RESEARCH STATEMENT

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1. Overview

Algebraic geometry can be viewed as the study of solutions of systems of polynomials in projective space, i.e. the study of algebraic varieties. One of the guiding problems in algebraic geometry is the classification of algebraic varieties. This is arduous task that will not be completed in the foreseeable future, but working towards this goal leads to lots of new mathematics. I am most interested in how notions of positivity (of divisors and more generally locally free sheaves) help in the classification of varieties.

As a first step in the classification of algebraic varieties, one looks for discrete invariants of a variety, for example, dimension, degree, genus, etc. The degree of a variety depends on how the variety is embedded into projective space, and this gives us our first glimpse into why "positivity" on a variety is important. Let X be a projective variety over the complex numbers and D a divisor on X, that is a linear combination $D = \sum n_i D_i$ where the n_i are integers and the $D_i \subset X$ are subschemes of codimension 1. We want to define what it means for D to be "positive." Perhaps the most appealing definition is to require that D be a hyperplane section under some projective embedding of X, so in particular, D gives an embedding $i : X \hookrightarrow \mathbb{P}^N$. If this is the case, we say that D is very ample. In practice, however, it is much easier to work with ample divisors, that is for some $m > 0$, mD is very ample. Analogously one can define amplitude for line bundles (rank one locally free sheaves) on X. The theory of ample line bundles was worked out in the fifties and early sixties and the fundamental conclusion is that ampleness can be characterized in three equivalent ways:

(1) (Geometric description) $\mathscr{L}$ is ample.
(2) (Cohomological description) For every coherent sheaf $\mathscr{F}$ there exists an m_0 such that $H^i(X, \mathscr{L}^m \otimes \mathscr{F}) = 0$ for all $m \geq m_0$ and $i > 0$, equivalently for every coherent sheaf $\mathscr{F}$ there exists an m_1 such that $\mathscr{L}^m \otimes \mathscr{F}$ is generated by global sections for $m \geq m_1$.
(3) (Numerical description) For every positive dimensional subvariety $V \subseteq X$, $\int_V c_1(\mathscr{L})^{\dim V} > 0$.

If a variety X is given without additional information, it is really difficult to find non-trivial ample line bundles, or for that matter, any non-trivial line bundles. The one we can expect to find is the canonical line bundle ω_X, the determinant of the cotangent bundle Ω_X. Let us consider the case of smooth algebraic curves, or compact Riemann surfaces, and let g be the genus. We have three distinct types of behavior:

- $g = 0$: $X = \mathbb{P}^1$ and $\omega_X \simeq \mathscr{O}_{\mathbb{P}^1}(-2)$ is anti-ample, i.e. the dual of ω_X is ample
- $g = 1$: X is a plane cubic (an elliptic curve) and $\omega_X = \mathscr{O}_X$, so neither ω_X nor its dual is ample
- $g \geq 2$: X is "of general type" and ω_X is ample.

In higher dimensions the situation is more complicated and one considers weaker notions of positivity, such as nef and big. Nefness is a numerical property which depends only on curves in the variety: a line bundle $\mathscr{L}$ on X is nef if $\int_C c_1(\mathscr{L}) \geq 0$ for every irreducible curve $C \subset X$. On the other hand, big is essentially a birational version of ample: let X be a smooth projective variety and $\mathscr{L}$ a line bundle on X; then for $m > 0$, $\mathscr{L}^m$ induces a rational map $\phi_m : X \dashrightarrow \mathbb{P}^N$ (this map is induced by a set of generators of the group of global sections of $\mathscr{L}^m$). If for all $m > 0$, $\mathscr{L}^m$ has no global sections, we say that $\mathscr{L}$ has Kodaira dimension $-\infty$, else the Kodiara dimension of $\mathscr{L}$, denoted $\kappa(\mathscr{L})$, is defined as $\kappa(\mathscr{L}) := \dim \phi_m(X)$ for $m \gg 0$. We say $\mathscr{L}$ is big if $\kappa(\mathscr{L}) = \dim X$. Of particular interest is the Kodaira dimension of the canonical bundle, dentoted $\kappa(X)$, and we say X is of general type if $\kappa(X) = \dim X$. Note that in the case of curves, the Kodaira dimension gives the same trichotomy: if $\kappa(X) < 0$, then X is $\mathbb{P}^1$; if $\kappa(X) = 0$, then X is an elliptic curve; if $\kappa(X) = 1$ then X is a curve of general type.

