Matching complexes on grids

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This manuscript is a supplement to, but not part of, the thesis *Simplicial Complexes of Graphs* [1]. For notation and basic concepts, we refer to the thesis.

1 Useful results

A simplicial complex is $VD(d)$ if the d-skeleton is vertex-decomposable (VD). Refer to the complex as $VD^+(d)$ if the complex is $VD(d)$ and admits a decision tree with all evasive sets of dimension d. Note that a $VD(d)$ simplicial complex has homotopical depth at least d and that a $VD^+(d)$ complex is homotopy equivalent to a wedge of spheres of dimension d [1].

For a graph G, $\mathsf{M}(G)$ is the simplicial complex of matchings contained in G.

Theorem 1.1 ([1, Th. 11.1]) *Let G be a graph on the vertex set V. Suppose that there is a partition $\{U_1, \ldots, U_t\}$ of V such that $|U_i| \le 3$ for each i and such that $G(U_i)$ is isomorphic to either K_1, K_2, K_3, or $\mathsf{Pa}_3 = ([3], \{12, 23\})$. Suppose further that whenever $G(U_i)$ is of the form $(\{a, b, c\}, \{ab, bc\})$ (thus isomorphic to Pa_3), the vertex b is not adjacent to any other vertices in G than a and c. Then $\mathsf{M}(G)$ is $VD(\nu)$, where*

$$\nu = \left\lceil \frac{|V| - t}{2} \right\rceil - 1.$$

In particular, this holds whenever $\{U_1, \ldots, U_t\}$ is a clique partition of G such that each U_i has size at most three. $\Box$

Theorem 1.2 ([1, Cor. 11.15]) *Let G be a graph on a vertex set V of size n such that $n \bmod 3 \in \{0, 1\}$. Suppose that there is a partition $\{U_1, \ldots, U_t\}$ of V such that $|U_i| = 3$ for each $i < t$ and $|U_t| = n \bmod 3$ and such that $G(U_i)$ is isomorphic to either K_3 or $\mathsf{Pa}_3 = ([3], \{12, 23\})$ for each $i < t$. Then $\mathsf{M}(G)$ has nonvanishing homology in dimension ν_n.* $\Box$

For $n \ge 0$, define Pa_n as the graph with edge set $\{i(i+1) : i \in [0, n-2]\}$; we define Pa_n as the empty graph if $n \in \{0, 1\}$. $\mathsf{M}(\mathsf{Pa}_n)$ is isomorphic to the complex of stable sets in Pa_{n-1}; Kozlov [3, Prop. 4.6] determined the homotopy type of this complex.

Proposition 1.3 ([1]) *Let $n \geq 0$ and $\nu_n = \lceil \frac{n-4}{3} \rceil$. Then $\mathsf{M}(\mathsf{Pa}_n)$ is $VD(\nu_n)$ and*

$$\mathsf{M}(\mathsf{Pa}_n) \simeq \begin{cases} \text{point} & \text{if } n \equiv 2 \pmod 3 \\ S^{\nu_n} & \text{if } n \not\equiv 2 \pmod 3). \ \square \end{cases}$$

2 Grids

The *d-dimensional grid* Grid_d is the infinite graph with vertex set $\mathbb{Z}^d$ and with an edge between every pair of vertices on Euclidean distance 1; a is adjacent to $a \pm \mathbf{e}_i$, where $\mathbf{e}_i$ is the i^{th} unit vector. We think about the direction of the d^{th} unit vector $\mathbf{e}_d = (0, \ldots, 0, 1)$ as "up"; edges of the form $\{a, a + \mathbf{e}_d\}$ are *vertical*, whereas other edges are *horizontal*. By convention, the 0-dimensional grid is a single vertex.

For a sequence $\mathsf{m} = (m_1, \ldots, m_d)$ of positive integers, define $\mathsf{Grid}(\mathsf{m})$ as the induced subgraph of Grid_d consisting of all vertices $(a_1, \ldots, a_d)$ such that $a_i \in [1, m_i]$ for $i \in [1, d]$. The number of vertices in $\mathsf{Grid}(\mathsf{m})$ is $\prod_{i=1}^{d} m_i$.

In this section, we review some enumerative results about the matching complex on $\mathsf{Grid}(\mathsf{m})$ and compute the depth of $\mathsf{M}(\mathsf{Grid}(\mathsf{m}))$ in the case that some m_i is divisible by three. First, we state a classical result about the number of perfect matchings in $\mathsf{Grid}(m, n)$.

Theorem 2.1 ([2, 5]) *Let m and n be integers, not both odd. Then the number of perfect matchings in $\mathsf{Grid}(m, n)$ equals the square root of*

$$\prod_{j=1}^{m} \prod_{k=1}^{n} \left(2 \cos\left(\frac{\pi j}{m+1} \right) + 2i \cos\left(\frac{\pi k}{n+1} \right) \right). \ \square$$

Using the techniques described in Propp [4, Sec. 4], one easily establishes the following result about the f-vector of $\mathsf{M}(\mathsf{Grid}(\mathsf{m}))$.

