ON A CHARACTER SUM PROBLEM OF H. COHN

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Abstract. Let f be a complex valued function on a finite field F such that $f(0) = 0$, $f(1) = 1$, and $|f(x)| = 1$ for $x \neq 0$. Cohn asked if it follows that f is a nontrivial multiplicative character provided that $\sum_{x \in F} f(x)\overline{f(x+h)} = -1$ for $h \neq 0$. We prove that this is the case for finite fields of prime cardinality under the assumption that the nonzero values of f are roots of unity.

1. Introduction

Let p be prime and let F_{p^k} be the finite field with p^k elements. Let $f : F_{p^k}^{\times} \to \mathbf{C}$ be a nontrivial multiplicative character, and extend f to a function on F_{p^k} by letting $f(0) = 0$. It is then easy to see that the following holds:

$$\sum_{x \in F_{p^k}} f(x)\overline{f(x+h)} = \begin{cases} -1 & \text{if } h \neq 0 \\ p^k - 1 & \text{if } h = 0 \end{cases} \tag{1.1}$$

Cohn asked (see p. 202 in [3]) if the converse is true in the following sense: if a function $f : F_{p^k} \to \mathbf{C}$ satisfies

$$f(0) = 0, \ f(1) = 1, \ \text{and } |f(x)| = 1 \text{ for } x \neq 0 \tag{1.2}$$

and equation 1.1, does it follow that f is a multiplicative character?

The problem has recently received some attention. In [2], Choi and Siu proved that the converse is not true for $k > 1$. One of the arguments given is quite pretty, and proceeds as follows: Let λ be a linear automorphism of F_{p^k} so that $\lambda(1) = 1$. If f satisfies 1.1 and 1.2, so does f composed with λ. Now, if f is an injective multiplicative character then the converse being true implies that f composed with λ must be an injective multiplicative character. On the other hand, a simple counting argument shows that the number of possible λ's is greater than the number of injective characters.

However, the case $k = 1$ remains unresolved. In [1], Biro proved that there are only finitely many functions satisfying equation 1.1 and 1.2

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for each p. Biro also solved the following "characteristic p" version of the problem ([1], Theorem 2):

Theorem (Biro)**.** *Let p be a prime, let F_p be the finite field with p elements, and $F \supset F_p$ any field of characteristic p. Assume that there is given an $a_i \in F$ for every $i \in F_p$ such that $a_0 = 0, a_1 = 1, a_i \neq 0$ for $i \neq 0$, and*

$$\sum_{i \in F_p^\times} \frac{a_{i+j}}{a_i} = -1$$

for every $j \in F_p^\times$. Then $a_i = i^A$ for every $i \in F_p$ with some $1 \leq A \leq p-2$.

Using this Biro deduces that the converse holds for functions taking values in $\{-1, 0, 1\}$.[1] In fact, if m is coprime to p, then the case of the nonzero values of f being m-th roots of unity can be deduced in a similar way: Let $\mathfrak{O}$ be the ring of integers in $\mathbf{Q}(e^{2\pi i/m})$, and let $\mathfrak{P} \subset \mathfrak{O}$ be a prime ideal lying above p. The result then follows from the theorem by letting $F = \mathfrak{O}/\mathfrak{P}$ and noting that m-th roots of unity are distinct modulo p. (Since $|f(x)| = 1$ for $x \neq 0$ we have $\overline{f(x)} = 1/f(x)$.)

The aim of this paper is to show that the converse is true for the case $k = 1$, under the additional assumption that the nonzero values of $f : F_p \to \mathbf{C}$ are m-th roots of unity, including the case $p|m$. We begin by giving a proof that does not depend on Biro's result for the case $(m, p) = 1$, and we then show how to modify the argument for the general case.

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2. Preliminaries

In what follows we assume that p is odd since the case $p = 2$ is trivial.

We will use the following conventions: if a function f takes values in $\mathbf{C}$ and $\sigma \in \mathrm{Aut}(\mathbf{C}/\mathbf{Q})$, then we let f^σ be the function defined by $f^\sigma(x) = \sigma(f(x))$. We regard $\psi(x) = e^{2\pi i x/p}$ as a nontrivial additive character of F_p. For an integer t, ψ_t will denote the character $\psi_t(x) = \psi(tx)$. By ζ_m we denote the m-th root of unity $\zeta_m = e^{2\pi i/m}$.

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[1] There appears to be several independent proofs of this result, see the introduction in [2].