2. Main Results of Ph.D. Thesis

Beginning in the early 1960s many mathematicians worked to generalize the above introduced notions of positivity to higher rank locally free sheaves (or vector bundles). Given a rank r vector bundle $\mathscr{E}$ on X, where $\dim X = n$, the idea is to pass to the associated projective bundle and reduce to the rank one case. More specifically, one forms the projective space bundle $\pi : \mathbb{P}(\mathscr{E}) \to X$, where $\dim \mathbb{P}(\mathscr{E}) = n + r - 1$, and $\mathbb{P}(\mathscr{E})$ has a tautological line bundle $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$ such that $\pi_* \mathscr{O}_{\mathbb{P}(\mathscr{E})}(1) = \mathscr{E}$ and $\pi^* \mathscr{E} \twoheadrightarrow \mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$. A vector bundle $\mathscr{E}$ on X is ample, respectively nef, if the tautological line bundle $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$ on $\mathbb{P}(\mathscr{E})$ is ample, respectively nef. This definition of an ample vector bundle captures many of the properties of an ample line bundle; for example, the amplitude of $\mathscr{E}$ can be characterized cohomologically.

In my thesis [Jab07] (see also [Jab09]) I introduced a new notion of positivity for vector bundles, which I call quasi-ample. A vector bundle $\mathscr{E}$ on X is quasi-ample if for every non-constant morphism $\gamma : C \to X$, where C is a complete nonsingular curve, $\gamma^* \mathscr{E}$ is ample on C. Clearly, one has the implications

$$\mathscr{E} \text{ is ample} \Rightarrow \mathscr{E} \text{ is quasi-ample } \Rightarrow \mathscr{E} \text{ is nef.}$$

More generally, if $U \subseteq X$ is an open subset, a vector bundle $\mathscr{E}$ on X is quasi-ample with respect to U if for every non-constant morphism $\gamma : C \to X$, where C is a complete nonsingular curve and $\gamma(C) \cap U \neq \emptyset$, $\gamma^* \mathscr{E}$ is ample on C. I prove that many desirable properties of ample vector bundles carry over to quasi-ample vector bundles.

As seen in the previous section, the positivity of the canonical bundle of a smooth projective variety is key in the classification of the variety. One can also try to classify the variety based on the positivity of the tangent bundle T_X or the cotangent bundle Ω_X. First consider the case where the tangent bundle is ample. As seen in the example of curves, $\mathbb{P}^1$ has an ample tangent bundle, and in fact the tangent bundle of $\mathbb{P}^n$ for any n is ample. A fundamental theorem of Mori states that $\mathbb{P}^n$ is the only smooth projective variety of dimension n with ample tangent bundle [Mor79]. Furthermore, his argument shows that if the anti-canonical line bundle is big and the tangent bundle is quasi-ample then it is indeed ample and hence $X \simeq \mathbb{P}^n$.

If X is a smooth projective variety one can also study the positivity of the cotangent bundle Ω_X. When Ω_X is ample, X has some very nice properties. Such varieties are hyperbolic[1] and the theme is that they exhibit strong forms of properties known or expected for hyperbolic varieties. For example, Lang has conjectured that X is hyperbolic if and only if all subvarieties of X are on general type – if we require the stronger condition that Ω_X is ample then all subvarieties of X are of general type. If we assume only that Ω_X is quasi-ample, X still has some nice properties, namely X does not contain rational or elliptic curves, and there do not exist non-constant maps $f : A \to X$ from an abelian variety A. Requiring that the cotangent bundle be ample is certainly a very strong property, and for a long time there were very few examples of such varieties, although they were expected to be reasonably abundant. Schneider [Sch86] showed that certain Kodaira surfaces,

