$ on choices, definition of defect divisors.* The independence of the numbers $P(\tau, D_i)$ on the choice of a particular open set and an adapted morphism allows us to define a "defect divisor" that measures additional fractional positivity along a $\mathcal{C}$-differential. In Section 4.B, we will extend this notion to sheaves of differentials. Our starting point is the following Corollary, which summarizes Observations 3.11 and 3.12.

Corollary and Definition 3.15. *Let* (X, D) *be a $\mathcal{C}$-pair and σ a section of* $\left(\mathrm{Sym}^{[d]} \Omega_X^p\right)(*\lceil D \rceil)$. *Further, consider an open set $U \subseteq X$ and an adapted morphism $\gamma : V \to U$. If $D_i \subseteq D$ is an irreducible component that intersects U and has finite $\mathcal{C}$-multiplicity and if $E \subset V$ is any divisor that dominates $D_i \cap U$, then $\gamma^{[*]}(\sigma)$ is a rational section of $\mathrm{Sym}^{[d]} \Omega_V^p(\log D_\gamma)$ whose pole order $P(\sigma, D_i)$ along E depends only on σ and on the component $D_i \subseteq D$, but not on the choice of U, the morphism γ or the particular divisor that dominates D_i.*

The section σ is in $\Gamma\left(U, \mathrm{Sym}_{\mathcal{C}}^{[d]} \Omega_X^p(\log D)\right)$ if and only if $P(\sigma, D_i) \leq 0$ for all components $D_i \subseteq D$ with finite $\mathcal{C}$-multiplicity. $\qquad\square$

Corollary 3.16. *To check the conditions spelled out in Definition 3.5, it suffices to consider a single covering by open sets $(U_\alpha)_{\alpha \in A}$ and for each U_α a single adapted morphism.* $\qquad\square$

Using the numbers P defined in 3.15, we define the defect divisor of a $\mathcal{C}$-differential.

Definition 3.17 (Defect divisor of a $\mathcal{C}$-differential)**.** *If (X, D) is a $\mathcal{C}$-pair, $D = \sum \frac{n_i - 1}{n_i} \cdot D_i$ and σ a section of $\mathrm{Sym}_{\mathcal{C}}^{[d]} \Omega_X^p(\log D)$, consider the following $\mathbb{Q}$-Weil divisor,*

$$R(\sigma) := \sum_{D_i \subseteq D \text{ with } n_i < \infty} \frac{-P(\sigma, D_i)}{n_i} \cdot D_i.$$

We call $R(\sigma)$ the defect divisor *of the section σ.*

Remark 3.18. The defect divisor $R(\sigma)$ is always effective. If two sections σ and τ of $\mathrm{Sym}_{\mathcal{C}}^{[d]} \Omega_X^p(\log D)$ differ only by multiplication with a nowhere-vanishing function, their defect divisors $R(\sigma)$ and $R(\tau)$ agree.

3.D.3. *The symmetric algebra of $\mathcal{C}$-differentials.* The special form of the generators for $\mathrm{Sym}_{\mathcal{C}}^{[d]}\big(\Omega_U^p(\log D)\big)$ found in Observation 3.10 makes it possible to interpret a tensor product of symmetric $\mathcal{C}$-differentials as a $\mathcal{C}$-differential. More precisely, we obtain the following multiplication morphisms.

Corollary 3.19. *Since $\lfloor a \rfloor + \lfloor b \rfloor \leq \lfloor a+b \rfloor$ for any pair of numbers a and b, the multiplication morphisms of symmetric differentials extend to multiplication morphisms of symmetric $\mathcal{C}$- differentials. More precisely, given any two numbers d_1, $d_2 \in \mathbb{N}$, we obtain sheaf morphisms*

$$\mathrm{Sym}_{\mathcal{C}}^{[d_1]} \Omega_X^p(\log D) \otimes \mathrm{Sym}_{\mathcal{C}}^{[d_2]} \Omega_X^p(\log D) \to \mathrm{Sym}_{\mathcal{C}}^{[d_1+d_2]} \Omega_X^p(\log D)$$

$$\mathrm{Sym}^{d_1}\left(\mathrm{Sym}_{\mathcal{C}}^{[d_2]} \Omega_X^p(\log D)\right) \to \mathrm{Sym}_{\mathcal{C}}^{[d_1 \cdot d_2]} \Omega_X^p(\log D)$$

that agree outside of $\mathrm{supp}(D)$ with the usual multiplication maps. $\qquad\square$

We obtain a symmetric algebra of $\mathcal{C}$-differentials, which will allow us to define a variant of the Kodaira-Iitaka dimension for sheaves of $\mathcal{C}$-differentials in Section 4.

Corollary 3.20 (Symmetric algebra of $\mathcal{C}$-differentials)**.** *With the multiplication morphisms of Corollary 3.19, the direct sum $\bigoplus_{d \geq 0} \mathrm{Sym}_{\mathcal{C}}^{[d]} \Omega_X^p(\log D)$ is a sheaf of $\mathscr{O}_X$-algebras.* $\quad\square$

3.D.4. *Behavior under subadapted morphisms.* Equation (3.8.1) immediately shows that the pull-back of $\mathcal{C}$-differentials under subadapted morphisms also become regular logarithmic differentials.

Corollary 3.21 (Behaviour under subadapted morphisms)**.** *Let (X, D) be a $\mathcal{C}$-pair and $\gamma : Y \to X$ a subadapted morphism. Similar to the setup of Definition 3.5, the natural pull-back morphism of differential forms extends to a morphism $\gamma^{[*]} \mathrm{Sym}_{\mathcal{C}}^{[d]} \Omega_X^p(\log D) \to \mathrm{Sym}^{[d]} \Omega_Y^p(\log D_\gamma)$.* $\qquad\square$

3.D.5. *A criterion for $\mathrm{Sym}_{\mathcal{C}}^{[m]} \Omega_X^p(\log D)|_F$ to be anti-nef.* In Remark 2.16 we considered the standard conormal sequence of adapted differentials for a smooth curve $F \subset X$. The following proposition gives a criterion for $\mathrm{Sym}_{\mathcal{C}}^{[m]} \Omega_X^p(\log D)|_F$ to be anti-nef, and will be an essential ingredient in the proof of Theorem 1.5.

Proposition 3.22. *Let $F \subset X$ be a smooth curve and assume that the following holds.*

(3.22.1) *The pair $\big(X, \lceil D \rceil\big)$ is snc along F, and F intersects $\mathrm{supp}(D)$ transversely.*

(3.22.2) *The normal bundle $N_{F/X}$ is nef.*

(3.22.3) *The $\mathbb{Q}$-divisor $-(K_F + D|_F)$ is nef.*

If $m \in \mathbb{N}^+$ is any number and $1 \leq p \leq \dim X$, then $\mathrm{Sym}_{\mathcal{C}}^{[m]} \Omega_X^p(\log D)|_F$ is anti-nef.

Proof. To start, observe that Condition (3.22.1) guarantees that $\mathrm{Sym}_{\mathcal{C}}^{[m]} \Omega_X^p(\log D)|_F$ is locally free along F. Let $H \subset X$ be a general hyperplane, and $\gamma : Y \to X$ be a cyclic adapted cover with extra branching along H. Let D_γ and H_γ be the divisors defined in Notation 2.8 and 2.10. Further, we consider the curve $\tilde{F} := \gamma^{-1}(F)$. Observe that $\tilde{F}$ is smooth, that Y is smooth along $\tilde{F}$, and that $\tilde{F}$ intersects $D_\gamma \cup H_\gamma$ transversely.

Since a sheaf is anti-nef if its pull-back under a finite map is anti-nef, it suffices to show that

$$\gamma^*\big(\mathrm{Sym}_{\mathcal{C}}^{[m]} \Omega_X^p(\log D)|_F\big) \subseteq \mathrm{Sym}^m\big(\Omega_Y^p(\log D_\gamma)_{\mathrm{adpt}}|_{\tilde{F}}\big)$$

is anti-nef. Since subsheaves and tensor powers of anti-nef sheaves are anti-nef, it suffices to see that $\Omega_Y^1(\log D_\gamma)_{\mathrm{adpt}}|_{\tilde{F}}$ is anti-nef. For that, recall the generalized conormal

sequence of Remark 2.16, which presents $\Omega_Y^1(\log D_\gamma)_{\mathrm{adpt}}|_{\tilde{F}}$ as an extension of two bundles, both of which are anti-nef by assumption.   □

4. SHEAVES OF $\mathcal{C}$-DIFFERENTIALS AND THEIR KODAIRA-IITAKA DIMENSIONS

Following [Cam08] closely, we define a variant of the Kodaira-Iitaka dimension for sheaves of $\mathcal{C}$-differentials in Section 4.A, where we also generalize the notion of "special" to $\mathcal{C}$-pairs. In Section 4.B we introduce the *defect divisor* of a sheaf, which helps in the computation of Kodaira-Iitaka dimensions.

Throughout the present Section 4, we consider a $\mathcal{C}$-pair (X, D) as in Definition 2.2 and let $\mathscr{F}$ be a reflexive sheaf of symmetric $\mathcal{C}$-differentials with inclusion

$$\iota : \mathscr{F} \hookrightarrow \mathrm{Sym}_{\mathcal{C}}^{[d]} \Omega_X^p(\log D).$$

We assume that $\mathscr{F}$ is saturated in $\mathrm{Sym}_{\mathcal{C}}^{[d]} \Omega_X^p(\log D)$, i.e., that the cokernel of ι is torsion free.

4.A. **Kodaira-Iitaka dimensions and special $\mathcal{C}$-pairs.** The usual definition of Kodaira-Iitaka dimension considers reflexive tensor powers of a given reflexive sheaf of rank one. In our setup, where $\mathscr{F}$ is a reflexive sheaf of symmetric $\mathcal{C}$-differentials, we aim to detect the fractional positivity encoded in the $\mathcal{C}$-pair by saturating the tensor product in $\mathrm{Sym}_{\mathcal{C}}^{[m \cdot d]} \Omega_X^p(\log D)$ before considering sections. The following notation is useful in the description of the process.

