

**Section number** 2 (Algebra).

**Name and affiliation of authors**

I. Arne Ring (arne.ring@epost.de), Hirschenberg 11, D-88471 Laupheim, Germany;

**Title of poster**

State Symmetries in Matrices and Vectors on Finite State Spaces.

**1991 Mathematics Subject Classification**

05E20 Group actions on designs, geometries and codes;

15A03 Vector spaces, linear dependence, rank;

20G20 Linear algebraic groups over the reals, the complexes, the quaternions.

**Key words**

state symmetry, linear algebra, vector space, symmetric group, general linear group, isotropy group.

*continued on next page*

### Abstract

Let  $C = \{1, 2, \dots, N\}$  be a finite set of natural numbers (the state space). Let  $\rho \in \mathcal{S}_N$  be a permutation of the elements of  $C$ , and let  $\mathbb{V} = \mathbb{R}^N$  be the vector space of real column vectors and  $\mathbb{M}$  be the vector space of real  $N \times N$  matrices.

A state symmetry of a vector or a matrix is defined as a permutation  $\rho$  for which holds  $v^\rho = v$ , where  $v^\rho = (v_i^\rho) := (v_{\rho(i)})$  (similarly  $M^\rho = M$  with  $M^\rho = (m_{i,j}^\rho) := (m_{\rho(i)\rho(j)})$ ), so that  $\rho$  is acting on  $\mathbb{V}$  and  $\mathbb{M}$ . If  $v^\rho = v$  (or  $M^\rho = M$ ) for a permutation  $\rho \neq id$ ,  $v$  (or  $M$ ) is called state symmetric. Hence,  $\mathcal{S}_v := \{\rho | v^\rho = v\}$  and  $\mathcal{S}_M := \{\rho | M^\rho = M\}$  are isotropy subgroups.

In addition,  $\mathbb{V}_\rho := \{v \in \mathbb{V} | v^\rho = v\}$  and  $\mathbb{M}_\rho := \{M \in \mathbb{M} | M^\rho = M\}$  are vector spaces. The dimension of  $\mathbb{V}_\rho$  is the number of disjoint cycles of  $\rho$ , and a canonical basis of this vector space can be derived directly. For  $\mathbb{M}_\rho$  the dimension is the sum of all  $GCD(n_1, n_2)$  where the sum is taken over all pairs of cycles  $c_1$  and  $c_2$  in  $\rho$  and  $n_r$  is the length of  $c_r$ .

The set  $G = \mathbb{M}_\rho \cap GL(N)$  of invertible matrices with state symmetry  $\rho$  is a group. Also this group acts on  $\mathbb{V}_\rho$  and  $\mathbb{M}_\rho$ .

If  $v^{(1)} = M v^{(0)}$  and  $v^{(2)} = M v^{(1)}$ , then the existence of a common state symmetry of the vectors  $v^{(0)}$  and  $v^{(1)}$  is generally not sufficient for the existence of a state symmetry of  $M$ , neither for the existence of a state symmetry of  $v^{(2)}$ .

However, the following hypothesis has been raised for vector sequences  $u = (v^{(i)})_{i=0, \dots, N}$ : Let  $\mathbb{W}_u := \{M | v^{(i+1)} = M v^{(i)} \text{ for } i = 0, \dots, N-1\}$  be the set of "generators" of  $u$ . If  $\rho$  is a proper state symmetry  $\rho$  of all vectors of  $u$ , then  $|\mathbb{W}_u| \geq 2$  and there is at least one matrix  $\hat{M} \in \mathbb{W}_u$  which is state symmetric. In this case, all elements of the infinite sequence  $u = (v^{(i)})_i$  are state symmetric.

Applications to simplify special Markow chains or structures of graphs will be presented.