CONVEXITY OF THE OPTIMAL STOPPING BOUNDARY FOR THE AMERICAN PUT OPTION

An American put option is a financial instrument which gives the owner the right to sell one asset of a certain stock at a fixed price at any time prior to some pre-determined expiration time. It is well-known that in the standard Black-Scholes model, the price (as a function of the present time and the current stock price) and the optimal strategy can be found by solving an optimal stopping problem. Alternatively, the price and the optimal stopping boundary can be characterized as the unique solution of a certain free boundary problem. Indeed, after a change of variables, the price $f(\cdot,\cdot)$ and the optimal stopping boundary $x(\cdot)$ solves the parabolic problem

$$\begin{cases} f_t = f_{xx} + (C-1)f_x - Cf & \text{if } x > x(t) \\ f = 1 - e^x & \text{if } x = x(t) \\ f_x = -e^x & \text{if } x = x(t) \\ f(0,x) = (1-e^x)^+. & \end{cases}$$

Here $C > 0$ is a parameter of the model for the stock price. It is known that the boundary $x(\cdot)$ is decreasing, that $x(0+) = 0$, and that $x(\infty) = \ln \frac{C}{C+1}$.

Our main result concerns the convexity of the free boundary. Friedman and Jensen [1] study the curvature of the free boundary in ice-melting problems (Stefan problems). Their methods rely upon studying the behavior of the level curves of the solution to a certain parabolic equation. Inspired by their work, we define the function

$$v := \frac{f_\tau}{f_x + e^x}.$$

We show that $v = -\dot{x}(t)$ for points at the boundary. Thus convexity of the free boundary follows if we show that $v(t, x(t))$ is decreasing in t. This is done by proving that all level curves of v starting at the boundary have to leave the continuation region at the origin, and by showing that the level curves leaving the continuation region at the origin are ordered decreasingly. When studying the behavior of the level curves of v, the maximum principle is used extensively.

Our methods are general in the sense that they can be applied also to more general equations.

References

[1] A. Friedman and R. Jensen, Convexity of the Free Boundary in the Stefan Problem and in the Dam Problem, *Arch. Rational Mech. Anal.* 67 (1977), 1-24.