[1]Fix an ample class h, then X is (algebraically) hyperbolic if there exists an $\epsilon > 0$ such that for every finite map $\gamma : C \to X$ from a smooth curve C, $2g(C) - 2 \geq \epsilon(C \cdot \gamma^* h)$

fibered by a smooth pencil $S \to C$ of curves have ample cotangent bundle. In my thesis I generalized his result in two respects - I allowed singular fibers, and I considered towers of morphisms.

Theorem 2.1. *Let*

$$X^n \xrightarrow{f_n} X^{n-1} \xrightarrow{f_{n-1}} X^{n-2} \xrightarrow{f_{n-2}} \cdots\cdots \xrightarrow{f_3} X^2 \xrightarrow{f_2} X^1$$

where each X^i is a smooth projective variety over $\mathbb{C}$ of dimension i, and each $f_i : X^i \to X^{i-1}$ is a smooth, projective morphism with $\mathrm{Var}(f_i) = i-1$. Then, Ω_{X^n} is nef and quasi-ample with respect to an open U_n and $\mathscr{O}_{\mathbb{P}(\Omega_{X^n})}(1)$ is a big line bundle on $\mathbb{P}(\Omega_{X^n})$.

The above U_i are constructed by removing, at each step, the points where the Kodaira-Spencer map $\alpha_i : H^0(X_p^{i+1}, f_{i+1}^* T_{X^i}|_{X_p^{i+1}}) \to H^1(X_p^{i+1}, T_{X_p^{i+1}})$ is not injective. I also considered towers of varieties

$$X^n \xrightarrow{f_n} X^{n-1} \xrightarrow{f_{n-1}} X^{n-2} \xrightarrow{f_{n-2}} \cdots\cdots \xrightarrow{f_3} X^2 \xrightarrow{f_2} X^1$$

where the f_i are not necessarily smooth, and showed that the sheaf of differential forms with logarithmic poles along D, $\Omega^1_{X^n}(\log D)$, is quasi-ample with respect to an open set, where D is a suitable divisor taking in account the singularities of the given morphisms. One of the key ingredients of these proofs is the positivity of $(f_i)_*(\omega^m_{X^i/X^{i-1}})$ [Kol87].

Furthermore, I showed that such towers of varieties exist. The existence of a projective surface mapping nonisotrivially to a curve (as in the hypothesis of Schneider's theorem) comes from the fact that moduli space of genus g curves, $\mathscr{M}_g$, contains complete curves. Starting with a curve of genus 2 and using a construction of Kodaira, I constructed a tower of varieties which satisfies the hypotheses of Theorem 2.1.

3. Families over Special Varieties

I next describe work with Stefan Kebekus [JK10b] and [JK10a]. This work is related to generalizations of Shafarevich Hyperbolicity. The classical setting is as follows: let B be a smooth projective curve of genus g, and Δ a divisor on B. Consider families over B, i.e. flat projective morphisms $f : X \to B$ with connected fibers, where X is a smooth projective variety. The family is called isotrivial if $X_a \simeq X_b$ for general $a, b \in B$, and is admissible if it is not isotrivial and $f : X \setminus f^{-1}(\Delta) \to B \setminus \Delta$ is smooth. In 1962, Shaferevich conjectured that for a fixed (B, Δ) and $q \geq 2$, there exists only finitely many isomorphism classes of admissible families of curves of genus q, and if $2g - 2 + \#\Delta \leq 0$, then there exist no such families. The conjecture was proven by Parshin [Par68] for $\Delta = \emptyset$ and by Arakelov [Ara71] in general. This conjecture has been generalized to higher dimensions and consists of three parts: boundedness, rigidity and hyperbolicity.

