12.5.12.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad , \quad 0 < x < \pi$$

$$\text{Villkor}: \begin{cases} \dfrac{\partial u}{\partial x}(0,y) = 0 \\ \dfrac{\partial u}{\partial x}(\pi,y) = 0 \\ u(x,0) = f(x) \\ u(x,y) \ \text{ begränsad då } y \to \infty \end{cases}$$

Separera variablerna: $u(x,y) = X(x)\,Y(y)$.

$$X''(x)Y(y) + X(x)Y''(y) = 0$$

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \text{konstant} = \lambda \quad .$$

$$\begin{cases} X''(x) - \lambda X(x) = 0 \\ Y''(y) + \lambda Y(y) = 0 \end{cases}$$

$\lambda > 0,\ \lambda = \mu^2,\ \mu \quad R.$

$$X''(x) - \mu^2 X(x) = 0$$

Lösningarna ges av $X(x) = A_1 e^{\mu x} + B_1 e^{-\mu x}$.

$$\lambda = 0$$

$$X''(x) = 0$$

$$X(x) = A_2 x + B_2$$

$$\lambda < 0,\ \lambda = -\mu^2,\ \ \mu \quad R.$$

$$X''(x) + \mu^2 X(x) = 0$$

$$X(x) = A_3 \cos \mu x + B_3 \sin \mu x$$

Substitutionen ger att randvillkoren kan skrivas

$$0 = \frac{\partial u}{\partial x}(0,y) = X'(0)Y(y)$$

$$0 = \frac{\partial u}{\partial x}(\pi,y) = X'(\pi)Y(y)$$

Dessa samband skall gälla för alla y .

Detta innebär att: $0 = X'(0)$, $0 = X'(\pi)$.

$$\lambda > 0$$

$$X'(x) = \mu(A_1 e^{\mu x} - B_1 e^{-\mu x})$$

$$0 = X'(0) = \mu(A_1 - B_1)$$

$$0 = X'(\pi) = \mu(A_1 e^{\mu\pi} - B_1 e^{-\mu\pi})$$

$$A_1 = B_1 = 0$$

Endast den triviala lösningen: $u = 0$.

$$\lambda = 0$$

$$X'(x) = A_2$$

$$\begin{cases} 0 = X'(0) = A_2 \\ 0 = X'(L) = A_2 \end{cases}$$

$$X(x) = B_2 \qquad Y(y) = C_2 y + D_2$$

$u(x,y)$ begränsad då $y \to \infty$ $\qquad C_2 = 0.$

$$\lambda < 0$$

$$X'(x) = \mu(-A_3 \sin \mu x + B_3 \cos \mu x)$$

$$\begin{cases} 0 = X'(0) = \mu(B_3) \\ 0 = X'(\pi) = \mu(-A_3 \sin \mu\pi + B_3 \cos \mu\pi) \end{cases}$$

$$B_3 = 0 \qquad A_3 \sin \mu\pi = 0$$

Icke triviala lösningar erhålles då: $\mu = n$.

$$X(x) = A_3 \cos nx$$

$$Y(y) = C_3 e^{ny} + D_3 e^{-ny}$$

$u(x,y)$ begränsad då $y \to \infty$ $\qquad C_3 = 0.$

$$Y(y) = D_3 e^{-ny}$$

$$u(x,y) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \; e^{-ny}$$

$$f(x) = u(x,0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$u(x,y) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad e^{-ny}$$