Notation 4.1 ($\mathcal{C}$-product sheaves, cf. [Cam08, Sect. 2.6]). Given a number $m \in \mathbb{N}^+$, Corollary 3.19 asserts that there exists a non-vanishing inclusion $\iota^m : \mathrm{Sym}^{[m]} \mathscr{F} \hookrightarrow \mathrm{Sym}_{\mathcal{C}}^{[m \cdot d]} \Omega_X^p(\log D)$. Let $\mathrm{Sym}_{\mathcal{C}}^{[m]} \mathscr{F}$ be the saturation of the image, i.e., the kernel of the associated map

$$\mathrm{Sym}_{\mathcal{C}}^{[m \cdot d]} \Omega_X^p(\log D) \to \mathrm{coker}(\iota^m) \Big/ \mathrm{tor}.$$

We call $\mathrm{Sym}_{\mathcal{C}}^{[m]} \mathscr{F}$ the *$\mathcal{C}$-product of $\mathscr{F}$*. There are inclusions

$$\mathrm{Sym}^{[m]} \mathscr{F} \hookrightarrow \mathrm{Sym}_{\mathcal{C}}^{[m]} \mathscr{F} \hookrightarrow \mathrm{Sym}_{\mathcal{C}}^{[m \cdot d]} \Omega_X^p(\log D).$$

Remark 4.2. The $\mathcal{C}$-product $\mathrm{Sym}_{\mathcal{C}}^{[m]} \mathscr{F}$ is a saturated subsheaf of a reflexive sheaf and therefore itself reflexive, by Proposition 3.2. If $\mathrm{rank}\,\mathscr{F} = 1$, this implies that the restriction of $\mathrm{Sym}_{\mathcal{C}}^{[m]} \mathscr{F}$ to the smooth locus of X is locally free, [OSS80, Lem. 1.1.15 on p. 154].

Definition 4.3 ($\mathcal{C}$-Kodaira-Iitaka dimension, cf. [Cam08, Sect. 2.7]). *If X is projective and* $\mathrm{rank}\,\mathscr{F} = 1$, *we consider the set*

$$M := \left\{ m \in \mathbb{N} \ \middle| \ h^0\big(X, \mathrm{Sym}_{\mathcal{C}}^{[m]} \mathscr{F}\big) > 0 \right\}.$$

If $M = \emptyset$, we say that the sheaf $\mathscr{F}$ has $\mathcal{C}$-Kodaira-Iitaka dimension minus infinity, $\kappa_{\mathcal{C}}(\mathscr{F}) = -\infty$. *Otherwise, by Remark 4.2, the restriction of $\mathrm{Sym}_{\mathcal{C}}^{[m]} \mathscr{F}$ to the smooth locus of X is locally free, and we consider the natural rational mapping*

$$\phi_m : X \dashrightarrow \mathbb{P}\left(H^0\big(X, \mathrm{Sym}_{\mathcal{C}}^{[m]} \mathscr{F}\big)^\vee \right), \text{ for each } m \in M.$$

Define the $\mathcal{C}$-Kodaira-Iitaka dimension as

$$\kappa_{\mathcal{C}}(\mathscr{F}) = \max_{m \in M} \left\{ \dim \overline{\phi_m(X)} \right\}.$$

Remark 4.4. If $D = \emptyset$, or if (X, D) is a logarithmic pair, it is clear from the construction and from the saturatedness assumption that $\mathrm{Sym}_{\mathcal{C}}^{[m]} \mathscr{F} \cong \mathrm{Sym}^{[m]} \mathscr{F}$ for all m, and that the $\mathcal{C}$-Kodaira-Iitaka dimension of $\mathscr{F}$ therefore equals the regular Kodaira-Iitaka dimension, $\kappa_{\mathcal{C}}(\mathscr{F}) = \kappa(\mathscr{F})$.

Remark 4.5 (Invariance of $\kappa_{\mathcal{C}}$ under $\mathcal{C}$-products). Using Remark 4.4, standard arguments show that if X is projective, then $\kappa_{\mathcal{C}}(\mathscr{F}) = \kappa_{\mathcal{C}}\left(\mathrm{Sym}_{\mathcal{C}}^{[m]} \mathscr{F}\right)$ for all positive m.

Warning 4.6. Unlike the standard Kodaira-Iitaka dimension, the $\mathcal{C}$-Kodaira-Iitaka dimension is defined only for subsheaves of $\mathrm{Sym}_{\mathcal{C}}^{[d]} \Omega_X^p(\log D)$. Its value is generally not an invariant of the sheaf alone and will often depend on the embedding.

Using the $\mathcal{C}$-Kodaira-Iitaka dimension instead of the standard definition, we have the following immediate generalization of Definition 1.2, which agrees with the old definition if (X, D) is a logarithmic pair.

Definition 4.7 (Special $\mathcal{C}$-pairs, cf. [Cam08, Def. 4.18 and Thm. 7.5]). *A $\mathcal{C}$-pair (X, D) is special if $\kappa_{\mathcal{C}}(\mathscr{F}) < p$ for any number $1 \le p \le \dim X$ and any saturated rank-one sheaf $\mathscr{F} \subseteq \mathrm{Sym}_{\mathcal{C}}^{[1]} \Omega_X^p(\log D)$.*

4.B. **Defect divisors for sheaves of $\mathcal{C}$-differentials.** If $\mathrm{rank}\, \mathscr{F} = 1$, then $\mathscr{F}|_{X_{\mathrm{reg}}}$ is locally free. If U_1 and $U_2 \subseteq X_{\mathrm{reg}}$ are open subsets of the smooth locus and if $\sigma_i \in \Gamma\left(U_i,\, \mathscr{F}\right)$ are generators of $\mathscr{F}|_{U_i}$, this implies that $\sigma_1|_{U_1 \cap U_2}$ and $\sigma_2|_{U_1 \cap U_2}$ differ only by multiplication with a nowhere-vanishing function. In particular, Remark 3.18 asserts that the defect divisors $R(\sigma_1)$ and $R(\sigma_2)$ agree on the overlap $U_1 \cap U_2$. The following definition therefore makes sense.

Definition 4.8 (Defect divisor and $\mathcal{C}$-divisor class of a sheaf of differentials). *If $\mathrm{rank}\, \mathscr{F} = 1$, let $R_{\mathscr{F}}$ be the unique $\mathbb{Q}$-Weil divisor on X such that for any open set $U \subseteq X_{\mathrm{reg}}$, and any generator $\sigma \in \Gamma\left(U,\, \mathscr{F}\right)$, we have $R_{\mathscr{F}} \cap U = R(\sigma)$. We call $R_{\mathscr{F}}$ the defect divisor of the sheaf $\mathscr{F}$.*

Recall that there exists, up to linear equivalence, a unique Weil divisor W such that $\mathscr{F} = \mathscr{O}_X(W)$. Let $\mathrm{Div}(\mathscr{F}) \in \mathrm{Cl}(X)$ be the associated element of the divisor class group. If X is $\mathbb{Q}$-factorial, we define the $\mathcal{C}$-divisor class of the sheaf $\mathscr{F}$, written $\mathrm{Div}_{\mathcal{C}}(\mathscr{F})$, as the $\mathbb{Q}$-linear equivalence class given by $\mathrm{Div}_{\mathcal{C}}(\mathscr{F}) := \mathrm{Div}(\mathscr{F}) + R_{\mathscr{F}}$.

Remark 4.9 (Pull-back of defect divisor under adapted morphisms). In the setup of Definition 4.8, if $U \subseteq X$ is any open set and $\gamma : V \to U$ any adapted morphism, it is clear from the definition that $\gamma^*(R_{\mathscr{F}})$ is an integral Weil divisor on V.

Remark 4.10 (Characterization of the defect divisor). In the setup of Definition 4.8, if $U \subseteq X$ is any open set and $\gamma : V \to U$ is any adapted morphism, Definition 3.5 of $\mathcal{C}$-differentials asserts that there exists an inclusion

$$i : \gamma^{[*]}(\mathscr{F}) \hookrightarrow \mathrm{Sym}^{[d]} \Omega_V^p(\log D_\gamma)$$

which factors into a sequence of inclusions,

$$(4.10.1) \qquad \gamma^{[*]}(\mathscr{F}) \longrightarrow \left(\gamma^*(\mathscr{F}) \otimes \mathscr{O}_V\left(\gamma^* R_{\mathscr{F}}\right)\right)^{**} \xrightarrow{\ j\ } \mathrm{Sym}^{[d]} \Omega_V^p(\log D_\gamma),$$

where the cokernel of j is torsion free in codimension one. The defect divisor $R_{\mathscr{F}}$ is uniquely determined by this property.

We next show that the defect divisor behaves nicely under $\mathcal{C}$-products.

Proposition 4.11 (Behaviour under $\mathcal{C}$-products)**.** *In the setup of Definition 4.8, if $m \in \mathbb{N}^+$ is any number, we have*

$$(4.11.1) \qquad \mathrm{Sym}_{\mathcal{C}}^{[m]} \mathscr{F} = \left(\mathscr{F}^{[m]} \otimes \mathscr{O}_X \big(\lfloor m \cdot R_{\mathscr{F}} \rfloor \big) \right)^{**}$$

$$(4.11.2) \qquad R_{\mathrm{Sym}_{\mathcal{C}}^{[m]} \mathscr{F}} = \underbrace{m \cdot R_{\mathscr{F}} - \lfloor m \cdot R_{\mathscr{F}} \rfloor}_{=:Q}$$

Proof. Let $U \subseteq X$ be any open set, and $\gamma : V \to U$ any finite adapted morphism. Then there exist open sets $U^\circ \subseteq U \cap X_{\mathrm{reg}}$ and $V^\circ \subseteq \gamma^{-1}(U^\circ)$ with $\mathrm{codim}_U U \setminus U^\circ = \mathrm{codim}_V V \setminus V^\circ \geq 2$ such that both the sheaf $\Omega_V^p(\log D_\gamma)$ and the cokernel of the injection

$$j : \gamma^*(\mathscr{F}) \otimes \mathscr{O}_{V^\circ} \big(\gamma^* R_{\mathscr{F}} \big) \hookrightarrow \mathrm{Sym}^d \Omega_V^p(\log D_\gamma)$$

are locally free on V°. Taking m^{th} symmetric products, the inclusion j yields an inclusion of sheaves on V°,

$$(4.11.3) \qquad j^m : \underbrace{\mathrm{Sym}^m \big(\gamma^*(\mathscr{F}) \otimes \mathscr{O}_{V^\circ} \big(\gamma^* R_{\mathscr{F}} \big) \big)}_{=:\mathscr{A}} \hookrightarrow \mathrm{Sym}^{m \cdot d} \Omega_V^p(\log D_\gamma),$$

with locally free cokernel. On V° and U°, respectively, the domain $\mathscr{A}$ of the map j^m can then be written as follows.

$$\begin{aligned}
\mathscr{A} &\cong \mathrm{Sym}^m \big(\gamma^*(\mathscr{F}) \big) \otimes \mathscr{O}_{V^\circ} \big(m \cdot \gamma^* R_{\mathscr{F}} \big) \\
(4.11.4) \qquad &\cong \gamma^* \big(\mathscr{F}^m \big) \otimes \mathscr{O}_{V^\circ} \big(\gamma^*(m \cdot R_{\mathscr{F}}) \big) \\
&\cong \gamma^* \big(\mathscr{F}^m \otimes \mathscr{O}_{U^\circ}(\lfloor m \cdot R_{\mathscr{F}} \rfloor) \big) \otimes \mathscr{O}_{V^\circ} \big(\gamma^* Q \big)
\end{aligned}$$

where Q is the $\mathbb{Q}$-divisor defined in (4.11.2) above. Since Q is effective, Inclusion (4.11.3) gives an inclusion of locally free sheaves on U°,

$$\mathscr{F}^m \otimes \mathscr{O}_{U^\circ}(\lfloor m \cdot R_{\mathscr{F}} \rfloor) \subseteq \mathrm{Sym}_{\mathcal{C}}^{[m]} \mathscr{F}.$$