We saw earlier that the Kodaira dimension of a variety is an important invariant in the classification of varieties; we now extend that notion to pairs (Y, D). Let Y be a smooth projective variety and D a simple normal crossing divisor[2] on Y. The pair (Y, D) is of log-general type if $\kappa(K_Y + D) = \dim(Y)$. Refining the distinction between "log-general type" and "other," Campana introduced the class of special varieties. For any $1 \leq p \leq \dim Y$ and for any rank one subsheaf $\mathscr{A} \subseteq \Omega^p_Y(\log D)$, Bogomolov-Sommese vanishing states that the Kodaira dimension of $\mathscr{A}$ is at most p, i.e. $\kappa(\mathscr{A}) \leq p$. A pair (Y, D) is called special if strict inequality holds. Clearly, if (Y, D) is a pair of log-general type, then (Y, D) is not special. As a less trivial example, Campana has shown that if $\kappa(Y) = 0$ or if Y is rationally connected, then (Y, D) for $D = 0$ is special [Cam04, 5.1, 2.28]. Conjecturally, special varieties have a number of good topological, geometrical and arithmetic properties. In particular, Campana has made the following conjecture.

[2] Roughly this means that D is reduced, the components of D are smooth and intersect "as transversely as possible."

Conjecture 3.1. [Cam08, Conj. 12.19] *Let Y be a smooth projective variety and D a simple normal crossing divisor on Y. If $f^\circ : X^\circ \to Y^\circ := Y \setminus D$ is a smooth family of canonically polarized complex varieties, then* $\mathrm{Var}(f^\circ) = 0$.

The variation of f°, $\mathrm{Var}(f^\circ)$, denotes the number of effective parameters of the birational equivalence classes of the fibers. If $\mathrm{Var}(f^0) = 0$ then $X_y \simeq X_t$ for general $y, t \in Y$; if $\mathrm{Var}(f^\circ) = \dim Y^\circ$, we say that f has maximum variation. With $f^\circ : X^\circ \to Y^\circ$ as above, this conjecture is related to the following conjectures, all of which can be seen as generalizations of the Shafarevich Hyperbolicity:

Conjecture 3.2. [Vie01] *If* $\mathrm{Var}(f^\circ) = \dim Y^0$*, then* $\kappa(Y^\circ) = \dim Y^\circ$.

Conjecture 3.3. [KK08a] *If* $\kappa(Y^\circ) = 0$*, then* f° *is isotrivial.*

Conjecture 3.4. [KK08a] *If* $\kappa(Y^\circ) = -\infty$*, then* $\mathrm{Var}(f^\circ) \leq \dim Y^\circ - 1$.

Conjecture 3.1 been proven by Kovács in the case of $\dim Y^\circ = 1$ [Kov96], [Kov00], and Conjectures 3.2-3.4 have been solved by Kebekus and Kovács in the case of $\dim Y^0 \leq 3$ [KK08a, KK08b, KK08c].

In [KK08a, KK08b, KK08c] an essential ingredient is the notion of a "Viehweg-Zuo" sheaf. Given $f^\circ : X^\circ \to Y^\circ$ as above, Viehweg and Zuo show that for some $m > 0$, the sheaf $\mathrm{Sym}^m \Omega^1_Y(\log D)$ contains a line bundle $\mathscr{A}$ of Kodaira dimension at least the variation of the family, i.e. $\kappa(\mathscr{A}) \geq \mathrm{Var}(f^\circ)$, [VZ02, 1.4(i)]. The Viehweg-Zuo sheaf $\mathscr{A}$ was crucial in the study of hyperbolicity properties of manifolds that appear as bases of families of maximal variation and has been used to show that any minimal model program of the pair (Y, D) factors the moduli map, [KK08a, KK08b, KK08c]. In spite of its importance, little is known about further properties of the sheaf $\mathscr{A}$. For example, it is unclear how the Viehweg-Zuo construction of positive sheaves of differentials behaves under base change. In [JK10b] we refine Veihweg and Zuo's result and show that the Viehweg-Zuo sheaf $\mathscr{A}$ comes from the coarse moduli space associated to $f^\circ : X^\circ \to Y^\circ$. As an immediate corollary we prove Campana's Conjecture 3.1 when Y° is a surface.