Let m be even and large enough so that all nonzero values of f are m-th roots of unity, and write $m = np^k$, where $(n, p) = 1$. Let $K = \mathbf{Q}(\zeta_n)$, $L = K(\zeta_p, \zeta_{p^k})$, and let $G = \mathrm{Gal}(L/\mathbf{Q})$, $H = \mathrm{Gal}(L/K)$ denote the Galois groups of $L/\mathbf{Q}$ and L/K. By $\mathfrak{O}_K$ and $\mathfrak{O}_L$ we will denote the ring of integers in K respectively L.

The "Gauss sum"

$$G(f, \psi) = \sum_{x=0}^{p-1} f(x)\psi(x)$$

is clearly an algebraic integer. As in the case of classical Gauss sums, the absolute value of $G(f, \psi)$ can easily be determined:

Lemma 1. *If f satisfies 1.1, then*

$$|G(f, \psi_t)| = \begin{cases} \sqrt{p} & \textit{if } t \not\equiv 0 \mod p, \\ 0 & \textit{if } t \equiv 0 \mod p. \end{cases}$$

Proof. We have

$$|G(f, \psi_t)|^2 = \sum_{x,y \in F_p} f(x)\overline{f(y)}\psi(t(x-y)) = \sum_{x,h \in F_p} f(x)\overline{f(x+h)}\psi(-th)$$

$$= \psi(0) \sum_{x \in F_p} f(x)\overline{f(x)} + \sum_{h \in F_p^\times} \psi(-th) \sum_{x \in F_p} f(x)\overline{f(x+h)}$$

$$= p - 1 - \sum_{h \in F_p^\times} \psi(-th) = \begin{cases} p & \text{if } t \not\equiv 0 \mod p, \\ 0 & \text{if } t \equiv 0 \mod p, \end{cases}$$

□

The action of complex conjugation on K is given by an element in G, and since G is abelian, equation 1.1 is G-invariant. I.e., if f satisfies 1.2, so does f^σ for all $\sigma \in G$. But if $\sigma \in G$ then $\sigma(G(f, \psi)) = G(f^\sigma, \psi_t)$, where $\sigma(\zeta_p) = \zeta_p^t$. Since f^σ also satisfies 1.1, we find that $|G(f^\sigma, \psi_t)| = p^{1/2}$, and hence the $\mathbf{Q}$-norm of $G(f, \psi)$ is a power of p. The factorization of the principal ideal $G(f, \psi)\mathfrak{O}_L$ thus consists only of prime ideals $\mathfrak{P}_L | p$.

It is well known that $\mathbf{Q}(\zeta_{p^k})/\mathbf{Q}$ is totally ramified over p, and that $\mathbf{Q}(\zeta_n)/\mathbf{Q}$ does not ramify at p if $(n, p) = 1$. Comparing ramification indices gives that if $\mathfrak{P}_K$ is a prime ideal in $\mathfrak{O}_K$ that divides p, then $\mathfrak{P}_K$ is totally ramified in L. In particular, if $\mathfrak{P}_L$ is any prime ideal in the ring of integers in $\mathfrak{O}_L$ that lies above p, then $\sigma(\mathfrak{P}_L) = \mathfrak{P}_L$ for all $\sigma \in H$.

Let $l = \max(1, k)$. Then H consists of elements σ_t such that

$$\sigma_t(\zeta_{p^l}) = \zeta_{p^l}^t, \quad \sigma_t(\zeta_n) = \zeta_n.$$

Choose t so that σ_t generates H. Applying σ_t to the principal ideal

$$G(f,\psi)\mathfrak{O}_L = \prod_{\mathfrak{P}_L | p} \mathfrak{P}_L{}^{\eta(\mathfrak{P}_L)}$$

we find that

$$\sigma_t(G(f,\psi)\mathfrak{O}_L) = \sigma_t(\prod_{\mathfrak{P}_L | p} \mathfrak{P}_L{}^{\eta(\mathfrak{P}_L)}) = \prod_{\mathfrak{P}_L | p} \mathfrak{P}_L{}^{\eta(\mathfrak{P}_L)} = G(f,\psi)\mathfrak{O}_L$$

and hence $\sigma_t(G(f,\psi)) = uG(f,\psi)$ for some unit u.