In particular, there exists an effective Cartier divisor P such that

$$\mathscr{F}^m \otimes \mathscr{O}_{U^\circ}(\lfloor m \cdot R_{\mathscr{F}} \rfloor) \otimes \mathscr{O}_{U^\circ}(P) = \mathrm{Sym}_{\mathcal{C}}^{[m]} \mathscr{F}.$$

Thus

$$\gamma^* \big(\mathscr{F}^m \otimes \mathscr{O}_{U^\circ}(\lfloor m \cdot R_{\mathscr{F}} \rfloor) \otimes \mathscr{O}_{U^\circ}(P) \big) \subseteq \mathrm{Sym}^{m \cdot d} \Omega_V^p(\log D_\gamma).$$

But since the cokernel of j^m is locally free, Equation (4.11.4) implies that $\gamma^*(P) \leq \gamma^*(Q)$. Since $\lfloor Q \rfloor = 0$, this is possible if and only if $P = 0$. This shows Assertion (4.11.1). Assertion (4.11.2) then follows from the characterization of the defect divisor given in Remark 4.10, Equation (4.11.4) and again from the fact that the cokernel of j^m is locally free. $\qquad \square$

As an immediate corollary, we can relate the $\mathcal{C}$-Kodaira-Iitaka dimension of a rank one subsheaf of $\mathrm{Sym}_{\mathcal{C}}^{[d]} \Omega_X^p(\log D)$ to the standard Kodaira-Iitaka dimension.

Corollary 4.12. *In the setup of Definition 4.8, if $m \in \mathbb{N}^+$ is any number, and $\gamma : Y \to X$ any adapted morphism, then there exists a sequence of inclusions as follows:*

$$\gamma^{[*]} \big(\mathrm{Sym}_{\mathcal{C}}^{[m]} \mathscr{F} \big) \longrightarrow \mathrm{Sym}^{[m]} \big(\gamma^*(\mathscr{F}) \otimes \mathscr{O}_Y \big(\gamma^* R_{\mathscr{F}} \big) \big) \longrightarrow \mathrm{Sym}^{[m \cdot d]} \Omega_Y^p(\log D_\gamma).$$

If X is are projective and if γ is proper, then $\kappa_{\mathcal{C}}(\mathscr{F}) \leq \kappa \left(\big(\gamma^*(\mathscr{F}) \otimes \mathscr{O}_Y \big(\gamma^* R_{\mathscr{F}} \big) \big)^{**} \right)$.

Proof. Substitute Equations (4.11.1) and (4.11.2) of Proposition 4.11 into the Sequence (4.10.1) to obtain the sequence of inclusions. The inequality of Kodaira-Iitaka dimensions follows immediately from the definition of $\kappa_\mathcal{C}$ and from the first inclusion. □

The following fact is another immediate consequence of Proposition 4.11 and of Remark 4.5.

Corollary 4.13. *If X is projective, and if $m \in \mathbb{N}^+$ is any number such that $m \cdot R_\mathscr{F}$ is an integral divisor, then $\kappa_\mathcal{C}(\mathscr{F}) = \kappa\big(m \cdot \mathrm{Div}_\mathcal{C}(\mathscr{F})\big)$.* □

5. The $\mathcal{C}$-pair associated with a fibration

If (Y, D) is a logarithmic pair, and $\pi : Y \to Z$ a fibration, we aim to describe the maximal divisor Δ on Z such that $\mathcal{C}$-differentials of the pair (Z, Δ) pull back to logarithmic differentials on (Y, D). Once Δ is found, we will see in Proposition 5.7 that any section in $\mathrm{Sym}^{[m]} \Omega_Y^p(\log D)$ which generically comes from Z is really the pull-back of a globally defined $\mathcal{C}$-differential from downstairs. The construction of Δ is originally found in slightly higher generality in [Cam08, Sec. 3.1], where the $\mathcal{C}$-pair (Z, Δ) is called the *base orbifolde* of the fibration. This section contains a short review of the construction, as well as detailed and self-contained proofs of all results required later.

In order to keep the technical apparatus reasonably small, we restrict ourselves to logarithmic pairs in this section, which is the case we need to handle in the proof of Theorem 1.5. The definitions and results of this section can be generalized in a straightforward manner to the case of arbitrary $\mathcal{C}$-pairs.

5.A. **Definition of the $\mathcal{C}$-base.** The following setup is maintained throughout the present Section 5.

Setup 5.1. *Let (Y, D) be a logarithmic pair, and $\pi : Y \to Z$ a proper, surjective morphism with connected fibers to a normal space.*

Notation 5.2 (Log discriminant locus). The *log discriminant locus* $S \subset Z$ is the smallest closed set S such that π is smooth away from S, and such that for any point $z \in Z \setminus S$, the fiber $Y_z := \pi^{-1}(z)$ is not contained in D, and the scheme-theoretic intersection $Y_z \cap D$ is an snc divisor in Y_z. We decompose

$$S = S_\mathrm{div} \cup S_\mathrm{small},$$

where S_div is a divisor, and $\mathrm{codim}_Z S_\mathrm{small} \geq 2$. The divisor S_div is always understood to be reduced.

Construction and Definition 5.3 ($\mathcal{C}$-base of the fibration, cf. [Cam08, Def 3.2]). Let $S_\mathrm{div} = \bigcup_i \Delta_i$ be the decomposition into irreducible components. We aim to attach multiplicities $a_i \in \mathbb{Q}^{\geq 0}$ to the components Δ_i, in order to define a $\mathcal{C}$-divisor $\Delta := \sum_i a_i \cdot \Delta_i$.

To this end, let $Z^\circ \subseteq Z$ be the maximal open subset such that π is equidimensional over Z°. Set $Y^\circ := \pi^{-1}(Z^\circ)$, and observe that all components Δ_i intersect Z° non-trivially. In particular, none of the divisors $\Delta_i^\circ := \Delta_i \cap Z^\circ$ is empty. Given one component Δ_i, the preimage $\pi^{-1}(\Delta_i^\circ)$ has support of pure codimension one in Y°, with decomposition into irreducible components

$$\mathrm{supp}\big(\pi^{-1}(\Delta_i^\circ)\big) = \bigcup_j E_{i,j}^\circ.$$

If for the given index i, all $E_{i,j}^\circ$ are contained in D, set $a_i := 1$. Otherwise, set

$$b_i := \min\{\text{multiplicity of } E_{i,j}^\circ \text{ in } \pi^{-1}(\Delta_i^\circ) \mid E_{i,j}^\circ \not\subset D\} \quad \text{and} \quad a_i := \frac{b_i - 1}{b_i}.$$

We obtain a divisor $\Delta := \sum_i a_i \cdot \Delta_i$ with $\operatorname{supp}(\Delta) \subseteq S_{\mathrm{div}}$. We call the $\mathcal{C}$-pair (Z, Δ) the $\mathcal{C}$-*base of the fibration* π.

The notion of the $\mathcal{C}$-base of a fibration is not very useful unless the fibration and the spaces have further properties, cf. Remark 5.5.2 below. We will therefore maintain the following assumptions throughout the remainder of the current Section 5.

Assumption 5.4. *In Setup 5.1, assume additionally that the following holds.*

(5.4.1) *The pair (Y, D) is snc. In particular, Y is smooth.*
(5.4.2) *The pair (Z, S_{div}) is snc. In particular, Z is smooth.*
(5.4.3) *Every irreducible divisor $E \subset Y$ with $\operatorname{codim}_Z \pi(E) \geq 2$ is contained in D.*

5.B. The pull-back map for $\mathcal{C}$-differentials and sheaves. Assumptions 5.4 guarantee that $\mathcal{C}$-differentials on (Z, Δ) can be pulled back to logarithmic differentials on (Y, D). In fact, a slightly stronger statement holds.

Proposition 5.5. *Under the Assumptions 5.4, decompose the divisor $D = D^h \cup D^v$ into the "horizontal" components D^h that dominate Z and the "vertical" components D^v that do not. With Δ as in Construction 5.3, the pull-back morphism of differentials extends to a map*[1]

$$(5.5.1) \qquad d\pi^m : \pi^{[*]}\big(\operatorname{Sym}_{\mathcal{C}}^{[m]} \Omega_Z^p(\log \Delta)\big) \to \operatorname{Sym}^{[m]} \Omega_Y^p(\log D^v)$$

for all numbers m and p.

Remark 5.5.2. For Proposition 5.5, it is essential to assume that the pair (Z, S_{div}) is snc. For an instructive example, let Z be a singular space, $\pi : Y \to Z$ a log desingulariazion of Z, let D be the π-exceptional locus, and take $m = 1$ and $p = \dim Z$. In this setting, the assertion of Proposition 5.5 holds if and only if the pair $(Z, \emptyset)$ is log canonical —this is actually the definition of log canonicity. We refer to [GKK08] for more general results in this context.

Proof of Proposition 5.5. Let $U \subseteq Z$ be an open set and let $\sigma \in \Gamma\big(U, \operatorname{Sym}_{\mathcal{C}}^{[m]} \Omega_Z^p(\log \Delta)\big)$ be any section. Its pull-back $\pi^{[*]}(\sigma)$ then gives a rational section of the sheaf $\operatorname{Sym}^{[m]} \Omega_Y^p(\log D^v)$, possibly with poles along the codimension-one components of $\pi^{-1}(S)$. We need to show that $\pi^{[*]}(\sigma)$ does in fact not have any poles. To this end, let $E \subseteq \pi^{-1}(S)$ be any irreducible component with $\operatorname{codim}_Y E = 1$. We will show $\pi^{[*]}(\sigma)$ does not have any poles along E.

If $E \subseteq D^v$, note that

$$\sigma \in \Gamma\big(U, \operatorname{Sym}_{\mathcal{C}}^{[m]} \Omega_Z^p(\log \Delta)\big) \subseteq \Gamma\big(U, \operatorname{Sym}^{[m]} \Omega_Z^p(\log S_{\mathrm{div}})\big).$$

Away from the small set in Y° where $\big(Y, \operatorname{supp} \pi^{-1}(S_{\mathrm{div}})\big)$ is not snc, the usual pull-back morphism for logarithmic differentials, $\pi^*\big(\Omega_Z^p(\log S_{\mathrm{div}})\big) \to \Omega_Y^p\big(\log \operatorname{supp} \pi^{-1}(S_{\mathrm{div}})\big)$ shows that $\pi^{[*]}(\sigma)$ has at most logarithmic poles along E. In particular, $\pi^{[*]}(\sigma)$ does not have any poles along E as a section of $\operatorname{Sym}^{[m]} \Omega_Y^p(\log D^v)$.

