CFSW Preconditioners for Solving Boundary Integral Equations on Polygons

D. Rostami Varnos Fadrani
fadrani@khayam.ut.ac.ir
Department of Mathematics and Computer Science,
Faculty of Science, University of Tehran
&
Institute for Studies in Theoretical Physics and Mathematics
P. O. Box 14155-6455 Tehran, Iran

Abstract. Fractional spline wavelet is extended to all fractional degrees $\alpha > \frac{-1}{2}$ for solving boundary integral equations. Its Galerkin discretization with N degrees of freedom on the boundary with fractional spline wavelet as basis function is analyzed. By using fractional spline wavelet bases and Galerkin method for these equations, a sparse matrix that can be mildly and moderately ill-conditioned is obtained. Since, truncation strategy is presented which allows to reduce the number of nonzero elements in the stiffness sparse matrix from $O(N^2)$ to $O(N \log N)$ entries. By introducing two operators which map every spare matrix to circulant sparse matrix, are obtained two classes of preconditioners that belong to a Banach space. Based on having some properties in the spectral theory for these classes of matrices, it can concluded that the circulant matrices are a good class of preconditioners for solving these equations. In general, we call them circulant fractional spline wavelet (CFSW) preconditioners. Therefore, two classes of stable algorithms are introduced for rapid numerical application.

Key words. wavelets, preconditioning, circulant operator, boundary integral equations.

AMS subject classifications. 65R20, 45L10

1 Introduction

Let us introduce the following boundary integral operators:

$$\begin{cases} T_1 f(x) = \int_\Gamma k(x,y) f(y) ds_y \\ T_1 : D_1 = H^{\frac{-1}{2}}(\Gamma) \mapsto R_1 = H^{\frac{1}{2}}(\Gamma), \end{cases} \tag{1}$$

$$\begin{cases} T_2 f(x) = \int_\Gamma \frac{\partial}{\partial n_y} k(x,y) f(y) ds_y \\ T_2 : D_2 = H^{\frac{1}{2}}(\Gamma) \mapsto R_2 = H^{\frac{1}{2}}(\Gamma), \end{cases} \tag{2}$$

$$\begin{cases} T_3 f(x) = \int_\Gamma \frac{\partial}{\partial n_x} k(x,y) f(y) ds_y \\ T_3 : D_3 = H^{\frac{-1}{2}}(\Gamma) \mapsto R_3 = H^{\frac{-1}{2}}(\Gamma), \end{cases} \tag{3}$$

and

$$\begin{cases} T_4 f(x) = -\frac{\partial}{\partial n_x} \int_\Gamma \frac{\partial}{\partial n_y} k(x,y) f(y) ds_y \\ T_4 : D_4 = H^{\frac{1}{2}}(\Gamma) \mapsto R_4 = H^{\frac{-1}{2}}(\Gamma), \end{cases} \tag{4}$$

where $x \in \Gamma$ and Γ is a polygon in $\mathbb{R}^2$. The above mappings are continuous in the Sobolve spaces (that are defined in [13]) and $k(x,y) := \log \|x-y\|, \quad x, y \in \mathbb{R}^2 \backslash \{0\}$.

All the considered boundary integral equations are integral equations of the first kind. For regular elliptic boundary value problems in bounded or unbounded domains, the method of reduction to boundary via Green's identities leads to equivalent boundary integral equations. Their discretization by finite element on the boundary manifold gives rise to the so-called boundary element method, which has become a standard tool in engineering practice by now (for a survey, we refer to [26] and the references there). While classically boundary element methods have been based mainly on the so-called equations of the second kind, it was shown more recently that a discretization based on the so-called equations of the first kind is also possible [21]. In particular, in conjunction with a Galerkin type discretization one obtains symmetric and positive definite stiffness matrices and approximations for unknown Cauchy data of the original boundary value problem. Distinct advantages of boundary integral equation based methods are the dimensional reduction of the computational domain by one and the discretization of exterior problems.

However, the resulting stiffness matrices are dense due to non-local nature of the boundary integral operators. This substantially increases the complexity of the matrix generation and of a matrix vector multiplication which is the basic step in an iterative solution of the linear system. In addition, for integral equations of the first kind, the condition of the stiffness matrices corresponding to standard finite element bases grows as the mesh width tends to zero, i.e. the matrices are usually ill conditioned.