In [JK10a] using a more advanced line of argumentation we show

Theorem 3.5. [JK10a] *Conjecture 3.1 is true if* $\dim Y^0 \leq 3$.

In addition to the Viehweg-Zuo sheaves, a second ingredient is the notion of a $\mathcal{C}$-pair (also called an *orbifoldes géométriques* by Campana). A $\mathcal{C}$-pair is a pair (Z, Δ) where Z is a normal variety and Δ is a $\mathbb{Q}-$divisor of the form $\Delta = \sum_i \frac{n_i-1}{n_i}\Delta_i$, where the Δ_i are irreducible and reduced Weil divisors on Z and $n_i \in \mathbb{N}^+ \cup \{\infty\}$. Following [Cam08] we consider sheaves of $\mathcal{C}$-differentials, $\mathrm{Sym}^{[m]}_{\mathcal{C}} \Omega^p_Z(\log \Delta) \subseteq \mathrm{Sym}^{[m]} \Omega^p_Z(\log\lceil\Delta\rceil)$. We show the existence of a slope filtration for $\mathcal{C}$-differentials and prove a variant of Bogomolov-Sommese Vanishing for $\mathcal{C}$-pairs of dimension less than or equal to three.

With these tools in place, the idea of the proof of Theorem 3.5 is as follows: If $\mathrm{Var}(f^\circ) > 0$, we reduce the problem to the case where we have a surjective morphism $\pi : (Y, D) \to (Z, \Delta)$, where (Z, Δ) is a $\mathcal{C}-$pair of dimension equal to $\mathrm{Var}(f)$, and the divisor Δ encodes information about D and the singular fibers of π. We then show that the Viehweg-Zuo sheaf $\mathscr{A} \subseteq \mathrm{Sym}^m \Omega^1_Y(\log D)$ induces a subsheaf $\mathscr{A}_Z \subseteq \mathrm{Sym}^{[m]}_{\mathcal{C}}(\log \Delta)$, whose $\mathcal{C}-$Kodaira dimension equals the dimension of Z, $\kappa_{\mathcal{C}}(\mathscr{A}_Z) = \dim Z$. Further analysis of this sheaf $\mathscr{A}_Z$ yields a contradiction.

4. Positivity on Toric Varieties

I next describe ongoing work with Sandra Di Rocco and Greg Smith [DRJS10], in which we investigate positivity of vector bundles on toric varieties. If X is a smooth toric variety and $\mathscr{L}$ is a line bundle on X then it is a classical result that $\mathscr{L}$ is ample if and only if $\mathscr{L}|_C$ is ample for every invariant curve C on X. In particular, on smooth toric varieties, quasi-ample and ample are equivalent for line bundles, and more so, one must only check positivity on finitely many curves. Moreover, the same statement is true for toric vector bundles. A toric (or equivariant) vector bundle on a toric variety X is a locally free sheaf of finite rank on X with a T-action on the corresponding geometric vector bundle $\mathbb{V}(\mathscr{E}) = \mathrm{Spec}(\mathrm{Sym}(\mathscr{E}))$ such that the projection $\phi : \mathbb{V}(\mathscr{E}) \to X$ is equivariant and the torus T acts linearly on the fibers of ϕ. If $\mathscr{E}$ is a toric vector bundle on a toric variety, then $\mathscr{E}$ is ample if and only if $\mathscr{E}|_C$ is ample for every invariant curve C on X [HMP10, 2.1]. Another classical result states that a line bundle on a smooth toric variety is ample if and only if it is very ample and it is nef if and only if it is generated by global sections. Hering, Mustaţă and Payne showed that a nef toric vector bundle on a smooth projective toric variety is not necessarily globally generated [HMP10, 4.15], but asked the following questions:

Question 4.1. *If X is a smooth projective toric variety and $\mathscr{E}$ is an ample toric vector bundle on X, then is $\mathscr{E}$ generated by global sections?*

Question 4.2. *If X is a smooth projective toric variety and $\mathscr{E}$ is an ample toric vector bundle on X, then is $\mathscr{E}$ very ample?*

Using this description of global sections of a toric vector bundle given by Klyachko, [Kly89], we conjecture that an ample toric vector bundle is both 1-jet spanned and 1-jet ample, as defined by Beltrametti, Di Rocco and Sommese, [BDRS99]. Since a 1-jet ample vector bundle is very ample, [BDRS99, 4.2], we then would arrive at an affirmative answer to both Questions 4.1 and 4.2.

Note that if X is a toric variety that is not smooth, there are ample equivariant vector bundles that are not globally generated [HMP10, 4.16], and if X is a smooth projective toric variety, a very ample vector bundle that is not toric, need not be 1-jet ample [BDRS99, 4.3].

5. Further Research

Concerning the positivity of vector bundles, I am interested in the following questions. Let X be a smooth projective variety:

Question 5.1. *If ω_X is quasi-ample, then is ω_X ample?*

If one assumes ω_X is big in addition to quasi-ample, then ω_X is indeed ample. This conjecture is one case of the following conjecture of Serrano [Ser95]: If $\mathscr{L}$ is a quasi-ample line bundle on X, a smooth projective variety of dimension n, then $\omega_X \otimes \mathscr{L}^m$ is ample for any $m > n + 1$. Serrano [Ser95] solved the case where $\dim X = 2$ and partly solved the case where $\dim X = 3$. In [CCP05], the case of $\dim X = 3$ is settled with the possible exception of X being a Calabi-Yau with $L \cdot c_2 = 0$, as well as some partial results to higher dimensions.

Question 5.2. *If X has a quasi-ample cotangent bundle Ω_X, is Ω_X ample?*

This is the natural extension of Question 5.1, although less is known. In general a quasi-ample vector bundle is not ample, as seen in an example of Mumford [Har70]. As noted earlier, if we consider the dual case, then X having a quasi-ample tangent bundle T_X implies that T_X is indeed ample, and so $X \simeq \mathbb{P}^n$.

As a first step, one could consider the case where Ω_X is quasi-ample and $\mathscr{O}_{\mathbb{P}(\Omega_X)}(1)$ is a big line bundle on $\mathbb{P}(\Omega_X)$ (or even the stronger condition that Ω_X is ample with respect to an open [Vie83], [Jab09]). For a general vector bundle, quasi-ample and ample with respect to an open is not enough to guarantee ampleness, as seen in Ramanujam's example [Har70]. Schneider [Sch86] shows, that if X a surface with quasi-ample cotangent bundle Ω_X, and the Chern classes of Ω_X satisfy a certain positivity property, then Ω_X is ample. This positivity of the Chern classes is related to, but stronger than $\mathscr{O}_{\mathbb{P}(\Omega_X)}(1)$ is a big line bundle on $\mathbb{P}(\Omega_X)$, and it would be interesting to extend his result to varieties of higher dimension.

In a slightly different direction, one also can investigate positivity of vector bundles on a specific class of varieties. As mentioned above if X is a smooth toric variety and $\mathscr{E}$ is an equivariant vector bundle on X, then $\mathscr{E}$ is ample if and only if $\mathscr{E}|_C$ is ample for every invariant curve C on X [HMP10, 2.1]. In their proof, Hering, Mustaţă and Payne show the following: Suppose X is a projective variety with the property that and there exist finitely many curves $C_1, \dots, C_l \subset X$ such that a vector bundle $\mathscr{E}$ is nef if and only $\mathscr{E}|_{C_i}$ is nef for $i = 1, \dots, l$. Then $\mathscr{E}$ is ample if and only if all $\mathscr{E}|_{C_i}$ are ample.

Question 5.3. *What other varieties satisfy this property?*

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