Since the absolute value of any complex embedding of $G(f,\psi)$ equals $\sqrt{p}$, we find that all conjugates of $u = \sigma(G(f,\psi))/G(f,\psi)$ has absolute value one. Hence u is in fact a root of unity, and there are integers a, b such that

$$\sigma_t(G(f,\psi)) = \zeta_{p^l}^a \zeta_n^b G(f,\psi). \tag{2.1}$$

3. The case $(m,p) = 1$

Since f is fixed by H we find that $\sigma_t(G(f,\psi)) = G(f,\psi_t)$, and equation 2.1 can, after the change of variable $x \to t^{-1}x$, be written as

$$\sum_{x=1}^{p-1} f(x)\psi(x) = \zeta_p^{-a}\zeta_n^{-b} \sum_{x=1}^{p-1} f(t^{-1}x)\psi(x). \tag{3.1}$$

Lemma 2. *If f takes values in n-th roots of unity for $x \not\equiv 0 \mod p$ and equation 3.1 holds then $a \equiv 0 \mod p$.*

Proof. From 3.1 we obtain that

$$\sum_{i=1}^{p-1} A_i \zeta_p^i = \sum_{i=0}^{p-1} B_i \zeta_p^i \tag{3.2}$$

where $A_i = f(i)$ and $B_i = \zeta_n^{-b} f(t^{-1}(i+a))$. (Note that $B_{p-a} = 0$.) Since $1 = -\sum_{i=1}^{p-1} \zeta_p^i$ we may rewrite 3.2 as

$$\sum_{i=1}^{p-1} A_i \zeta_p^i = \sum_{i=1}^{p-1} (B_i - B_0)\zeta_p^i. \tag{3.3}$$

The elements $\{\zeta_p, \zeta_p^2, \zeta_p^3, \ldots \zeta_p^{p-1}\}$ are linearly independent over K, hence $A_i = B_i - B_0$. From lemma 1 we have $\sum_{x=0}^{p-1} f(x) = 0$, which implies that $\sum_{i=1}^{p-1} A_i = 0$, as well as $\sum_{i=0}^{p-1} B_i = 0$. Therefore,

$$0 = \sum_{i=1}^{p-1} A_i = \sum_{i=1}^{p-1}(B_i - B_0) = \sum_{i=0}^{p-1} B_i - pB_0 = -pB_0.$$

But $B_0 = \zeta_n^{-b} f(t^{-1}(0+a))$ which is nonzero unless $a \equiv 0 \mod p$. $\square$

Thus

$$\sum_{x=1}^{p-1} f(x)\psi(x) = \zeta_n^{-b} \sum_{x=1}^{p-1} f(t^{-1}x)\psi(x) \tag{3.4}$$

and the linear independence of $\{\zeta_p, \zeta_p^2, \zeta_p^3, \dots \zeta_p^{p-1}\}$ over K implies that

$$f(t^{-1}x) = f(x)\zeta_n^b$$

for all $x \neq 0$. Thus

$$f(t^{-k}) = f(t^{-(k-1)})\zeta_n^b = \dots = f(1)\zeta_n^{kb} = \zeta_n^{kb}.$$

Taking $k = p-1$ we find that ζ_n^b is a $(p-1)$-th root of unity, and that f is a multiplicative character.

4. The general case

In this case $m = np^k$ where $(n,p) = 1$ and $k > 0$. We will need the following:

Lemma 3. *If $a_i \in K$ and $\sum_{i=0}^{p^k-1} a_i \zeta_{p^k}^i \in K(\zeta_p)$ then*

$$\sum_{i=0}^{p^k-1} a_i \zeta_{p^k}^i = \sum_{j=0}^{p-1} a_{p^{k-1}j} \zeta_p^j \tag{4.1}$$

Proof. We may assume that $k > 1$. The minimal polynomial for ζ_{p^k} (over K as well as over $\mathbf{Q}$) is given by

$$\frac{x^{p^k} - 1}{x^{p^{k-1}} - 1} = 1 + x^{p^{k-1}} + x^{2p^{k-1}} + \dots + x^{(p-1)p^{k-1}}.$$

Hence, by letting $\tilde{i} \in [0, p^{k-1} - 1]$ be a representative of i modulo p^{k-1}, we can rewrite the left hand side of equation 4.1 as

$$\sum_{i=0}^{(p-1)p^{k-1}-1} (a_i - a_{(p-1)p^{k-1}+\tilde{i}})\zeta_{p^k}^i$$

with no further relations among the $\zeta_{p^k}^i$'s, and thus

$$\sum_{i=0}^{(p-1)p^{k-1}-1} (a_i - a_{(p-1)p^{k-1}+\tilde{i}})\zeta_{p^k}^i \in K(\zeta_p)$$

if and only if $a_i - a_{(p-1)p^{k-1}+\tilde{i}} = 0$ for all i not congruent to zero modulo p^{k-1}. □