A substantial reduction in the complexity of the calculation of the stiffness matrix and of the multiplication of the stiffness matrix with a vector is possible by multi-pole expansions of the potentials proposed in [18] and the panel clustering proposed independently in [10]. Another avenue to overcome the above mentioned drawbacks of boundary integral equation of integral operators which, through the choice of a special, so-called wavelet basis, yields numerically sparse stiffness matrices, i.e. they are still densely populated, but most of their entries are so small that they can be neglected without affecting the overall accuracy of the discretization (so-called truncation of stiffness matrices). For Galerkin discretization of rather general integral operators of order zero it was shown in [2] that generation of the stiffness matrix and the matrix vector multiplication can be achieved with essentially optimal complexity (i.e. up to possibly a logarithmic term) up to any prescribed fixed accuracy in [2]. These observations motivate the use of wavelet bases in boundary element methods. There, however, one is less interested in truncating the stiffness matrices in order to achieve an a-prior fixed accuracy as in [2], but rather in truncation which does not decrease the asymptotic convergence rate of the overall boundary element scheme. This question was addressed recently in the periodic setting for a wide class of boundary integral operators and basis functions for general Galerkin-Petrov schemes in [7]. It was shown there in particular, that truncation schemes are possible which preserve the asymptotic rate of convergence of the Galerkin-Petrov scheme with optimal complexity. In addition, it was shown that for operators of nonzero order, a simple diagonal preconditioning of the multi-scale discretization renders the condition number of the truncated stiffness matrices bounded, i.e. the above mentioned ill conditioning can be completely avoided. In the present paper we use fractional spline wavelet basis instead of wavelet basis. Experimental results will be shown that the optimal complexity for fractional spline wavelet base may be better than other base. But, we remark that our aim is to implement of new algorithms base on circulant fractional spline wavelet. The present paper is devoted to the analysis of fractional spline wavelet based symmetric Galerkin schemes for boundary integral equations of the first kind on polygonal in $\mathbb{R}^2$. On the other hand, the goal is to know how to precondition effectively, both in the case of numerical linear algebra (where one usually thinks of a finite dimensional problem to be solved) and in function spaces where the preconditioning corresponds to software which approximately solves the original problem. Therefore, outlines of this paper is as follows: In section 2 the fractional spline wavelet basis is introduced base and we introduced the Galerkin method for obtaining a system. For this result it is essential that we considered Galerkin discretization of these integral equations. In section 3, for speeding up the convergence non-stationary methods, we introduce two classes of preconditioners. We also discuss properties of these classes of circulant preconditioners in sections 3 and 4. Moreover, we confine ourselves to part stability plays in connection with the finite section method in section 4 (see theorem 4). Finally, in section 5, is shown that the classes of preconditioners to implement and in the numerical test cases considered leads to very significant improvement in accuracy.

2 Galerkin Boundary Element Methods by FSP

Let Γ be a bounded, polygonal and non-intersecting boundary with N_0 straight sides Γ_j and vertices $V = \{P_j^0\}_{j=1,\ldots,N_0}$. The domain $\Omega \subset \mathbb{R}^2$ is one of the components of $\mathbb{R}^2 \backslash \Gamma$, i. e., either the bounded interior or the unbounded exterior of the curve Γ. The unit normal vector on the boundary Γ which points from Ω to $\Omega^c := \mathbb{R}^2 \backslash \overline{\Omega}$ is denoted by n (it is defined almost everywhere on Γ). We can parameterize the boundary Γ by a 1-periodic function

$\Phi : [0,1] \mapsto \Gamma$ satisfying

$$\Phi(\frac{i}{N_0}) = P_i^0 \qquad\qquad i = 1, ..., N_0,$$

where the components of Φ are linear polynomials on each of the integrals $[\frac{i-1}{N_0}, \frac{i}{N_0}]$, $\quad i = 1, ..., N_0$.

For the approximation of the solutions of the above equations we choose a positive integer N and use the uniform mesh with the mesh points $\Phi(\frac{i}{NN_0})$, $j = 0, ..., NN_0$. Then we define the space S_N^α (where $\alpha > \frac{-1}{2}$) of fractional spline wavelet functions with degree α. The generating basis of this space (on the mesh) is defined as follows (see [24] and [25]):

Consider the Fractional Spline Wavelet (FSW) bases

$$\left[\begin{array}{l} \psi_+^{\alpha+i}(x/2) = \sum_{k\in Z} \frac{(-1)^k}{2^{\alpha+i}} \sum_{l\in Z} \left(\begin{array}{c} \alpha+i+1 \\ l \end{array}\right) \beta_*^{2\alpha+i+1}(l+k-1)\beta_+^{\alpha+i}(x-k), \\ i = 0, 1, 2, ... \end{array}\right. \tag{5}$$

where $\beta^\alpha(x)$ is a fractional B-spline with degree α, also, we define the fractional causal B-splines by taking $(\alpha+1)$th fractional difference of the one side power function

$$\beta_+^\alpha(x) = \frac{\Delta_+^{\alpha+1} x_+^\alpha}{\Gamma(\alpha+1)} = \frac{1}{\Gamma(\alpha+1)} \sum_{k=0}^{+\infty} \left(\begin{array}{c} \alpha+1 \\ k \end{array}\right) (x-k)_+^\alpha$$