---

[1]Since $\Omega_Y^p(\log D^v)$ is locally free, we could write $\operatorname{Sym}^m \Omega_Y^p(\log D^v)$ instead of the more complicated $\operatorname{Sym}^{[m]} \Omega_Y^p(\log D^v)$. We have chosen to keep the square brackets throughout in order to be consistent with the notation used in the remainder of this paper.

It remains to consider the case where $E \not\subset D^v$. In this case, Assumptions 5.4 guarantee that E dominates a component of S_{div}. For simplicity of notation, we may remove from Z all other irreducible components of S, and also the small set where π is not equidimensional. We can then assume without loss of generality that $S = \pi(E)$, and that the restricted morphism $\pi|_{Y \setminus D}$ is surjective and equidimensional. By construction of Δ, the morphism $\pi|_{Y \setminus D}$ is then subadapted, in the sense of Definition 2.7. In particular, Corollary 3.21 shows that $(\pi|_{Y \setminus D})^{[*]}(\sigma)$ is a section of $\mathrm{Sym}^{[m]} \Omega^p_{Y \setminus D}$ without any poles along $E \cap (Y \setminus D)$. $\qquad\square$

As an immediate corollary we see that the $\mathcal{C}$-base of the fibration π is special if the logarithmic pair (Y, D) is special.

Corollary 5.6. *Under the Assumptions 5.4, if the logarithmic pair (Y, D) is special in the sense of Definition 1.2, then the $\mathcal{C}$-pair (Z, Δ) is special in the generalized sense of Definition 4.7.* $\qquad\square$

5.C. **The push-forward map for $\mathcal{C}$-differentials and sheaves.** To properly formulate the assumption that a section in $\mathrm{Sym}^{[m]} \Omega^p_Y (\log D)$ comes from Z "generically", consider the sheaf $\mathscr{B} \subseteq \mathrm{Sym}^{[m]} \Omega^p_Y (\log D)$, defined to be the saturation of the image of the map $d\pi^m$ introduced in (5.5.1). The following proposition then says that any section in $\mathscr{B}$ comes from a globally defined section on Z.

Proposition 5.7. *Under the Assumptions 5.4, if $\mathscr{B} \subseteq \mathrm{Sym}^{[m]} \Omega^p_Y (\log D)$ is the saturation of the image of the map $d\pi^m$ introduced in* (5.5.1), *then the natural injection*

$$\iota : \mathrm{Sym}^{[m]}_{\mathcal{C}} \Omega^p_Z (\log \Delta) \to \pi_*(\mathscr{B})$$

is isomorphic for all numbers m and p.

Remark 5.7.1. Since the morphism π is log smooth over $Z^\circ := Z \setminus S$, the standard sequence of logarithmic differentials on the preimage set $Y^\circ := \pi^{-1}(Z^\circ)$,

$$0 \longrightarrow \pi^*(\Omega^1_Z)|_{Y^\circ} \overset{d\pi|_{Y^\circ}}{\longrightarrow} \Omega^1_Y(\log D)|_{Y^\circ} \longrightarrow \Omega^1_{Y/Z}|_{Y^\circ} \otimes \mathscr{O}_{Y^\circ}(D) \longrightarrow 0,$$

shows that the cokernel of $d\pi|_{Y^\circ}$ is torsion free on Y° and that the image of $d\pi|_{Y^\circ}$ is saturated in $\Omega^1_Y(\log D)|_{Y^\circ}$. By [Har77, II, Ex. 5.16], the same holds for p-forms and their symmetric products. By Proposition 3.3, the sheaves $\mathscr{B}$ and $\pi^{[*]}\big(\mathrm{Sym}^{[m]}_{\mathcal{C}} \Omega^p_Z (\log \Delta)\big)$ therefore agree along Y°.

Proof. Since $\mathrm{Sym}^{[m]}_{\mathcal{C}} \Omega^p_Z (\log \Delta)$ is reflexive and $\pi_*(\mathscr{B})$ is the push-forward of a torsion-free sheaf, hence torsion free, it suffices to prove surjectivity of ι away from any given small set. We can therefore assume without loss of generality throughout the proof that π is equidimensional and that $S_{\mathrm{small}} = \emptyset$.

Let $U \subseteq Z$ be any open set and let $\sigma \in \Gamma\big(U, \pi_*(\mathscr{B})\big)$ be any section. By Remark 5.7.1, the sheaves $\mathscr{B}$ and $\pi^*\big(\mathrm{Sym}^{[m]}_{\mathcal{C}} \Omega^p_Z (\log \Delta)\big)$ agree along $\pi^{-1}(U \cap Z^\circ)$. Since $\pi_*(\mathscr{O}_Y) = \mathscr{O}_Z$, the section σ therefore induces a section

$$\sigma' \in \Gamma\big(U \cap Z^\circ, \mathrm{Sym}^{[m]}_{\mathcal{C}} \Omega^p_Z (\log \Delta)\big).$$

The sections σ and σ' define saturated subsheaves

$$\mathscr{A} \subseteq \mathscr{B}|_{\pi^{-1}(U)} \quad \text{and} \quad \mathscr{A}' \subseteq \mathrm{Sym}^{[m]}_{\mathcal{C}} \Omega^p_Z (\log \Delta)|_U,$$

together with an inclusion $d\pi^m : \pi^*(\mathscr{A}') \to \mathscr{A}$. We need to show that the obvious injective map

(5.7.2) $$d\pi^m : \Gamma(U, \mathscr{A}') \to \Gamma(\pi^{-1}(U), \mathscr{A})$$

is surjective.

As in Construction 5.3, decompose $S_{\mathrm{div}} = \cup \Delta_i$ into irreducible components. For any given index i, let $E_{i,j} \subset Y$ be those divisors that dominate Δ_i. Observe that $\mathrm{Sym}^{[m]} \Omega_Y^p(\log D)$ and $\mathrm{Sym}_{\mathcal{C}}^{[m]} \Omega_Z^p(\log \Delta)$ are both locally free. In particular, the saturated subsheaves $\mathscr{A}$ and $\mathscr{A}'$ are reflexive of rank one, hence invertible, [OSS80, Lem. 1.1.15], and there exist non-negative numbers $c_{i,j}$ such that

$$\mathscr{A} \cong \pi^*(\mathscr{A}') \otimes \mathscr{O}_Y\left(\sum c_{i,j} E_{i,j}\right).$$

With this notation, surjectivity of (5.7.2) is an immediate consequence of the following claim.

Claim 5.7.3. For any index i with $\Delta_i \cap U \neq \emptyset$, there exists an index j such that $E_{i,j}$ appears in $\pi^*(S_{\mathrm{div}})$ with multiplicity strictly larger than $c_{i,j}$.

Application of Claim 5.7.3. Assume that Claim 5.7.3 holds true. We can view σ' as a $\mathcal{C}$-differential with poles along the Δ_i,

$$\sigma' \in \Gamma\left(U, \left(\mathrm{Sym}_{\mathcal{C}}^{[m]} \Omega_Z^p(\log \Delta)\right) \otimes \mathscr{O}_Z(m_i \Delta_i)\right).$$

We need to show that all numbers m_i are zero. Observe that the section σ can be seen as a rational section in $\pi^*(\mathscr{A}')$ whose pole order along any component $E_{i,j}$ is at least m_i times the multiplicity of $E_{i,j}$ in $\pi^*(\Delta_i)$. With Claim 5.7.3, this is possible if and only if $m_i = 0$ for all indices i. In particular, σ lies in the image of the map (5.7.2). Proposition 5.7 is thus shown once Claim 5.7.3 is established.

Proof of Claim 5.7.3. To prove Claim 5.7.3, let any index i be given.

If $a_i = 1$, let j be any other index. By definition of a_i, the divisor $E_{i,j}$ is then contained in D. Let $y \in Y$ be a general point of $E_{i,j}$ and set $z := \pi(y)$. Claim 5.7.3 then reduces to the standard fact that near z and y, respectively, the pull-back of a local generator of $\Omega_Z^p(\log \Delta_i)$ gives a non-vanishing section in $\Omega_Y^p(\log E_{i,j})$. It follows that $c_{i,j} = 0$ for all j, proving Claim 5.7.3 in this case.

If $a_i < 1$, then there exists an index j such that $E_{i,j} \not\subset D$, such that b_i is the multiplicity of $E_{i,j}$ in $\pi^*(\Delta_i)$, and $a_i = \frac{b_i-1}{b_i}$. As above, let $y \in Y$ be a general point of $E_{i,j}$ and set $z := \pi(y)$. Thus, if we set

$$U^\circ := U \setminus \bigcup_{i' \neq i} \Delta_{i'} \quad \text{and} \quad V^\circ := \pi^{-1}(U^\circ) \setminus \bigcup_{j' \neq j} E_{i,j'},$$

then $y \in V^\circ$, $z \in U^\circ$, and the morphism $\pi^\circ := \pi|_{V^\circ}$ is adapted. Now, if the claim was false and $b_i \leq c_{i,j}$, we obtain a morphism

$$(\pi^\circ)^*(\mathscr{A}' \otimes \mathscr{O}_{U^\circ}(\Delta_i)) \to \mathscr{A}|_{V^\circ} \subseteq \mathrm{Sym}^{[m]} \Omega_{V^\circ}^p(\log D).$$

By Definition 3.5 of $\mathcal{C}$-differentials and by Corollary 3.16, this says that $\mathscr{A}' \otimes \mathscr{O}_{U^\circ}(\Delta_i)$ is a subsheaf of $\mathrm{Sym}_{\mathcal{C}}^{[m]} \Omega_{U^\circ}^p(\log \Delta)$, contradicting the assumption that $\mathscr{A}'$ is saturated in $\mathrm{Sym}_{\mathcal{C}}^{[m]} \Omega_{U^\circ}^p(\log \Delta)$.   □

We end this section with a discussion of push-forward properties of subsheaves of $\mathscr{B}$. Given a saturated subsheaf $\mathscr{A} \subseteq \mathscr{B}$ of rank one on Y, with non-negative Kodaira-Iitaka dimension, we can construct a reflexive rank one subsheaf $\mathscr{A}_Z \subseteq \mathrm{Sym}_{\mathcal{C}}^{[m]} \Omega_Z^p(\log \Delta)$ on the

base of the fibration, whose $\mathcal{C}$-Kodaira-Iitaka dimension agrees with the standard Kodaira-Iitaka dimension of $\mathscr{A}$. This sheaf will be used in the proof of Theorem 1.5.