Recall from equation 2.1 (note that $l = k$ since $k \geq 1$) that

$$\sigma_t(G(f,\psi)) = \zeta_{p^k}^a \zeta_n^b G(f,\psi).$$

Let $\tilde{G} = \zeta_{p^k}^s G(f,\psi)$ where $\sigma_t(\zeta_{p^k}^s)/\zeta_{p^k}^s = \zeta_{p^k}^{-a}$. (Such an s exists as $\sigma_t(\zeta_{p^k}^s)/\zeta_{p^k}^s = \zeta_{p^k}^{(t-1)s}$, and $t \not\equiv 1 \mod p$ since σ_t generates H.) We then have

$$\sigma_t(\tilde{G}) = \sigma_t(\zeta_{p^k}^s G(f,\psi))$$
$$= \sigma_t(\zeta_{p^k}^s)\sigma_t(G(f,\psi)) = \sigma_t(\zeta_{p^k}^s)\zeta_{p^k}^a \zeta_n^b G(f,\psi)) = \zeta_n^b \tilde{G}.$$

The following lemma shows that $\tilde{G}$ must transform by a nontrivial n-th root of unity:

Lemma 4. *There is no integer s such that $\zeta_{p^k}^s G(f,\psi) \in K$.*

Proof. We first assume that $\zeta_{p^k}^s = 1$. Let $G(f,\psi)\mathfrak{O}_L = \prod_{\mathfrak{P}_L|p} \mathfrak{P}_L{}^{\eta(\mathfrak{P}_L)}$ be the factorization of the principal ideal $G(f,\psi)\mathfrak{O}_L$. Since p does not ramify in K, we have $p\mathfrak{O}_K = \prod_{\mathfrak{P}_K|p} \mathfrak{P}_K$, and hence $p\mathfrak{O}_L = \prod_{\mathfrak{P}_L|p} \mathfrak{P}_L{}^e$ where e is the ramification index of $\mathfrak{P}_K$ in L.

Since $\psi(x) = \zeta_p^x$ is congruent to 1 modulo $\mathfrak{P}_L$ for all x, we find that

$$G(f,\psi) = \sum_{x=0}^{p-1} f(x)\psi(x) \equiv \sum_{x=1}^{p-1} f(x) \mod \mathfrak{P}_L$$

for all $\mathfrak{P}_L|p$. Now, since $f(0) = 0$, we have $\sum_{x=1}^{p-1} f(x) = G(f,\psi_0)$ and by lemma 1, $G(f,\psi_0) = 0$. Thus $G(f,\psi) \in \mathfrak{P}_L$ for all $\mathfrak{P}_L|p$, i.e., $\eta(\mathfrak{P}_L) > 0$ for all $\mathfrak{P}_L|p$. But if $G(f,\psi) \in K$ then $e|\eta(\mathfrak{P}_L)$ for all $\mathfrak{P}_L|p$, and since complex conjugation permutes the set of primes of $\mathfrak{O}_L$ that lies above p, and

$$p = G(f,\psi)\overline{G(f,\psi)},$$

we get that $\mathfrak{P}_L{}^{2e}|p\mathfrak{O}_L$ for all $\mathfrak{P}_L$, contradicting that the ramification index is e.

For the general case, the previous argument carries through by noting that $\zeta_{p^k}^s$ is a unit (and thus multiplication of $G(f,\psi)$ by $\zeta_{p^k}^s$ does not change the ideal factorization) and that $G(f,\psi) \in \mathfrak{P}_L$ if and only if $\zeta_{p^k} G(f,\psi) \in \mathfrak{P}_L$. □

Since σ_t has order $p^{k-1}(p-1)$ and $(n,p) = 1$ we find that ζ_n^b must be a nontrivial $(p-1)$-th root of unity. Hence there exists a nontrivial multiplicative character χ of $F_p^\times$ such that $\chi(t^{-1}) = \zeta_n^b$. But

$\sigma_t(G(\chi,\psi)) = G(\chi,\psi_t) = \chi(t^{-1})G(\chi,\psi)$ and thus

$$\delta = \frac{\tilde{G}}{G(\chi,\psi)}$$

is σ_t-invariant and hence an element of K. Moreover, $|\delta| = 1$ (for all complex embeddings) since $|\tilde{G}| = |G(\chi,\psi)| = p^{1/2}$.