and

$$\beta_*^\alpha(x) = \frac{\Delta_+^{\alpha+1} |x|_*^\alpha}{\Gamma(\alpha+1)} = \frac{1}{\Gamma(\alpha+1)} \sum_{k\in Z} \left(\begin{array}{c} \alpha+1 \\ k + \frac{\alpha+1}{2} \end{array}\right) |x-k|_*^\alpha,$$

where $\alpha > \frac{-1}{2}$ in order to ensure square integrability. These functions interpolate the usual polynomial B-splines; these are recovered for α integer. They are "causal" in the sense that their support belong to $\mathbb{R}^+$. Here, we assume that the fractional has been extended to non-integer α by $\alpha! = \Gamma(\alpha+1)$ using Euler's gamma function. Here $\Delta_+^\alpha f(x) = \sum_{k=0}^{+\infty} (-1)^\alpha \left(\begin{array}{c} \alpha+1 \\ k \end{array}\right) f(x-k)$,

$$x_+^\alpha = \begin{cases} x^\alpha & \alpha \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and } |x|_*^\alpha = \begin{cases} \frac{|x|^\alpha}{-2\sin(\frac{\pi}{2})} & \text{if} \quad \alpha \text{ not even} \\ \frac{x^{2n}\log x}{(-1)^{1+n}\pi} & \text{if} \quad \alpha = 2n \text{ (even)} \end{cases}.$$

As S_N^α is subspace of $L^2(\Gamma)$, we can approximate the above boundary integral equations with Galerkin method. It is clear that $\psi_+^{\alpha+i}$ is a Riesz basis for $L^2(\Gamma)$ [25]. As the degree α increases, the functions converge to modulated Gaussian which are known to optimally time-frequency localized in sense of Eisenberh uncertainty principle. This limit behavior can be inferred from the general convergence theorem in [1]. We use test and trial functions in these finite dimensional subspace and obtain the following Galerkin approximations.

In general, we can use the fractional spline wavelet bases for representation of linear operators. Hence, let $f \in D_i$, $g \in R_i$ and T_i is a linear operator such that

$$T_i f = g \qquad i = 1, 2, 3, 4. \tag{6}$$

First, we have the standard forms which starts with the decomposition of f and g in a wavelet fractional basis, by using P_n as projection method:

$$P_n f = \sum_{j=0} < f, \psi_+^{\alpha+j} > \psi_+^{\alpha+j},$$

$$P_n g = \sum_{j=0} < g, \psi_+^{\alpha+j} > \psi_+^{\alpha+j}.$$

Hence, we have:

$$T_i P_n f = \sum_{j=0} < f, \psi_+^{\alpha+j} > T_i \psi_+^{\alpha+j}.$$

Also, we have

$$T_i \psi = \sum_{j=0} < T_i \psi, \psi_+^{\alpha+j} > \psi_+^{\alpha+j},$$

which gives:

$$T_i P_n f = \sum_{j=0} \sum_{k=0} < T_i \psi_+^{\alpha+k}, \psi_+^{\alpha+j} >< f, \psi_+^{\alpha+k} > \psi_+^{\alpha+j} = \sum_{k=0} < g, \psi_+^{\alpha+k} > \psi_+^{\alpha+k}. \tag{7}$$

Remark. In general, for $\Omega \subseteq \mathbb{R}^n$ such that $n \geq 2$, we can write $P_n f = \sum a_{j_1 \dots j_n} \Pi_{k=1}^{j_n} \psi_+^{\alpha_k}$ where $\otimes$ is the tensor product for producing the space of $V_{j_1} \otimes ... \otimes V_{j_n}$ and $a_{j_1 \dots j_n}$ are unknown coefficients [5], [16].

Therefore, (7) can be depicted by on **infinity matrix** whose coefficients are

$$a_{kj} =< T_i \psi_+^{\alpha+k}, \psi_+^{\alpha+j} > \psi_+^{\alpha+j},$$

and the following system is obtained:

$$A_n x = b. \tag{8}$$

Where

$$A_n = (a_{kj} =< T_i \psi_+^{\alpha+k}, \psi_+^{\alpha+j} > \psi_+^{\alpha+j})_{k,j=0}^{n}, \tag{9}$$

and

$$x = (< f, \psi_+^{\alpha+k} >), \quad b = (< g, \psi_+^{\alpha+k} > \psi_+^{\alpha+k}). \tag{10}$$

Therefore, we observe that the equation (6) can have recourse to the finite systems that is (8) we call this transformation finite section method. Let $\{A_n\}_{n=1}^{\infty}$ be a sequence on $n \times n$

matrices A_n. This sequence is said to be stable if there is an n_0 such that the matrices A_n are invertible for all $n \geq n_0$ and

$$\sup_{n \geq n_0} \|A_n^{-1}\| < \infty.$$

Using the convention to put $\|T_i\| = \infty$ if T_i is not invertible, we can say that $\{A_n\}_{n=1}^{\infty}$ is a stable sequence if and only if

$$\limsup_{n\to\infty} \|A_n^{-1}\| < \infty,$$

stability plays a central role in asymptotic linear algebra and numerical analysis. The our aim is to confine ourselves to part stability that is plays in connection with the finite section method.