Corollary 5.8. *In the setup of Proposition 5.7, let $\mathscr{A} \subseteq \mathscr{B}$ be a saturated subsheaf of rank one with $\kappa(\mathscr{A}) \geq 0$. Then there exists a saturated, reflexive subsheaf $\mathscr{A}_Z \subseteq \mathrm{Sym}_{\mathcal{C}}^{[m]} \Omega_Z^p(\log \Delta)$ of rank one such that $d\pi^m\left(\pi^*(\mathscr{A}_Z)\right) \subseteq \mathscr{A}$ and $\kappa_{\mathcal{C}}(\mathscr{A}_Z) = \kappa(\mathscr{A})$.*

Proof. If $F \subset Y$ is a general π-fiber, Remark 5.7.1 implies that the restriction $\mathscr{B}|_F$ is trivial. Since a tensor product of the restriction $\mathscr{A}|_F \subseteq \mathscr{B}|_F$ has a non-trivial section by assumption, this implies that $\mathscr{A}|_F$ is also trivial. In particular, the sheaf $\pi_*(\mathscr{A})$ is generically of rank one. Consider the inclusion

$$\pi_*(\mathscr{A}) \subseteq \pi_*(\mathscr{B}) \cong \mathrm{Sym}_{\mathcal{C}}^{[m]} \Omega_Z^p(\log \Delta)$$

and let $\mathscr{A}_Z$ be the saturation of $\pi_*(\mathscr{A})$ in $\mathrm{Sym}_{\mathcal{C}}^{[m]} \Omega_Z^p(\log \Delta)$. It is clear that $d\pi^m\left(\pi^*(\mathscr{A}_Z)\right) \subseteq \mathscr{A}$ holds generically, and since $\mathscr{A}$ is saturated, this inclusion will hold everywhere.

It remains to show that $\kappa_{\mathcal{C}}(\mathscr{A}_Z) = \kappa(\mathscr{A})$. The inequality $\kappa_{\mathcal{C}}(\mathscr{A}_Z) \leq \kappa(\mathscr{A})$ is clear. To prove that $\kappa_{\mathcal{C}}(\mathscr{A}_Z) \geq \kappa(\mathscr{A})$, note that if m' is any number, if $\mathscr{B}'$ is the saturation of the image

$$d\pi^{m \cdot m'} : \pi^*\left(\mathrm{Sym}_{\mathcal{C}}^{[m \cdot m']} \Omega_Z^p(\log \Delta)\right) \to \mathrm{Sym}^{[m \cdot m']} \Omega_Y^p(\log D^v)$$

and $\sigma \in \Gamma\left(Y, \mathscr{A}^{\otimes m'}\right)$ any section, then the inclusion $\mathscr{A}^{\otimes m'} \subseteq \mathscr{B}'$ shows that σ induces a section $\sigma' \in \Gamma\left(Z, \mathrm{Sym}_{\mathcal{C}}^{[m]} \Omega_Z^p(\log \Delta)\right)$ which, away from S_{div}, lies in $\mathscr{A}_Z^{[m']} \subseteq \mathrm{Sym}_{\mathcal{C}}^{[m]} \Omega_Z^p(\log \Delta)$. It follows that σ' is a section in the saturation of $\mathscr{A}_Z^{[m']}$ which, by definition, is precisely $\mathrm{Sym}_{\mathcal{C}}^{[m']} \mathscr{A}_Z$. In summary, we obtain an injection

$$\Gamma\left(Y, \mathscr{A}^{\otimes m'}\right) \to \Gamma\left(Z, \mathrm{Sym}_{\mathcal{C}}^{[m']} \mathscr{A}_Z\right).$$

This shows the equality of Kodaira-Iitaka dimensions. $\square$

PART II. **FRACTIONAL POSITIVITY**

6. THE SLOPE FILTRATION FOR $\mathcal{C}$-DIFFERENTIALS

The results of the following two sections are new to the best of our knowledge. In this section we discuss a weak variant of the Harder-Narasimhan filtration that works on sheaves of $\mathcal{C}$-differentials and takes the extra fractional positivity of these sheaves into account.

If X is a normal polarized variety, $\mathscr{F}$ a reflexive sheaf with slope $\mu(\mathscr{F}) \leq 0$ and $\mathscr{A} \subset \mathscr{F}$ a subsheaf with positive slope, it is clear that the maximally destabilizing subsheaf of $\mathscr{F}$ is a proper subsheaf of positive slope. In particular, there exists a number $p < \mathrm{rank}\, \mathscr{F}$, and a rank-one subsheaf $\mathscr{B} \subset \bigwedge^{[p]} \mathscr{F}$ that is likewise of positive slope $\mu(\mathscr{B}) > 0$. The following proposition gives a similar, but slightly stronger result when $\mathscr{F}$ is replaced with the sheaf of $\mathcal{C}$-differentials.

Proposition 6.1. *Let (X, D) be a $\mathcal{C}$-pair of dimension n, as in Definition 2.2. Assume that X is projective and $\mathbb{Q}$-factorial, and let A be an ample Cartier divisor. If $(K_X + D).A^{n-1} \leq 0$ and if there exists a number m and a reflexive sheaf $\mathscr{A} \subseteq \mathrm{Sym}_{\mathcal{C}}^{[m]} \Omega_X^1(\log D)$ of rank one with $c_1(\mathscr{A}).A^{n-1} > 0$, then there exists a number $p < \dim X$ and reflexive sheaf $\mathscr{B} \subset \mathrm{Sym}_{\mathcal{C}}^{[1]} \Omega_X^p(\log D)$ of rank one with $\mathrm{Div}_{\mathcal{C}}(\mathscr{B}).A^{n-1} > 0$.*

Proof. Let $H \subset X$ be a general hyperplane section, and $\gamma : Y \to X$ an adapted cover with extra branching along H and cyclic Galois group G, as in Proposition 2.9 on page 6. We use Notation 2.10 throughout the proof. Further, let $H_{1,Y}, \ldots, H_{n-1,Y} \in |\gamma^* A|$ be general elements, and consider the associated complete intersection curve

$$C_Y := H_{1,Y} \cap \cdots \cap H_{n-1,Y}.$$

Since Proposition 6.1 remains invariant if we replace A with a positive multiple, we may assume without loss of generality that the Mehta-Ramanathan theorem [Fle84, Thm. 1.2] holds for C_Y, i.e. that taking the Harder-Narasimhan filtration of the sheaf $\Omega_Y^{[1]}(\log D_\gamma)_{\mathrm{adpt}}$ of adapted differentials commutes with restriction to C_Y.

Recall from Remark 2.15 that

$$c_1\left(\Omega_Y^{[1]}(\log D_\gamma)_{\mathrm{adpt}}\right) = c_1\left(\gamma^*(K_X + D)\right).$$

In particular, we have that $c_1\left(\Omega_Y^{[1]}(\log D_\gamma)_{\mathrm{adpt}}\right).C_Y = \gamma^*\left((K_X + D).A^{n-1}\right) \leq 0$. On the other hand, it follows immediately from the definition of $\mathcal{C}$-differentials that there exists an inclusion

$$\gamma^{[*]}(\mathscr{A}) \hookrightarrow \mathrm{Sym}^{[m]} \Omega_Y^{[1]}(\log D_\gamma)_{\mathrm{adpt}}.$$

By assumption, we have that $c_1\left(\gamma^{[*]}(\mathscr{A})\right).C_Y = \gamma^*\left(c_1(\mathscr{A}).A^{n-1}\right) > 0$. In particular, it follows that the vector bundle $\Omega_Y^{[1]}(\log D_\gamma)_{\mathrm{adpt}}|_{C_Y}$ has negative degree, but is not anti-nef. Thus, the maximally destabilizing subsheaf $\mathscr{C}_Y \subset \Omega_Y^{[1]}(\log D_\gamma)_{\mathrm{adpt}}$ has positive slope, $c_1(\mathscr{C}_Y).C_Y > 0$. It follows that $p := \mathrm{rank}\,\mathscr{C}_Y < \dim Y = \dim X$, and that $\mathscr{B}_Y := \det \mathscr{C}_Y$ is a reflexive subsheaf $\mathscr{B}_Y \subset \Omega_Y^{[p]}(\log D_\gamma)_{\mathrm{adpt}}$ of rank one and positive slope.

As a next step, we will construct a sheaf $\mathscr{B} \subset \mathrm{Sym}_{\mathcal{C}}^{[1]} \Omega_X^p(\log D)$ on X. To this end, observe that the line bundle $\mathscr{O}_Y(\gamma^* A)$ is invariant under the action of the cyclic Galois group G on the Picard group. Since the sheaf $\Omega_Y^{[p]}(\log D_\gamma)_{\mathrm{adpt}}$ is also stable under the action of G, it follows immediately from the uniqueness of the maximally destabilizing sheaf that $\mathscr{C}_Y$ and $\mathscr{B}_Y$ are likewise G-stable. If we set $X^\circ := X_{\mathrm{reg}} \setminus \mathrm{supp}(D)$, then

$$\Omega_Y^{[1]}(\log D_\gamma)_{\mathrm{adpt}}|_{\gamma^{-1}(X^\circ)} = \gamma^{[*]}\left(\mathrm{Sym}_{\mathcal{C}}^{[1]} \Omega_Y^1(\log D)|_{X^\circ}\right).$$

Using the G-invariance of $\mathscr{B}_Y$ we obtain a sheaf on X°, say $\mathscr{B}^\circ \subset \mathrm{Sym}_{\mathcal{C}}^{[1]} \Omega_X^p(\log D)|_{X^\circ}$, such that $\gamma^{[*]}(\mathscr{B}^\circ) = \mathscr{B}_Y|_{\gamma^{-1}(X^\circ)}$. Let $\mathscr{B}$ be the maximal extension[2] of $\mathscr{B}^\circ$ in $\mathrm{Sym}_{\mathcal{C}}^{[1]} \Omega_X^p(\log D)$, i.e., the kernel of the natural map

$$\mathrm{Sym}_{\mathcal{C}}^{[1]} \Omega_X^p(\log D) \to \mathrm{Sym}_{\mathcal{C}}^{[1]} \Omega_X^p(\log D)|_{\gamma^{-1}(X^\circ)} \Big/ \mathscr{B}^\circ.$$

It is then clear that $\mathscr{B}$ is reflexive of rank one. In particular, $\mathscr{B}$ is locally free wherever X is smooth.

It remains to show that $\mathrm{Div}_{\mathcal{C}}(\mathscr{B}).A^{n-1} > 0$. To this end, recall from Remark 4.10 that there is an inclusion

$$(6.1.1) \qquad\qquad \left(\gamma^*(\mathscr{B}) \otimes \mathscr{O}_Y(\gamma^* R_{\mathscr{B}})\right)^{**} \hookrightarrow \Omega_Y^{[p]}(\log D_\gamma)$$

whose cokernel is torsion free in codimension one. Since the left hand side of (6.1.1) agrees with $\mathscr{B}_Y$ generically, reflexivity then implies that

$$(6.1.2) \qquad\qquad \mathscr{B}_Y \cong \left(\gamma^*(\mathscr{B}) \otimes \mathscr{O}_Y(\gamma^* R_{\mathscr{B}})\right)^{**}.$$

---

[2]We refer to [Gro60, I.9.4] for a general discussion of the maximal extension, or *prolongement canonique* of subsheaves.