Theorem 2.2 *Let $d \geq 1$ and let $\mathsf{m}' = (m_1, \ldots, m_{d-1})$ be a sequence of positive integers. Then the generating function*

$$G_{\mathsf{m}'}(q, t) = \sum_{n \geq 1} f(\mathsf{M}(\mathsf{Grid}(\mathsf{m}', n)), q) t^n$$

for the f-polynomial of $\mathsf{M}(\mathsf{Grid}(\mathsf{m}', n))$ is a rational function in q and t. In particular, the generating function

$$-G_{\mathsf{m}'}(-1, t) = \sum_{n \geq 1} \tilde{\chi}(\mathsf{M}(\mathsf{Grid}(\mathsf{m}', n))) t^n$$

for the reduced Euler characteristic of $\mathsf{M}(\mathsf{Grid}(\mathsf{m}', n))$ is a rational function in t. $\square$

We now consider topological properties of $\mathsf{M}(\mathsf{Grid}(\mathsf{m}))$. It is probably very hard to determine the homotopy type of this complex. We confine ourselves to computing the depth and the connectivity degree in certain cases.

Lemma 2.3 *Let* $\mathsf{m} = (m_1, \ldots, m_d)$ *be a sequence of positive integers such that* $n = \prod_{i=1}^{d} m_i$ *is congruent to* 0 *or* 1 *modulo* 3*. Then the shifted connectivity degree of* $\mathsf{M}(\mathsf{Grid}(\mathsf{m}))$ *is at most* $\lceil (n-4)/3 \rceil$*. In fact,* $\mathsf{M}(\mathsf{Grid}(\mathsf{m}))$ *has nonvanishing homology in dimension* $\lceil (n-4)/3$*.*

Proof. It is well-known and easy to prove that $\mathsf{M}(\mathsf{Grid}(\mathsf{m}))$ contains a Hamiltonian path. Partition this path into $3\lfloor n/3 \rfloor$ short paths of length three and, if $n \bmod 3 = 1$, an additional single vertex. This partition satisfies the conditions of Corollary 1.2; hence we are done. □

Theorem 2.4 *Let* $\mathsf{m} = (m_1, \ldots, m_d)$ *be a sequence of positive integers such that* $n = \prod_{i=1}^{d} m_i$ *is divisible by three. Then* $\mathsf{M}(\mathsf{Grid}(\mathsf{m}))$ *is* $VD(n/3-1)$*.*

Proof. By symmetry, we may assume that m_d is divisible by three. Write $\Sigma = \mathsf{M}(\mathsf{Grid}(\mathsf{m}))$. Let Y be the set of edges $\{a, a+\mathbf{e}_i\}$ such that $a = (a_1, \ldots, a_d)$ satisfies $a_d \equiv 2 \pmod 3$ and such that $i \in [1, d-1]$; thus all edges in Y are horizontal. We want to show that $\Sigma(B, Y \setminus B)$ is $VD(n/3-1)$ for each $B \subseteq Y$. Obviously, $\Sigma(B, Y \setminus B)$ is empty whenever the edges in B do not form a matching; thus assume that the edges in B do form a matching.

Let U be the set of vertices not contained in any edge in B; $|U| = n - 2|B|$. Let G be the induced subgraph of $\mathsf{Grid}(m_1, \ldots, m_d) \setminus (Y \setminus B)$ on the vertex set U. Partition U as

$$\{U_{\mathsf{a}',k} = U_{a_1,\ldots,a_{d-1},k} : a_i \in [1, m_i], k \in [1, m_d/3]\}$$

in the following manner; see Figure 1 for an illustration.

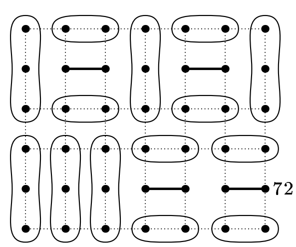

Figure 1: The sets $U_{\mathsf{a}',k}$ in the case that $(m,n) = (7,6)$ and the edges in B are $\{\{42,52\},\{62,72\},\{25,35\},\{55,65\}\}$. Edges remaining to be checked are marked with dotted lines.

- If $(\mathsf{a}', 3k-1) \in U$, then define

$$U_{\mathsf{a}',k} = \{(\mathsf{a}', r) : r \in \{3k-2, 3k-1, 3k\}\}.$$

We have that $G(U_{\mathsf{a}',k})$ is isomorphic to P_3 with $(\mathsf{a}', 3k-1)$ as the middle vertex. Moreover, $(\mathsf{a}', 3k-1)$ is not adjacent to any other vertices than $(\mathsf{a}', 3k-2)$ and $(\mathsf{a}', 3k)$ in G; all other edges incident to $(\mathsf{a}', 3k-1)$ belong to $Y \setminus B$.