On the other hand, Meyer [14], [15] and other authors [11], [16] have shown that integral operators and Laplacian operators have the representation of sparse matrix if the wavelet bases are used. Therefore, it can be assumed that the matrix of (9) is a sparse matrix. An advantage for fast wavelet transforms on A_n, is that it requires $O(n)$ or $O(n\log(n))$ operations. We claim that if circulant operators are used then the above order will reduce since we know that convolution integral and elliptic operators have representations of sparse Toeplitz matrices (see [8], [12] and [17]). Also, if a powerful method such as Krylov methods is used for solving (8), then we have a super-linear convergence rate. A history of non-stationary iterative methods (for matrices) is given in D. Young [28], and a recent survey can be found in R. Freund, G. Golub and N. Nachtigal [28]. The very recent methods include GMRES by Y. Saad and M. H. Schultz [28], QMR by R. Freund and Nachtigal [28], CG, see P. Sonneveld [28], and different variants of GMRES due to H. A. van der Vorst and C. Vuik [28] (partly motivated by the "EN-update" in T. Eirola and O. Nevanlinna [28]). Here, we remark that in (8), A_n may be mildly and and moderately ill-conditioned [27]. Is there the convergence of Krylov subspace methods? for solving (8) by a non-stationary method (see [19], [20] and [21]). One standard way for speeding up the convergence rate of the Krylov methods is to apply a preconditioner so that it causes to cluster the spectrum. But, how can we make a preconditioner such that the spectrum of matrix is clustered? Also, we have to present some properties for convergence analysis. For answering to these questions the following sections are provided.

3 Circulant Fractional Spline Wavelet (CFSW) Preconditioners

We know that $C_n =$ Circ $[c_0, c_1, ..., c_n] = [c_{i-j(\bmod n+1)}]_{i,j=0}^{n}$ is called a circulant matrix and $T_n = [t_{j-k}]_{i,j=0}^{n}$ is called a Toeplitz matrix. Also, if n is a fixed integer, then

$F = \frac{1}{\sqrt{n}}(\xi^{(i-1)(j-1)})_{i,j=0}^{n}$ is called the Fourier matrix of order n that $\xi = exp(\frac{-2\pi i}{n})$ also we assume that $\alpha > \frac{-1}{2}$.

Let c be an operator that maps every $n \times n$ matrix A_n to a circulant matrix C_n. It is also assumed that this operator is onto and 1-1. We denote the Banach algebra of all $n \times n$ matrices over the complex field with $(M_{n\times n}, \|.\|)$ and we consider the subalgebra of all circulant matrices $(C_{n\times n}, \|.\|)$ as an inverse closed, commutative algebra. Also, we assume that $(S_{n\times n}, \|.\|)$ is the subalgebra of all sparse matrices. Let's consider the following definitions:

Definition 1. The CFSW operator c is called the optimal CFSW (OCFSW) operator if for finding the circulant matrix, we minimize $\|C_n - A_n\|$ over $(C_{n\times n}, \|.\|)$.

Definition 2. The CFSW operator c is called the super-optimal (SCFSW) operator if for finding the circulant matrix, we minimize $\|I - C_n^{-1}A_n\|$ over $(C_{n\times n}, \|.\|)$.

For any A_n in $S_{n\times n}$, let $\delta(A_n)$ denote the diagonal matrix whose diagonal is equal diagonal of the matrix A_n. We first give two methods for finding $c(A_n)$.

Theorem 1. Let $A_n = (a_{ij}) \in S_{n\times n}$ and $c(A_n)$ be the minimize of $\|C_n - A_n\|$ over all $C_n \in C_{n\times n}$. Then $c(A_n)$ is uniquely determined A_n. Moreover,
(i) $c(A_n)$ is given by

$$c(A_n) = \sum_{j=0}^{n-1}\left(\frac{1}{n}\sum_{p-q=j(modn)} a_{pq}\right)Q^j, \tag{11}$$

where Q is the $n \times n$ circulant matrix

$$Q = \begin{pmatrix} 0 & & & & 1 \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ 0 & & & 1 & 0 \end{pmatrix}, \tag{12}$$

(ii) $c(A_n)$ is also given by

$$c(A_n) = F^*\delta(FA_nF^*)F, \tag{13}$$

where F is the Fourier matrix and * denotes conjugate transposition.