We observe that the sheaf $\mathscr{B}_Y$ is locally free along the general curve C_Y because the construction of $\mathscr{B}_Y$ does not depend on the choice of C_Y. The Isomorphism (6.1.2) then implies the following:

$$
\begin{aligned}
\gamma^*\big(\mathrm{Div}_{\mathcal{C}}(\mathscr{B}).A^{n-1}\big) &= \gamma^*\big((c_1(\mathscr{B}) + c_1(R_{\mathscr{B}})).A^{n-1}\big) && \text{Def. of } \mathrm{Div}_{\mathcal{C}} \\
&= c_1(\mathscr{B}_Y).\big(\gamma^*(A)\big)^{n-1} && \text{Isom. (6.1.2)} \\
&= c_1(\mathscr{B}_Y).C_Y > 0. && \text{Choice of } C_Y
\end{aligned}
$$

It follows that $\mathrm{Div}_{\mathcal{C}}(\mathscr{B}).A^{n-1} > 0$, as claimed. $\qquad\square$

7. BOGOMOLOV-SOMMESE VANISHING FOR $\mathcal{C}$-PAIRS

In this section we generalize the classical Bogomolov-Sommese Vanishing Theorem 1.1 to sheaves of $\mathcal{C}$-differentials on $\mathcal{C}$-pairs with log canonical singularities. To do so, we must restrict ourselves to the case where X is a projective, $\mathbb{Q}$-factorial, and $\dim X \leq 3$. The restriction on the dimension is necessary to apply the Bogomolov-Sommese vanishing theorem for log canonical threefold pairs, [GKK08, Thm. 1.4].

Proposition 7.1 (Bogomolov-Sommese vanishing for 3-dimensional $\mathcal{C}$-pairs)**.** *Let (X, D) be a $\mathcal{C}$-pair, as in Definition 2.2. Assume that X is projective and $\mathbb{Q}$-factorial, that $\dim X \leq 3$ and that the pair (X, D) is log canonical. If $1 \leq p \leq \dim X$ is any number and if $\mathscr{A} \subseteq \mathrm{Sym}_{\mathcal{C}}^{[1]} \Omega_X^p(\log D)$ is a reflexive sheaf of rank one, then $\kappa_{\mathcal{C}}(\mathscr{A}) \leq p$.*

Proof. Let $\mathscr{A} \subseteq \mathrm{Sym}_{\mathcal{C}}^{[1]} \Omega_X^p(\log D)$ be any given reflexive sheaf of rank one. In order to show that $\kappa_{\mathcal{C}}(\mathscr{A}) \leq p$, let $H \subset X$ be a general hyperplane section, and let $\gamma : Y \to X$ be an adapted cover with extra branching along H and cyclic Galois group G, as in Proposition 2.9 on page 6.

As a first step, we show that the pair (Y, D_γ) is log canonical. Since H is general, [KM98, 5.17] implies that

$$
\mathrm{discrep}(X, D + H) = \min\{0, \mathrm{discrep}(X, D)\}.
$$

Since (X, D) is log canonical, $\mathrm{discrep}(X, D + H) \geq -1$, so $(X, D + H)$ is also log canonical. By [KM98, 2.27], the pair $\big(X, D + \frac{N-1}{N}H\big)$ is then log canonical as well, where N is the least common multiple of those $\mathcal{C}$-multiplicities that are not infinity, as in Proposition 2.9. Next, recall from Lemma 2.11 that the log canonical divisor of (Y, D_γ) is expressed as follows,

$$
K_Y + D_\gamma = \gamma^*(K_X + D) + (N-1) \cdot H_\gamma = \gamma^*\left(K_X + D + \frac{N-1}{N}H\right).
$$

Then since $(X, D + \frac{N-1}{N}H)$ is log canonical, so is (Y, D_γ), [KM98, 5.20].

As a next step, recall from Remark 4.9 that the pull-back $\gamma^*(R_{\mathscr{A}})$ of the defect divisor is an integral divisor on Y, and consider the sheaf

$$
\mathscr{B} := \big(\gamma^*(\mathscr{A}) \otimes \mathscr{O}_Y(\gamma^* R_{\mathscr{A}})\big)^{**}.
$$

We have seen in Corollary 4.12 that $\kappa(\mathscr{B}) \geq \kappa_{\mathcal{C}}(\mathscr{A})$, and that there exists an inclusion

$$
\mathscr{B} \hookrightarrow \mathrm{Sym}^{[1]} \Omega_Y^p(\log D_\gamma) = \Omega_Y^{[p]}(\log D_\gamma).
$$

If we show that $\mathscr{B}$ is $\mathbb{Q}$-Cartier, then the Bogomolov-Sommese Vanishing Theorem for log canonical threefold pairs, [GKK08, Thm. 1.4], applies to show that $\kappa(\mathscr{B}) \leq p$. This will yield the claim. To show that $\mathscr{B}$ is $\mathbb{Q}$-Cartier, recall that X is $\mathbb{Q}$-factorial. Since

X is normal, and $\mathscr{A}$ is reflexive of rank one, there exists a divisor D on X such that $\mathscr{A} \cong \mathscr{O}_X(D)$. It follows that

$$\mathscr{B} \cong \mathscr{O}_Y\big(\gamma^*(D + R_{\mathscr{A}})\big).$$

Since a suitable multiple of the $\mathbb{Q}$-divisor $D + R_{\mathscr{A}}$ is Cartier, it follows that $\mathscr{B}$ is $\mathbb{Q}$-Cartier, as claimed. This ends the proof. $\square$

Combining Propositions 6.1 and 7.1, we obtain a useful criterion that can be used to show that $\mathbb{Q}$-Fano $\mathcal{C}$-pairs (X, D) with ample anticanonical class $-(K_X + D)$ have Picard number $\rho(X) > 1$. This will be an essential ingredient in the proof of Theorem 1.5.

Corollary 7.2. *Let (X, D) be a $\mathcal{C}$-pair, as in Definition 2.2. Assume that X is projective and $\mathbb{Q}$-factorial, that $\dim X = n \leq 3$ and that the pair (X, D) is log canonical. Let A be an ample Cartier divisor. If $(K_X + D).A^{n-1} \leq 0$ and if there exists a number m and a reflexive sheaf $\mathscr{A} \subseteq \operatorname{Sym}_{\mathcal{C}}^{[m]} \Omega_X^1(\log D)$ of rank one with $c_1(\mathscr{A}).A^{n-1} > 0$, then $\rho(X) > 1$.*

Proof. Suppose to the contrary that $\rho(X) = 1$. Given $\mathscr{A} \subseteq \operatorname{Sym}_{\mathcal{C}}^{[m]} \Omega_X^1(\log D)$ of rank one with $c_1(\mathscr{A}).A^{n-1} > 0$, let $\mathscr{B} \subset \operatorname{Sym}_{\mathcal{C}}^{[1]} \Omega_X^p(\log D)$ be the reflexive rank one sheaf constructed in Proposition 6.1, where $p < n$. The assumptions that $\rho(X) = 1$ and X is $\mathbb{Q}$-factorial imply that $\mathscr{B}$ is $\mathbb{Q}$-Cartier and a $\mathbb{Q}$-ample sheaf of p-forms. In particular, by Corollary 4.13, $\kappa_{\mathcal{C}}(\mathscr{B}) = n$. But by Proposition 7.1, we know that $\kappa_{\mathcal{C}}(\mathscr{B}) \leq p < n$, a contradiction. It follows that $\rho(X) > 1$. $\square$

PART III. **PROOF OF CAMPANA'S CONJECTURE IN DIMENSION 3**

8. SETUP FOR THE PROOF OF THEOREM 1.5

We prove Theorem 1.5 in the remainder of the paper. The following assumptions are maintained throughout the proof.

Assumption 8.1. *Let $f^\circ : X^\circ \to Y^\circ$ be a smooth projective family of canonically polarized manifolds of relative dimension n, over a smooth quasi-projective base of dimension $\dim Y^\circ \leq 3$. We assume that the family is not isotrivial, $\operatorname{Var}(f^\circ) > 0$, and let $\mu : Y^\circ \to \mathfrak{M}$ be the associated map to the coarse moduli space, whose existence is shown, e.g. in [Vie95, Thm. 1.11]. Arguing by contradiction, we assume that Y° is a special variety.*

Remark 8.2. Since Y° is special, it is not of log-general type. By [KK08b, Thm. 1.1], this already implies that the variation of f° cannot be maximal, i.e., $\operatorname{Var}(f^\circ) < \dim Y^\circ$.

We also fix a smooth projective compactification Y of Y° such that $D := Y \setminus Y^\circ$ is a divisor with simple normal crossings. Furthermore, we fix a compactification $\overline{\mathfrak{M}}$ of $\mathfrak{M}$ and let $\mu^{(0)} : Y \dashrightarrow \overline{\mathfrak{M}}$ be the associated rational map.

9. VIEHWEG-ZUO SHEAVES ON (Y, D)

9.A. **Existence of differentials coming from the moduli space.** Under the assumptions spelled out in Section 8, Viehweg and Zuo have shown in [VZ02, Thm. 1.4(i)] that Y° carries many logarithmic pluri-differentials. More precisely, they prove the fundamental result that there exists a number $m > 0$ and an invertible sheaf $\mathscr{A} \subseteq \operatorname{Sym}^m \Omega_Y^1(\log D)$ whose Kodaira-Iitaka dimension is at least the variation of the family, $\kappa(\mathscr{A}) \geq \operatorname{Var}(f^\circ)$.

We recall a refinement of Viehweg and Zuo's theorem which asserts that the "Viehweg-Zuo sheaf" $\mathscr{A}$ really comes from the coarse moduli space $\mathfrak{M}$. To formulate this result precisely, we use the following notation.

Notation 9.1. Consider the subsheaf $\mathscr{B} \subseteq \Omega^1_Y(\log D)$, defined on presheaf level as follows: if $U \subset Y$ is any open set and $\sigma \in \Gamma(U, \Omega^1_Y(\log D))$ any section, then $\sigma \in \Gamma(U, \mathscr{B})$ if and only if the restriction $\sigma|_{U'}$ is in the image of the differential map $d\mu|_{U'} : \mu^*(\Omega^1_{\mathfrak{M}})|_{U'} \to \Omega^1_{U'}$, where $U' \subseteq U \cap Y^\circ$ is the open subset where the moduli map μ has maximal rank.

