12.4.9.

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + 2\beta \frac{\partial u}{\partial t}\ , \quad 0 < \beta < 1\ , \quad t > 0$$

$$\text{Randvillkor : } \begin{cases} u(0,t) = 0 \\ u(\pi,t) = 0 \end{cases}$$

$$\text{Begynnelsevillkor : } \begin{cases} u(x,0) = f(x) \\ \dfrac{\partial u}{\partial t}(x,0) = 0 \end{cases}$$

Separera variablerna: $u(x,t) = X(x)T(t)$.

$$X''(x)T(t) = X(x)T''(t) + 2\beta X(x)T'(t)$$

Dividera med $X(x)T(t)$.

$$\frac{X''(x)}{X(x)} = \frac{T''(t) + 2\beta T'(t)}{T(t)} = \text{konstant} = \lambda \quad .$$

$$\begin{cases} X''(x) - \lambda X(x) = 0 \\ T''(t) + 2\beta T'(t) - \lambda T(t) = 0 \end{cases}$$

Bestäm först de lösningar som uppfyller

differentialekvationen och randvillkoren .

$$\lambda > 0,\ \lambda = \mu^2,\ \mu \quad R.$$

$$X''(x) - \mu^2 X(x) = 0$$

Lösningarna ges av $X(x) = A_1 e^{\mu x} + B_1 e^{-\mu x}$.

$$\lambda = 0$$

$$X''(x) = 0$$

$$X(x) = A_2 x + B_2$$

$\lambda < 0,\ \lambda = -\mu^2,\ \ \mu \quad R.$

$$X''(x) + \mu^2 X(x) = 0$$

$$X(x) = A_3 \cos \mu x + B_3 \sin \mu x$$

Substitutionen ger att randvillkoren kan skrivas

$0 = u(0, t) = \mathrm{X}(0)\mathrm{T}(\mathrm{t})$

$0 = u(\pi, t) = \mathrm{X}(\pi)\mathrm{T}(\mathrm{t})$

Dessa samband skall gälla för alla t .

Detta innebär att: $0 = X(0)$, $0 = X(\pi)$.

$\lambda > 0$

$$\begin{cases} 0 = X(0) = A_1 - B_1 \\ 0 = X(\pi) = A_1 e^{\mu\pi} - B_1 e^{-\mu\pi} \end{cases}$$

$$A_1 = B_1 = 0$$

Endast den triviala lösningen: $u = 0$.

$$\lambda = 0$$

$$\begin{cases} 0 = X(0) = B_2 \\ 0 = X(\pi) = A_2\pi + B_2 \end{cases}$$

$$A_2 = B_2 = 0$$

Endast den triviala lösningen: $u = 0$.

$$\lambda < 0$$

$$\begin{cases} 0 = X(0) = A_3 \\ 0 = X(\pi) = A_3 \cos \mu\pi + B_3 \sin \mu\pi \end{cases}$$

$$A_3 = 0 \qquad B_3 \sin \mu\pi = 0$$

Icke triviala lösningar erhålles då: $\mu = n$.

$$X(x) = B_3 \sin nx$$

$$T''(t) + 2\beta T'(t) + n^2 T(t) = 0$$

$$(D^2 + 2\beta D + n^2)T(t) = 0$$

$$((D+\beta)^2 + n^2 - \beta^2)T(t) = 0$$

$$T(t) = e^{-\beta t}(C_3 \cos\sqrt{n^2 - \beta^2}\,t + D_3 \sin\sqrt{n^2 - \beta^2}\,t)$$

$$X(x)T(t) = e^{-\beta t}(B_3 C_3 \cos\sqrt{n^2 - \beta^2}\,t + B_3 D_3 \sin\sqrt{n^2 - \beta^2}\,t)\sin nx$$

$$u(x,t) = \sum_{n=1}^{\infty} e^{-\beta t}(a_n \cos\sqrt{n^2 - \beta^2}\,t + b_n \sin\sqrt{n^2 - \beta^2}\,t)\sin nx$$

$$\frac{\partial u(x,t)}{\partial t} = \sum_{n=1}^{\infty} -\beta e^{-\beta t}(a_n \cos\sqrt{n^2-\beta^2}\,t + b_n \sin\sqrt{n^2-\beta^2}\,t)\sin nx +$$

$$+\sum_{n=1}^{\infty} \sqrt{n^2-\beta^2}\,e^{-\beta t}(-a_n \sin\sqrt{n^2-\beta^2}\,t + b_n \cos\sqrt{n^2-\beta^2}\,t)\sin nx$$

Begynnelsevillkoren ger:

$$\begin{cases} f(x) = u(x,0) = \displaystyle\sum_{n=1}^{\infty} a_n \sin nx \\ 0 = \dfrac{\partial u}{\partial t}(x,0) = \displaystyle\sum_{n=1}^{\infty} (\sqrt{n^2-\beta^2}\,b_n - \beta a_n)\sin nx \end{cases}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$$b_n = \frac{\beta a_n}{\sqrt{n^2-\beta^2}} = \frac{\beta}{\sqrt{n^2-\beta^2}}\frac{2}{\pi}\int_0^{L} f(x)\sin nx\,dx$$

$$u(x,t) = \sum_{n=1}^{\infty} e^{-\beta t} a_n\left(\cos\sqrt{n^2-\beta^2}\,t + \frac{\beta}{\sqrt{n^2-\beta^2}}\sin\sqrt{n^2-\beta^2}\,t\right)\sin nx$$