Proof. For the proof of (i) (see [22], [23]). For (ii), we first note that any circulant matrix C_n can be expressed as $F^*\Lambda_n F$, where Λ_n is a diagonal matrix containing the eigenvalues of C_n; see Davis [6]. Since the Frobenius norm is unitary- invariant, we have:

$$\|C_n - A_n\|_F = \|F^*\Lambda_n F - A_n\|_F = \|\Lambda_n - F^* A_n F\|_F.$$

Thus, the problem of minimizing $\|C_n - A_n\|_F$ over $C_{n\times n}$ is equivalent to the problem of minimizing $\|\Lambda_n - F^* A_n F\|_F$ over all diagonal matrices. Since Λ_n can only affect the diagonal entries of FA_nF^*, we see that the solution for the latter problem is $\Lambda_n = \delta(FA_nF^*)$. Hence, $F^*\delta(FA_nF^*)F$ is minimizer of $\|C_n - A_n\|_F$. It is clear from the argument above that Λ_n and hence, $c(A_n)$ are uniquely determined by A_n. □

We remark that by (14), the jth entry in the first column of $c(A_n)$ is given by

$$[c(A_n)]_{j0} = \frac{1}{n} \sum_{p-q=j(\bmod n)} a_{pq} = \frac{1}{n} \text{ tr } (A_n Q^{-1}), \quad j = 0, 1, ..., n-1 \tag{14}$$

where tr (.) denotes the trace. By (13), the eigenvalues of $c(A_n)$ are given by $\delta(FA_nF^*)$. The following lemma is on the algebraic properties of the CFSW operator.

Lemma 1.

(i) For all $A_n, B_n \in S_{n\times n}$ and α, β complex scalars, $c(\alpha A_n + \beta B_n) = \alpha c(A_n) + \beta(B_n)$. Moreover, for all $A_n \in S_{n\times n}$, $c^2(A_n) = c(c(A_n)) = c(A_n)$. Thus c is a linear projection operator.

(ii) Let $A_n \in S_{n\times n}$ then tr $(c(A_n))$ = tr $(A_n) = \sum_{j=0}^{n-1} \lambda_j(A_n)$, where $\lambda_j(A_n)$ are the eigenvalues of A_n.

(iii) For all $A_n \in S_{n\times n}$, we have: $c(A_n^*) = c(A_n)^*$.

(iv) Let $A_n \in S_{n\times n}$ and $c_n \in C_{n\times n}$. Then
$c(C_n A_n) = C_n . c(A_n)$,
$c(A_n C_n) = c(A_n) . C_n$.

Proof. The proofs of (i) and (ii) are trivial; therefore we omit them. By using (14) and the fact that $\delta(A_n^*) = (\delta(A_n))^*$. One can easily prove (iii). For the proof of (iv), see theorem 2 in [3] and [17]. □

Next we are going to give some geometric properties of the CFSW operator. For all $A_n, B_n \in S_{n\times n}$, let $< A_n, B_n >= (1/n)$ tr $(A_n B_n^*)$. Obviously $< A_n, B_n >_F$ is an inner product in $S_{n\times n}$ and $< A_n, A_n >= (1/n)\|A_n\|_F^2$. It is easy to show that $\{Q^j |\ \ j = 0, ..., n-1\}$, where Q is given in (14), is an orthonormal basis of $(C_{n\times n}, \|.\|_F)$. We show below that $A_n - c(A_n)$ is perpendicular to the space $C_{n\times n}$.

Lemma 2. Let $A_n \in S_{n\times n}$, then we have:
(i) $< A_n - c(A_n), C_n >_F = 0$ for all $C_n \in C_{n\times n}$,
(ii) $< A_n, c(A_n) >= \frac{1}{n\|c(A_n)\|_F^2}$,
(iii) $\|A_n - c(A_n)\|_F^2 = \|A_n\|_F^2 - \|c(A_n)\|_F^2$.

Proof. For (i), since $\{Q^j\}_{j=0}^{n-1}$ is an orthonormal basis of $C_{n\times n}$, it suffices to show that $< A_n - c(A_n), Q^j >_F = 0$ for $j = 0, ..., n-1$. However, by (14) and Lemma 1(i), we have:

$$< A_n - c(A_n), Q^j >_F = \frac{1}{n} \text{ tr } ([A_n - c(A_n)]Q^{-j}) =$$

$$= \frac{1}{n} \text{ tr } (A_n Q^{-j}) - \frac{1}{n} \text{ tr } (c(A_n)Q^{-j}) =$$

$$= [c(A_n)]_{j0} - [c(c(A_n))]_{j0} =$$

$$= [c(A_n)]_{j0} - [c(A_n)]_{j0} = 0.$$

Now (ii) follows directly from (i). For (iii), we have, by parts (i) and (ii) above,

$$\|A_n - c(A_n)\|_F^2 = n < A_n - c(A_n), A_n - c(A_n) >=$$

$$= n < A_n - c(A_n), A_n >_F = n < A_n, A_n >_F - n < c(A_n), A_n >_F = \|A_n\|_F^2 - \|c(A_n)\|_F^2. \square$$

4 Spectral Properties of CFSW Operators

In this section, we discuss some spectral properties of the CFSW operators and so, we prove the stability of using preconditioners. The following theorem was first proved for the real scalar field in Tryshnikov [22]. His proof uses (12), and this proof here uses (14).