Remark 9.2. By construction, it is clear that the sheaf $\mathscr{B}$ is a saturated subsheaf of $\Omega^1_Y(\log D)$. We say that $\mathscr{B}$ is the saturation of $\mathrm{Image}(d\mu)$ in $\Omega^1_Y(\log D)$.

The refinement of Viehweg-Zuo's result is then formulated as follows.

Theorem 9.3 (Existence of differentials coming from the moduli space, [JK09, Thm. 1.5])**.** *There exists a number $m > 0$ and an invertible subsheaf $\mathscr{A} \subseteq \mathrm{Sym}^m \mathscr{B}$ whose Kodaira-Iitaka dimension is at least the variation of the family, $\kappa(\mathscr{A}) \geq \mathrm{Var}(f^\circ)$.* $\qquad\square$

9.B. **Pushing down Viehweg-Zuo sheaves.** In the course of the proof, we will often need to compare Viehweg-Zuo sheaves on different birational models of a given pair. The following elementary lemma shows that Viehweg-Zuo sheaves can be pushed down to minimal models, and that the Kodaira-Iitaka dimension does not decrease in the process.

Lemma 9.4. *Let (Z, Δ) be a $\mathcal{C}$-pair and $\mathscr{A} \subseteq \mathrm{Sym}^{[m]}_{\mathcal{C}} \Omega^p_Z(\log \Delta)$ a reflexive rank one sheaf for some $m, p > 0$. Let $\lambda : Z \dashrightarrow Z'$ be a birational map whose inverse image does not contract any divisor. If Z' is normal and Δ' is the cycle-theoretic image of Δ, then there exists a reflexive rank one sheaf $\mathscr{A}' \subseteq \mathrm{Sym}^{[m]}_{\mathcal{C}} \Omega^p_{Z'}(\log \Delta')$ with $\kappa_{\mathcal{C}}(\mathscr{A}') \geq \kappa_{\mathcal{C}}(\mathscr{A})$.*

Remark 9.4.1. Since λ is birational, it is clear that any number which appears as a coefficient in the divisor Δ', also appears as a coefficient in Δ. Consequently, (Z', Δ') is again a $\mathcal{C}$-pair.
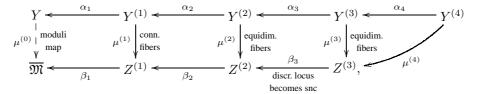
Proof of Lemma 9.4. The assumption that λ^{-1} does not contract any divisor and the normality of Z' guarantee that $\lambda^{-1} : Z' \dashrightarrow Z$ is a well-defined embedding over an open subset $U \subseteq Z'$ whose complement $\overline{Z}' := Z' \setminus U$ has codimension $\mathrm{codim}_{Z'} \overline{Z}' \geq 2$, cf. Zariski's main theorem [Har77, V 5.2]. In particular, $\Delta'|_U = (\lambda^{-1}|_U)^{-1}(\Delta)$. Let $\iota : U \hookrightarrow Z'$ denote the inclusion and set $\mathscr{A}' := \iota_*((\lambda^{-1}|_U)^*\mathscr{A})$. Since $\mathrm{codim}_{Z'} \overline{Z}' \geq 2$, the sheaf $\mathscr{A}'$ is reflexive and agrees with $\mathscr{A}$ on the open set where λ^{-1} is an isomorphism. By reflexivity, we obtain an inclusion of sheaves, $\mathscr{A}' \subseteq \mathrm{Sym}^{[m]}_{\mathcal{C}} \Omega^1_{Z'}(\log \Delta')$. Likewise, we obtain that $\mathrm{Sym}^{[d]}_{\mathcal{C}} \mathscr{A}' \cong \iota_*((\lambda^{-1}|_U)^* \mathrm{Sym}^{[d]}_{\mathcal{C}} \mathscr{A})$ for all $d > 0$. This gives $h^0(Z', \mathrm{Sym}^{[d]}_{\mathcal{C}} \mathscr{A}') \geq h^0(Z, \mathrm{Sym}^{[d]}_{\mathcal{C}} \mathscr{A})$ for all d, hence $\kappa_{\mathcal{C}}(\mathscr{A}') \geq \kappa_{\mathcal{C}}(\mathscr{A})$. $\qquad\square$

As an immediate corollary, we get that the property of being special is inherited by preimages of birational morphisms of pairs.

Corollary 9.5. *Let (Z, Δ) be a $\mathcal{C}$-pair, and let $\lambda : Z \dashrightarrow Z'$ be a birational morphism whose inverse does not contract a divisor. Assume that Z' is normal, and let Δ' be the cycle-theoretic image of Δ. If the $\mathcal{C}$-pair (Z', Δ') is special in the sense of Definition 4.7, and if $E \subset Z$ is any λ-exceptional effective $\mathbb{Q}$-divisor such that $(Z, \Delta + E)$ is a $\mathcal{C}$-pair, then $(Z, \Delta + E)$ is also special.*

Proof. Let $\mathscr{A} \subseteq \text{Sym}_{\mathcal{C}}^{[1]} \Omega_Z^p(\log \Delta + E)$ be a reflexive rank one sheaf for some $p > 0$. Then by Lemma 9.4, there exists a reflexive rank one sheaf $\mathscr{A}' \subseteq \text{Sym}_{\mathcal{C}}^{[m]} \Omega_{Z'}^p(\log \Delta')$ with $\kappa_{\mathcal{C}}(\mathscr{A}') \geq \kappa_{\mathcal{C}}(\mathscr{A})$. But since (Z', Δ') is special, we have that $p > \kappa_{\mathcal{C}}(\mathscr{A}') \geq \kappa_{\mathcal{C}}(\mathscr{A})$. $\square$

10. SIMPLIFICATION: FACTORIZATION OF THE MODULI MAP

In order to simplify the setup of the proof, we aim to replace the pair (Y, D) with a pair that is somewhat easier to manage. To this end, we will now construct a commutative diagram of morphisms between normal varieties,

where β_2, β_3 and all α_i are birational morphisms, and where $Z^{(3)}$ and $Y^{(4)}$ are smooth.

10.A. **Construction of $\mu^{(1)}$ and $Z^{(1)}$.** If necessary, blow up Y outside of Y°, in order to obtain a variety $Y^{(1)}$ which is smooth and where the associated map $Y^{(1)} \dashrightarrow \overline{\mathfrak{M}}$ becomes a morphism. The factorization via a normal space $Z^{(1)}$ is then obtained by Stein factorization.

10.B. **Construction of $Z^{(2)}$ and $Y^{(2)}$.** The map $\mu^{(1)}$ induces a natural, generically injective map from $Z^{(1)}$ into the Chow variety of $Y^{(1)}$,

$$\gamma : Z^{(1)} \dashrightarrow \text{Chow}(Y^{(1)}), \quad z \mapsto (\mu^{(1)})^{-1}(z).$$

Consider a blow-up $\beta_2 : Z^{(2)} \to Z^{(1)}$ such that the composition $\gamma \circ \beta_2 : Z^{(2)} \dashrightarrow \text{Chow}(Y^{(1)})$ becomes a morphism and such that $Z^{(2)}$ is smooth. Let $Y^{(2)}$ be the normalization of the pull-back of the universal family over $\text{Chow}(Y^{(1)})$. Since the normalization morphism is finite, the fiber dimension does not change, and the resulting map $\mu^{(2)}$ will have connected fibers, all of pure dimension $\dim Y^{(2)} - \dim Z^{(2)}$.

10.C. **Construction of $Z^{(3)}$ and $Y^{(3)}$.** Set $D^{(2)} := \text{supp}(\alpha_1^{-1} \circ \alpha_2^{-1}(D))$. Decompose $D^{(2)}$ into "horizontal" components that dominate $Z^{(2)}$ and "vertical" components that do not,

$$D^{(2)} := D^{(2,h)} \cup D^{(2,v)},$$

and set $D_Z := \mu^{(2)}(D^{(2,v)})$. Further, let $\Delta^{(2)} \subset Z^{(2)}$ be the discriminant locus of $\mu^{(2)}$. Recall from Notation 5.2 that this is the smallest closed subset such that $\mu^{(2)}$ is smooth over $Z^{(2)} \setminus \Delta^{(2)}$, and such that the scheme-theoretic intersection $D^{(2)} \cap (\mu^{(2)})^{-1}(z)$ is a proper snc divisor in the fiber $(\mu^{(2)})^{-1}(z)$, for all $z \in Z^{(2)} \setminus \Delta^{(2)}$. Let $\beta_3 : Z^{(3)} \to Z^{(2)}$ be a blow-up such that $Z^{(3)}$ is smooth, and the preimages $\beta_3^{-1}(\Delta^{(2)})$, $\beta_3^{-1}(D_Z)$ and $\beta_3^{-1}(\Delta^{(2)} \cup D_Z)$ are all divisors with snc support. Let $Y^{(3)}$ be the normalization of $Y^{(2)} \times_{Z^{(2)}} Z^{(3)}$. The induced morphism $\mu^{(3)}$ will again have connected, equidimensional fibers of pure dimension. Finally, set $\Delta^{(3)} := \text{supp} \beta_3^{-1}(\Delta^{(2)} \cup D_Z)$.

10.D. **Construction of $Y^{(4)}$.** Set $D^{(3)} := \text{supp}(\alpha_1 \circ \alpha_2 \circ \alpha_3)^{-1}(D)$, let $\alpha_4 : Y^{(4)} \to Y^{(3)}$ be a log resolution of the pair $(Y^{(3)}, D^{(3)})$ and set $D^{(4)} := \alpha_4^{-1}(D^{(3)})$.

10.E. **Extension of the boundary.** If $\alpha := \alpha_1 \circ \alpha_3 \circ \alpha_3 \circ \alpha_4$, we obtain a birational morphism $\alpha : Y^{(4)} \to Y$ where $D = \alpha_*(D^{(4)})$. The obvious fiber product yields a family of canonically polarized varieties over $Y^{(4)} \setminus D^{(4)}$ such that $\mu^{(4)}$ factors the moduli map, and such that the associated map $Z^{(3)} \to \overline{\mathfrak{M}}$ is generically finite.

To simplify the argumentation further and to define a meaningful $\mathcal{C}$-base of the fibration $\mu^{(4)}$, we will now extend the boundary $D^{(4)}$ slightly. To this end, let

$$E^{(4)} \subseteq (\mu^{(4)})^{-1}(\Delta^{(3)})$$

be the union of the irreducible components $E' \subseteq (\mu^{(4)})^{-1}(\Delta^{(3)})$ which are α_4-exceptional and not contained in $D^{(4)}$. By definition of log-resolution, the logarithmic pair $(Y^{(4)}, D^{(4)} + E^{(4)})$ is snc, and Corollary 9.5 asserts that the pair is special.

Remark 10.1. Since $\mu^{(3)}$ is equidimensional, any α_4-exceptional divisor is also $\mu^{(4)}$-exceptional. By construction of $E^{(4)}$, this implies that any $\mu^{(4)}$-exceptional divisor is contained in $D^{(4)} + E^{(4)}$.

