PERIODIC NAVIE-STOKES SOLUTIONS WITH CRYSTALLOGRAPHIC SYMMETRIES

O.I. Bogoyavlenskij

Steklov Mathematical Institute, Moscow, Russia

We study periodic solutions to the Navier-Stokes equations (NSE) with arbitrary vector periods $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$. Let Λ be the lattice of periods $\mathbf{p} = n_1\mathbf{p}_1 + n_2\mathbf{p}_2 + n_3\mathbf{p}_3$, $n_i \in \mathbb{Z}$, and Λ^* be the reciprocal lattice of vectors $\mathbf{k} = n_1\mathbf{k}_1 + n_2\mathbf{k}_2 + n_3\mathbf{k}_3$ where $\mathbf{k}_i \cdot \mathbf{p}_j = 2\pi\delta_{ij}$. Functions $\mathbf{V}(t, \mathbf{x}) = \sum \mathbf{V}_\mathbf{k}(t) \exp(i\mathbf{k} \cdot \mathbf{x})$, $\nabla p(t, \mathbf{x}) = \mathbf{p}_0(t) + \sum p_\mathbf{k}(t) \exp(i\mathbf{k} \cdot \mathbf{x})$ are periodic with periods $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$. Here $\mathbf{k} \in \Lambda^*$, $\mathbf{V}_{-\mathbf{k}} = \overline{\mathbf{V}_\mathbf{k}}$, $p_{-\mathbf{k}} = \overline{p_\mathbf{k}}$, and $\mathbf{V}_\mathbf{k} \cdot \mathbf{k} = 0$. For periodic solutions, the NSE reduce to the dynamical system

$$\dot{\mathbf{V}}_\mathbf{n} = -\mathbf{n}^2\nu\mathbf{V}_\mathbf{n} + \frac{i}{\mathbf{n}^2}\mathbf{n} \times \left(\mathbf{n} \times \sum_{\mathbf{k}\in\Lambda^*} (\mathbf{V}_\mathbf{k} \cdot \mathbf{n})\, \mathbf{V}_{\mathbf{n}-\mathbf{k}}\right), \quad \dot{\mathbf{V}}_0 = -\frac{1}{\rho}\mathbf{p}_0, \qquad (1)$$

where vectors $\mathbf{k}, \mathbf{n} \in \Lambda^*$. We show that dynamical systems (1) for different triples of periods $\mathbf{p}_j$ generically are not equivalent to each other: the moduli space of non-equivalent systems (1) has dimension 6.

We construct exact NSE solutions with crystallographic symmetry groups G that have pure rotational point groups $\Gamma \subset SO(3)$. There are 52 such groups G among 219 nonisomorphic crystallographic groups in three dimensions. The point group Γ can be either cyclic C_n, or dihedral D_n, $n = 2, 3, 4, 6$, or tetrahedral T or octahedral group O. The constructed exact solutions depend upon all four variables t, x_1, x_2, x_3.

We obtain complete classification of periodic solutions with pairwise non-interacting Fourier modes. For such solutions, the non-zero Fourier components $\mathbf{V}_\mathbf{k}(t) \in \mathbb{C}^3$ correspond to vectors $\mathbf{k}$ of the reciprocal lattice Λ^* that necessarily belong either to the spheres $\mathbf{k}^2 = \alpha^2$, or to the circumferences $\mathbf{k} \cdot \mathbf{e} = 0$, $\mathbf{k}^2 = \alpha^2$, or to the straight lines $\mathbf{k} = \lambda\mathbf{n}$ or to the planes $\mathbf{k} \cdot \mathbf{e} = 0$, where $\mathbf{k}$, $\mathbf{n} \in \Lambda^*$ and $\mathbf{e} \in \Lambda$.

The system (1) has an infinite-dimensional Lie group of symmetries $G = H(\Lambda)\dot{\times}\mathbf{A}$ where $H(\Lambda)$ is the holohedry group of the lattice Λ (or Λ^*) and $\mathbf{A}$ is the abelian Lie group of vector-valued functions $\mathbf{S}(t)$. Applying the symmetry transforms $\tilde{\mathbf{V}}_\mathbf{k} = \exp(i\mathbf{k} \cdot \mathbf{S}(t))Q\mathbf{V}_{Q^{-1}(\mathbf{k})}(t)$, $\tilde{\mathbf{V}}_0 = Q\mathbf{V}_0(t) - \dot{\mathbf{S}}(t)$, $\tilde{p}_\mathbf{k} = \exp(i\mathbf{k} \cdot \mathbf{S}(t))p_{Q^{-1}(\mathbf{k})}(t)$, $\tilde{\mathbf{p}}_0 = Q\mathbf{p}_0(t) + \rho\ddot{\mathbf{S}}(t)$, where $Q \in H(\Lambda)$, we demonstrate the non-uniqueness of solutions to the Cauchy problem for the periodic NSE with $\mathbf{V}_0(t) \neq 0$.

References:

1. Bogoyavlenskij, O.I.: Infinite families of exact periodic solutions to the Navier-Stokes equations. *Moscow Mathematical Journal.* **3**, N2, 1-10 (2003).

2. Bogoyavlenskij, O.I.: Exact unsteady solutions to the Navier-Stokes and viscous MHD equations. *Physics Letters A* **307**, 281-286 (2003).