Theorem 2. If $A_n \in S_{n\times n}$ is Hermitian, then $c(A_n)$ is Hermitian. Moreover, we have:

$$\lambda_{min}(A_n) \le \lambda_{min}(c(A_n)) \le \lambda_{max}(c(A_n)) \le \lambda_{max}(A_n),$$

where $\lambda_{max}(.)$ and $\lambda_{min}(.)$ denote the largest and the smallest eigenvalues respectively. In particular, if A_n is positive definite, then $c(A_n)$ is also positive definite.

Proof. By Lemma 1(iii), it is clear that $c(A_n)$ is Hermitian. By (9), we know that the eigenvalues of $c(A_n)$ are given by $\delta(FA_nF^*)$. Suppose that $\delta(FA_nF^*) = diag(\lambda_0, ..., \lambda_{n-1})$ with $\lambda_j = \lambda_{min}(c(A_n))$ and $\lambda_k = \lambda_{max}(c(A_n))$. Let e_j and e_k denote the jth and the kth unit vectors respectively. Since A_n is Hermitian, we have:

$$\lambda_{max}(c(A_n)) = \lambda_k = \frac{e_k^* F A_n F^* e_k}{e_k^* e_k} \leq$$

$$\leq \max_{x \neq 0} \frac{x^* F A_n F^* x}{x^* x} = \max_{x \neq 0} \frac{x^* A_n x}{x^* x} = \lambda_{max}(A_n).$$

Similarly,

$$\lambda_{min}(A_n) = \min_{x \neq 0} \frac{x^* A_n x}{x^* x} = \min_{x \neq 0} \frac{x^* F A_n F^* x}{x^* x} \leq$$

$$\leq \frac{e_j^* F A_n F^* e_j}{e_j^* e_j} = \lambda_j = \lambda_{min}(c(A_n)).$$

From the inequality above, we can easily see that $c(A_n)$ is positive definite when A_n is positive definite.□

Lemma 3. For all $A_n \in S_{n\times n}$, $c(A_nA_n^*) - c(A_n)c(A_n^*)$ is a positive semi-definite matrix.

Proof. Let $A_n = (a_{jk})$ and $[F]_{jk} = (\frac{1}{\sqrt{n}})\xi_j^k$, where $\xi_j = e^{-2\pi ij/n}$. Let

$$D_n = c(A_nA_n^*) - c(A_n)c(A_n^*) =$$

$$= F^* \left[\delta(FA_nA_n^*F^*) - \delta(FA_nF^*)\delta(FA_n^*F^*)\right],$$

then for all $k = 0, ..., n-1$ we have:

$$[\delta(FA_nA_n^*F^*)]_{kk} = [\delta((FA_n)(FA_n)^*)]_{kk} =$$

$$= \frac{1}{n}\sum_{q=0}^{n-1}\left(\sum_{p=0}^{n-1} a_{pq}\xi_k^p\right)\overline{\left(\sum_{p=0}^{n-1} a_{pq}\xi_k^p\right)},$$

and

$$[\delta(FA_nF^*)\delta(FA_n^*F^*)]_{kk} = \left(\frac{1}{n}\sum_{p=0}^{n-1}\sum_{q=0}^{n-1} a_{pq}\xi_k^{p-q}\right)\overline{\left(\frac{1}{n}\sum_{p=0}^{n-1}\sum_{q=0}^{n-1} a_{pq}\xi_k^{p-q}\right)}.$$

Hence, the kth eigenvalue of D_n is given by

$$\lambda_k(D_k) = \frac{1}{n}\sum_{q=0}^{n-1}\left|\sum_{p=0}^{n-1} a_{pq}\xi_k^p\right|^2 - \left|\frac{1}{n}\sum_{p=0}^{n-1}\sum_{q=0}^{n-1} a_{a_{pq}\xi_k^{p-q}}\right|^2.$$

Since

$$\left|\frac{1}{n}\sum_{p=0}^{n-1}\sum_{q=0}^{n-1} a_{a_{pq}\xi_k^{p-q}}\right| \leq \frac{1}{n}\sum_{q=0}^{n-1}\left|\sum_{p=0}^{n-1} a_{pq}\xi_k^p\right| |\xi_k^{-q}| = \frac{1}{n}\sum_{q=0}^{n-1}\left|\sum_{p=0}^{n-1} a_{pq}\xi_k^p\right|,$$

we have:

$$\lambda_k(D_n) \geq \frac{1}{n}\sum_{q=0}^{n-1}\left|\sum_{p=0}^{n-1} a_{pq}\xi_k^p\right|^2 - \left(\frac{1}{n}\sum_{q=0}^{n-1}\left|\sum_{p=0}^{n-1} a_{pq}\xi_k^p\right|\right)^2.$$