10.F. **Summary, Simplification.** Replacing (Y, D) by the pair $(Y^{(4)}, D^{(4)} + E^{(4)})$, if necessary, we can assume without loss of generality for the remainder of the proof that the following holds.

(10.1.1) The moduli map $\mu^\circ : Y^\circ \to \mathfrak{M}$ extends to a morphism $\mu : Y \to \overline{\mathfrak{M}}$.
(10.1.2) There exists a morphism $\pi : Y \to Z$ to a smooth variety Z of positive dimension which factors the moduli map as follows

$$Y \xrightarrow[\text{conn. fibers}]{\pi} Z \xrightarrow[\text{generically finite}]{} \overline{\mathfrak{M}}.$$

with μ above the arrows.

(10.1.3) If $E \subset Y$ is a divisor with $\operatorname{codim}_Z \pi(E) \geq 2$, then $E \subseteq D$.
(10.1.4) There exists an snc divisor $\Delta_{\mathrm{red}} \subset Z$ such that for any point $z \in Z \setminus \Delta_{\mathrm{red}}$, the fiber $Y_z := \pi^{-1}(z)$ is smooth, not contained in D, and the scheme-theoretic intersection $Y_z \cap D$ is a reduced snc divisor in Y_z.

Remark 10.2. Condition (10.1.4) guarantees that the codimension-one part of the discriminant locus of π is an snc divisor in Z. Together with Remark 10.1 or Condition (10.1.3), this implies that the morphism π satisfies the Assumptions of 5.4, which guarantee the existence of a $\mathcal{C}$-base with good pull-back and push-forward properties for $\mathcal{C}$-differentials. We are therefore free to use the results of Sections 5.B and 5.C in our setting.

11. PROOF OF THEOREM 1.5

Let (Z, Δ) be the $\mathcal{C}$-base of the fibration π, as constructed in Section 5, Construction 5.3. By construction, it is clear that $\operatorname{supp}(\Delta) \subseteq \Delta_{\mathrm{red}}$, where $\Delta_{\mathrm{red}} \subset Z$ is the divisor introduced in Section 10.F above. In particular, the divisor Δ has snc support, and the pair (Z, Δ) is dlt, [KM98, Cor. 2.35 and Def. 2.37]. Since the logarithmic pair (Y, D) is special by assumption, Corollary 5.6 implies that (Z, Δ) is a special $\mathcal{C}$-pair in the sense of Definition 4.7.

Next, let $\mathscr{B} \subseteq \Omega_Y^1(\log D)$ be the sheaf introduced in Notation 9.1 above. By Theorem 9.3 there exists an invertible, saturated sheaf

$$\mathscr{A} \subseteq \operatorname{Sym}^m \mathscr{B} \subseteq \operatorname{Sym}^m \Omega_Y^1(\log D)$$

with $\kappa(\mathscr{A}) \geq \mathrm{Var}(f^\circ) = \dim Z$. Since Z is generically finite over $\overline{\mathfrak{M}}$, the sheaf $\mathscr{B}$ is also the saturation of the image of

$$d\pi : \pi^*(\Omega_Z^1) \to \Omega_Y^1(\log D).$$

Corollary 5.8 thus asserts that $\mathscr{A}$ descends to a reflexive subsheaf $\mathscr{A}_Z \subseteq \mathrm{Sym}_{\mathcal{C}}^{[m]} \Omega_Z^1(\log \Delta)$ of rank one, with $\kappa_{\mathcal{C}}(\mathscr{A}_Z) = \dim Z$.

11.A. Case: $\dim Z = 1$. Since Z is a curve, $\mathrm{Sym}_{\mathcal{C}}^{[m]} \Omega_Z^1(\log \Delta)$ is of rank one and therefore equals $\mathscr{A}_Z$. Recall from Remark 4.5 that this asserts that $\kappa_{\mathcal{C}}\big(\mathrm{Sym}_{\mathcal{C}}^{[1]} \Omega_Z^1(\log \Delta)\big) = 1$, contradicting the fact that the $\mathcal{C}$-pair (Z, Δ) is special. This ends the proof in case $\dim Z = 1$.

11.B. Case: $\dim Z = 2$. Applying the the minimal model program to the dlt pair (Z, Δ), we obtain a birational morphism[3] $\lambda : Z \to Z_\lambda$. Set $\Delta_\lambda := \lambda_*(\Delta)$, and recall that Z_λ is $\mathbb{Q}$-factorial, that the pair $(Z_\lambda, \Delta_\lambda)$ is dlt and that it does not admit divisorial contractions.

Let $\mathscr{A}_\lambda \subset \mathrm{Sym}_{\mathcal{C}}^{[m]} \Omega_{Z_\lambda}^1(\log \Delta_\lambda)$ be the Viehweg-Zuo sheaf associated to $\mathscr{A}_Z \subset \mathrm{Sym}_{\mathcal{C}}^{[m]} \Omega_Z^1(\log \Delta)$, as given by Lemma 9.4, and note that $\kappa_{\mathcal{C}}(\mathscr{A}_\lambda) = \dim Z = 2$. For convenience of argumentation, we consider the possibilities for $\kappa(K_Z + \Delta)$ separately.

11.B.1. *Sub-case: $\kappa(K_Z + \Delta) = -\infty$.* In this case, the pair $(Z_\lambda, \Delta_\lambda)$ is either $\mathbb{Q}$-Fano and has Picard number $\rho(Z_\lambda) = 1$, or $(Z_\lambda, \Delta_\lambda)$ admits an extremal contraction of fiber type and has the structure of a proper Mori fiber space.

The case $\rho(Z_\lambda) = 1$, however, is ruled out by Corollary 7.2: if $\rho(Z_\lambda) = 1$, then $K_{Z_\lambda} + \Delta_\lambda$ is anti-ample. If $A \subset Z_\lambda$ is a general hyperplane section, this gives $(K_{Z_\lambda} + \Delta_\lambda).A < 0$. Corollary 7.2 then asserts that $\rho(Z_\lambda) > 1$, contrary to our assumption.

We thus obtain that $\rho(Z_\lambda) > 1$, and that there exists a fiber-type contraction $\pi : Z_\lambda \to B$, where B is a curve. If F is a general fiber of π, then $F \simeq \mathbb{P}^1$, F is entirely contained in the snc locus of $(Z_\lambda, \Delta_\lambda)$, and F intersects Δ_λ transversely. Since the normal bundle N_{F/Z_λ} is trivial and $-(K_F + \Delta_\lambda|_F)$ is nef, Proposition 3.22 asserts that $\mathrm{Sym}_{\mathcal{C}}^{[m']} \Omega_{Z_\lambda}^1(\log \Delta_\lambda)|_F$ is anti-nef, for all numbers $m' \in \mathbb{N}^+$. It follows that

$$\mathrm{Sym}_{\mathcal{C}}^{[m']} \mathscr{A}_\lambda|_F \subset \mathrm{Sym}_{\mathcal{C}}^{[m' \cdot m]} \Omega_{Z_\lambda}^1(\log \Delta_\lambda)|_F$$

is a subsheaf of an anti-nef bundle, hence anti-nef for all $m' \in \mathbb{N}^+$. This clearly contradicts $\kappa_{\mathcal{C}}(\mathscr{A}_\lambda) = \dim Z = 2$.

11.B.2. *Sub-case: $\kappa(K_Z + \Delta) = 0$.* In this case, the classical Abundance Theorem [KM98, Sect. 3.13] asserts that there exists a number $n \in \mathbb{N}^+$ such that

$$(11.0.1) \qquad \mathscr{O}_{Z_\lambda}\big(n \cdot (K_{Z_\lambda} + \Delta_\lambda)\big) \cong \mathscr{O}_{Z_\lambda}.$$

If the boundary divisor Δ_λ is empty, then the $\mathcal{C}$-pair $(Z_\lambda, \Delta_\lambda)$ is a logarithmic pair for trivial reasons, and [KK08b, Prop. 9.1] implies that $\kappa(\mathscr{A}_\lambda) \leq 0$, a contradiction. It follows that Δ_λ is *not* empty.

For sufficiently small $\varepsilon_0 \in \mathbb{Q}^+$, we can therefore consider the dlt pair $(Z_\lambda, (1 - \varepsilon_0)\Delta_\lambda)$. Equation (11.0.1) implies that $-\big(K_{Z_\lambda} + (1 - \varepsilon_0)\Delta_\lambda\big)$ is $\mathbb{Q}$-effective. In particular, we have that $\kappa\big(K_{Z_\lambda} + (1 - \varepsilon_0)\Delta_\lambda\big) = -\infty$. We can therefore run the minimal model program of the pair $\big(Z_\lambda, (1 - \varepsilon_0)\Delta_\lambda\big)$, in order to obtain a birational morphism $\mu : Z_\lambda \to Z_\mu$ to a

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[3]Since Z is a surface, the minimal model program does not involve flips.

normal, $\mathbb{Q}$-factorial variety. Set $\Delta_\mu := \mu_*(\Delta_\lambda)$. As before, Lemma 9.4 gives the existence of a Viehweg-Zuo sheaf $\mathscr{A}_\mu \subseteq \mathrm{Sym}_{\mathcal{C}}^m \, \Omega^1_{Z_\mu}(\log \Delta_\mu)$ with $\kappa_{\mathcal{C}}(\mathscr{A}_\mu) = 2$.

To continue, observe that the map μ is also a minimal model program of the pair $(Z_\lambda, (1-\varepsilon)\Delta_\lambda)$, for any sufficiently small number $\varepsilon \in \mathbb{Q}$. In particular, the pair $\big(Z_\mu, (1-\varepsilon)\Delta_\mu\big)$ is dlt for all ε, its Kodaira dimension is $\kappa\big(K_{Z_\mu} + (1-\varepsilon)\Delta_\mu\big) = -\infty$, and the pair (Z_μ, Δ_μ) is hence dlc [KK08b, 9.4], in particular log canonical. In this setting, the arguments of the previous Section 11.B.1 apply verbatim.

11.B.3. *Sub-case:* $\kappa(K_Z + \Delta) > 0$. The Abundance Theorem guarantees the existence of a regular Iitaka-fibration $\pi : Z_\lambda \to B$, such that $K_{Z_\lambda} + \Delta_\lambda$ is trivial on the general fiber F. The same argumentation as in Section 11.B.1 applies to show that $\mathrm{Sym}_{\mathcal{C}}^{[m']} \mathscr{A}_\lambda$ is anti-nef for all $m' \in \mathbb{N}^+$, contradicting $\kappa_{\mathcal{C}}(\mathscr{A}_\lambda) = \dim Z = 2$. This finishes the proof in the case $\dim Z = 2$ and ends the proof of Theorem 1.5. $\qquad\square$

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