- If $(\mathsf{a}', 3k-1)$ is adjacent to $(\mathsf{a}', 3k-1) + \mathbf{e}_i$ in B, then define

$$U_{\mathsf{a}',k} = \{(\mathsf{a}', 3k-2), (\mathsf{a}', 3k-2) + \mathbf{e}_i\}.$$

  Note that $G(U_{\mathsf{a}',k})$ is isomorphic to K_2.

- If $(\mathsf{a}', 3k-1)$ is adjacent to $(\mathsf{a}', 3k-1) - \mathbf{e}_i$ in B, then define

$$U_{\mathsf{a}',k} = \{(\mathsf{a}', 3k), (\mathsf{a}', 3k) - \mathbf{e}_i\}.$$

  Again, $G(U_{\mathsf{a}',k})$ is isomorphic to K_2.

One readily verifies that each vertex is contained in exactly one set in the partition. The given partition satisfies the conditions of Theorem 1.1, which implies that $\mathsf{M}(G)$ is $VD(\alpha)$, where $\alpha = \frac{|U|-n/3}{2} - 1$; the number of sets in the partition is $\prod_{i=1}^{d-1} m_i \cdot (m_d/3) = n/3$. Since $\Sigma(B, Y \setminus B) = \{B\} * \mathsf{M}(G)$, $\Sigma(B, Y \setminus B)$ is hence $\mathrm{VD}(\beta)$, where

$$\beta = \alpha + |B| = \frac{|U| - n/3}{2} - 1 + \frac{n - |U|}{2} = \frac{n}{3} - 1.$$

By properties of decision trees [1], we are done. □

Corollary 2.5 *Let* $\mathsf{m} = (m_1, \dots, m_d)$ *be a sequence of positive integers such that* $n = \prod_{i=1}^{d} m_i$ *is divisible by three. Then the shifted connectivity degree and homotopical depth of* $\mathsf{M}(\mathsf{Grid}(\mathsf{m}))$ *equals* $n/3 - 1$. □

We do not know the connectivity degree and depth of $\mathsf{M}(\mathsf{Grid}(\mathsf{m}))$ in the case that n is not divisible by three.

Conjecture 2.6 *Let* $\mathsf{m} = (m_1, \dots, m_d)$ *be a sequence of positive integers and write* $n = \prod_{i=1}^{d} m_i$. *Then the homotopical depth of* $\mathsf{M}(\mathsf{Grid}(\mathsf{m}))$ *equals* $\lceil \frac{n-4}{3} \rceil$.

The conjecture is true in the special case $d = 2$ and $m_1 \leq 2$:

Proposition 2.7 *Let* m_1 *and* m_2 *be positive integers such that* $m_1 \leq 2$. *Then the homotopical depth of* $\mathsf{M}(\mathsf{Grid}(m_1, m_2))$ *equals* $\lceil (m_1 m_2 - 4)/3 \rceil$.

Proof. We already know that the proposition is true if m_2 is divisible by three. If $m_1 = 1$ and $m_2 = n$, then we obtain Pa_n; by Proposition 1.3, the depth of this complex is at least $\nu_n = \lceil (n-4)/3 \rceil$. One readily verifies that Pa_n contains a maximal face of dimension exactly ν_n, which implies that the depth equals ν_n.

The remaining case is that $m_1 = 2$ and $m_2 = 3q + r$, where $q \geq 0$ and $r \in \{1, 2\}$. Write $m_2' = 3q$ and proceed in exactly the same manner as in the proof of Theorem 2.4 with the subcomplex $\mathsf{M}(\mathsf{Grid}(2, m_2'))$. Extend the family $\{U_{a',k}\}$ with the set $\{(1, m_2'+1), (2, m_2'+1)\}$ and also the set $\{(1, m_2'+2), (2, m_2'+2)\}$ if $r = 2$. Using the same argument as in the proof of Theorem 2.4, we conclude that $\mathsf{M}(\mathsf{Grid}(2, m_2))$ is $\mathrm{VD}(\beta)$, where

$$\beta \geq \frac{2m_2 - 2q - r}{2} - 1 = \frac{2m_2 - r/2}{3} - 1 \geq \frac{2m_2 - 4}{3}.$$

To prove that the depth is exactly $\lceil (2m_2 - 4)/3 \rceil$, let σ be the face containing the edges $\{(1, 3k-2), (1, 3k-1)\}$ and $\{(2, 3k-1), (2, 3k)\}$ for each $k \in [1, m_2'/3]$ and also the edge $\{(1, m_2'+i), (2, m_2'+i)\}$ for $1 \leq i \leq r$. Then σ is a maximal face in $\mathsf{M}(\mathsf{Grid}(2, m_2))$ and contains $\lceil (2m_2 - 1)/3 \rceil$ edges as desired. □

References

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