Let $d_{qk} = \frac{1}{n}|\sum_{q=0}^{n-1} d_{qk}^2 a_{pq}\xi_k^p|$; then by the Cauchy-Schwartz inequality, we have:

$$\lambda_k(D_n) \geq n\sum_{q=0}^{n-1} d_{qk}^2 - \left(\sum_{q=0}^{n-1} d_{qk}\right)^2 \geq 0, \quad k = 0, ..., n-1.$$

Thus D_n is positive semi-definite. □

Theorem 3. For all $n \geq 1$, we have:

(i) $\|c\|_1 = \sup_{\|A_n\|_1=1}\|c(A_n)\|_1 = 1,$

(ii) $\|c\|_\infty = \sup_{\|A_n\|_\infty=1}\|c(A_n)\|_\infty = 1,$

(iii) $\|c\|_F = \sup_{\|A_n\|_F=1}\|c(A_n)\|_F = 1,$

(iv) $\|c\|_2 = \sup_{\|A_n\|_2=1}\|c(A_n)\|_2 = 1.$

Proof. To prove (i), we first note that if $A_n = I$, then $\|c(A_n)\|_1 = \|I\|_1 = 1$. For general A_n in $S_{n\times n}$, we have by (11)

$$\|c(A_n)\|_1 = \sum_{j=0}^{n-1}\left|\frac{1}{n}\sum_{p-q=j(\text{ mod n})} a_{pq}\right| \leq \frac{1}{n}\sum_{j=0}^{n-1}\sum_{p-q=j(\text{ mod n})}|a_{pq}| =$$

$$= \frac{1}{n}\sum_{i=0}^{n-1}\sum_{k=0}^{n-1}|a_{ik}| \leq \frac{1}{n}.n.\|A_n\|_1.$$

Hence, $\|c\|_1 = 1$ for all n. By a similar argument, we can prove (ii). To prove (ii), we notice that if $A_n = (\frac{1}{n})\|I\|_F = 1$.

For general A_n in $S_{n\times n}$ by Lemma 2(iii), we have:

$$\|c(A_n)\|_F^2 = \|A_n\|_F^2 - \|A_n - c(A_n)\|_F^2 \leq \|A_n\|_F^2.$$

Thus $\|c(A_n)\|_F \leq \|A_n\|_F$. Therefore, $\|A_n\|_F = 1$ for all n.

To prove (iv), by Lemma 1(iii), Lemma 3, and Theorem 2, we have:

$$\|c(A_n)\|_2^2 = \lambda_{\max}(c(A_n)^*c(A_n)) = \lambda_{\max}(c(A_n^*)c(A_n)) \leq$$

$$\leq \lambda_{\max}(c(A_n^*A_n)) \leq \lambda_{\max}(A_n^*A_n) = \|A_n\|_2^2,$$

for all A_n in $S_{n\times n}$. Since $\|c(I)\|_2 = \|I\|_2 = 1, \quad \|c\|_2 = 1.$ □

We can repeat the above statements for the SCFW preconditioner. But some important questions have remained. Does a SCFW exist? and how can we make it?

Instead minimizing $\|I - C_n^{-1}A_n\|_F$, $A_n \in S_{n\times n}$, we consider the problem of minimizing $\|I - \hat{C}_n A_n\|_F$ over all nonsingular $\hat{C}_n$ in $C_{n\times n}$. Letting $\hat{C}_n = F^*\Lambda_n F$, we have:

$$\|I - \hat{C}_n A_n\|_F = \|I - F^*\Lambda_n F A_n\|_F = \|I - \Lambda_n F A_n F^*\|_F =$$
$$= \text{ tr } (I - \Lambda_n F A_n F^* - F A_n^* F^* \Lambda_n^* + \Lambda_n F A_n A_n^* F^* \Lambda_n^*) =$$
$$= \text{ tr } (I - \Lambda_n \delta(F A_n F^*) - \delta(F A_n^* F^*)\Lambda_n^* + \Lambda_n \delta(F A_n A_n^* F^*)\Lambda_n^*)\,.$$

Let Λ_n, $\delta(FA_nF^*)$ and $\delta(FA_nA_n^*F^*)$ be given by diag $(\lambda_0, ..., \lambda_{n-1})$, diag $(u_0, ..., u_{n-1})$ and diag $(w_0, ..., w_{n-1})$ respectively. Therefore,we have:

$$\min \|I - \hat{C}_n A_n\|_F = \min\{ \text{ tr } (I - \Lambda_n \delta(F A_n F^*) - \delta(F A_n F^*)\Lambda_n^*) + \Lambda_n \delta(F A_n F^*)\Lambda_n^*\} =$$
$$= \min_{\lambda_0,\dots,\lambda_{n-1}} \sum_{n=0}^{n-1} (I - \lambda_k u_k - \bar{u}_k \bar{\lambda}_k + \lambda_k w_k \bar{\lambda}_k)\,.$$