Write $f(x) = f_1(x)f_2(x)$ where $f_1(x)$ takes values in p^k-th roots of unity and $f_2(x)$ takes values in n-th roots of unity. We will show that $f_1(x)$ must be constant.

Lemma 5. *Let*

$$a_i = \sum_{x:\zeta_{p^k}^s f_1(x)\psi(x)=\zeta_{p^k}^i} f_2(x)$$

If

$$\zeta_{p^k}^s \sum_{x=1}^{p-1} f(x)\psi(x) = \delta \sum_{x=1}^{p-1} \chi(x)\psi(x), \tag{4.2}$$

then $|a_i| = 0$ *unless* $i = p^{k-1}j$ *for* $j = 1, 2, \ldots, p-1$, *in which case* $|a_i| = 1$. *In particular,* $\zeta_{p^k}^s f_1(x)\psi(x)$ *ranges over all nontrivial p-th roots of unity.*

Proof. Collecting terms in 4.2 according to the values of $\zeta_{p^k}^s f_1(x)\psi(x)$, we obtain

$$\sum_{i=0}^{p^k-1} a_i\zeta_{p^k}^i = \delta \sum_{i=1}^{p-1} \chi(i)\zeta_p^i \in K(\zeta_p). \tag{4.3}$$

Clearly $a_i \in K$ and $a_i \neq 0$ for at most $p-1$ values of i. Letting $A_i = a_{p^{k-1}i}$ we may, by lemma 3, write equation 4.3 as

$$\sum_{i=0}^{p-1} A_i\zeta_p^i = \delta \sum_{i=1}^{p-1} \chi(i)\zeta_p^i.$$

Since $1 = -\sum_{i=1}^{p-1} \zeta_p^i$ we get that

$$\sum_{i=1}^{p-1}(A_i - A_0)\zeta_p^i = \sum_{i=0}^{p-1} A_i\zeta_p^i = \delta \sum_{i=1}^{p-1} \chi(i)\zeta_p^i$$

and hence $A_i - A_0 = \delta\chi(i)$ for all i.

Since $a_i \neq 0$ for at most $p-1$ values of i, $A_0 \neq 0$ implies that $A_j = 0$ for some $j \neq 0$, and thus $|A_0| = |\delta\chi(j) - A_j| = 1$. Since

$$0 = \delta \sum_{i=1}^{p-1} \chi(i) = \sum_{i=1}^{p-1}(A_i - A_0) = \sum_{i=0}^{p-1} A_i - pA_0,$$

we find that $|\sum_{i=0}^{p-1} A_i| = p|A_0| = p$. On the other hand, $|\sum_{i=0}^{p-1} A_i| \leq \sum_{x=1}^{p-1} |f_2(x)| = p-1$. Thus $A_0 = 0$, and it follows that $A_i = \delta\chi(i)$ for $i \neq 0$. In other words, $a_{p^{k-1}j} = A_j = \delta\chi(j)$ for $j = 1, 2, \ldots, p-1$, and since there are at most $p-1$ nonzero values among the a_i's, the remaining ones must all be equal to zero. □

Now, the lemma gives that $\zeta_{p^k}^s f_1(1)\psi(1) = \zeta_{p^k}^s \zeta_p$ is a p-th root of unity, hence p^{k-1} must divide s, and the nonzero values of $f_1(x)\psi(x)$ are thus distinct p-th roots of unity. Replacing ψ by ψ_r, for $r \not\equiv 0 \mod p$, in the previous argument gives that $f_1(x)\psi(rx)$ also ranges over distinct p-th roots of unity. On the other hand, if $f_1(x)$ is not constant, then there exists $r \not\equiv 0 \mod p$ such that the set $\{f_1(x)\psi_r(x)\}_{x=1}^{p-1}$ contains strictly less than $p-1$ elements. (If $f_1(x_1) \neq f_1(x_2)$, write $f_1(x_1) = \zeta_p^{y_1}$, $f_1(x_2) = \zeta_p^{y_2}$ and take $r \equiv -(y_2 - y_1)(x_2 - x_1)^{-1} \mod p$.) Hence $f_1(x)$ must be constant, and since $f_1(1) = 1$, we find that the nonzero values of $f(x)$ are in fact n-th roots of unity. The result has thus been reduced to the case $(m, p) = 1$.

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