Notice that by (14) and Lemma 3, $w_k \geq u_k \bar{u}_k$ for all $k = 0, ..., n-1$. Hence, for all complex scalars λ_k, $k = 0, .., n-1$, the terms $1 - \lambda_k u_k - \bar{u}_k \bar{\lambda}_k + \lambda_k w_k \bar{\lambda}_k$ are nonnegative. Differentiating them with respect to the real and imaginary parts of λ_k and setting the derivative to zero, we get

$$\lambda_k = \frac{\bar{u}_k}{w_k}, \qquad k = 0, ..., n-1.$$

Since A_n and $c(A_n)$ are nonsingular, both w_k and u_k are nonzero. Hence, λ_k also nonzero. Thus the minimizer of $\|I - \hat{C}_n A_n\|_F$ is nonsingular and gives by

$$\hat{C}_n = F^*\Lambda_n F = F^*\delta(F A_n F^*)[\delta(F A_n A_n^* F^*)]^{-1} F =$$
$$= [F^*\delta(F A_n^* F^*)F^*][F^*\delta(F A_n A_n^* F^*)F]^{-1} =$$
$$= c(A_n^*)c(A_n A_n^*)^{-1}.$$

Therefore, the SCFSW preconditioner is given by

$$\hat{C}_n^{-1} = c(A_n A_n^*)c(A_n^*)^{-1}.$$

Let us S_i be an operator so that (6) is transferred to

$$S_i T_i = S_i g.$$

Therefore, for stability, we can state the following theorem:

Theorem 4. Let $\{A_n\}$ be a sequence of $n \times n$ matrices for $\alpha > \frac{-1}{2}$ and suppose there is an operator S_iT_i such that $C_n^{-1}A_n \to S_iT_i$ and $(C_n^{-1}A_n)^* \to (S_iT_i)^*$ strongly. If $\{C_n^{-1}A_n\}$ is stable, then S_iT_i is necessarily invertible and

$$\|(S_iT_i)^{-1}\| \geq \liminf_{n\to\infty} \|A_n^{-1}C_n\|.$$

Proof. Suppose $\|A_n^{-1}C_n\| \geq M$ for infinitely many n and P_n be the following projection:

$$P_n : l^2 \to l^2, \qquad (x_0, x_1, x_2, ...) \to (x_0, x_1, x_2, ..., x_{n-1}, 0, 0, ...).$$

For $x \in l^2$ and these n,

$$\|P_n x\| = \|A_n^{-1}C_nC_n^{-1}A_nP_nx\| \leq M\|C_n^{-1}A_nP_nx\|,$$

$$\|P_nx\| = \|((C_n^{-1}A_n)^*)^{-1}(C_n^{-1}A)^*P_nx\| \leq M\|(C_n^{-1}A_n)^*P_nx\|$$

and passing to the limit $n \to \infty$, we get

$$\|x\| \leq M\|S_iT_ix\|, \qquad \|x\| \leq M\|(S_iT_i)^*x\|,$$

which implies that S_iT_i is invertible and $\|(S_iT_i)^{-1}\| \leq M$. Here S_i can be as a continues CFSW operator. □

5 Numerical Results

This section is dedicated to numerical examples in order to confirm our theory. We have implemented our computing using programs written in the symbolic language of Matlab Version 6.0 on the Connection Machine 200. This computer was configured with 16,384 bit serial processor, 512 floating point processors, and a Vax 6320 front end.

The PCG was used to solve (8) with different $\alpha > \frac{-1}{2}$ as a non-stationary method. As the stopping criterion a relative residual reduction of $\epsilon_N = 10^{-8}$ is used. To solve the linear systems involved, we used a preconditioned CG method. These preconditioners denote with $OCFSW$ and $SCFSW$ described in definitions 1 and 2 respectively. Also, in the following tables *Iter.* denotes the number of iterations, *Sec.* is the computing time needed for solution process, $R.$=$\|\widetilde{f_N} - f\|_{L^2(\Gamma)}$ is residual of solution f, also, $\widetilde{f_N}$ is approximated by one of the algorithms and $R.\ W$ denotes the residual without precondition. On the other hand, the $R.\ WW$ denotes the a residual for an algorithm that is provided without preconditioner on wavelet functions and Galerkin method [7].

Now we consider (6) in the L-shaped domain described by the nodes (0,0), (0.25, 0), (0.25, 0.25), (-0.25, 0.25), (-0.25,-0.25), (0,-0.25) as Γ with $k(x,y) = \log\|x-y\|$ and the exact solution be $f(y) = y_1^2 - y_2^2$. Therefore, we carry out four examples base on the defined operators (1), (2), (3) and (4). So, the following Tables for $\alpha = \frac{1}{2}$ are obtained:

Table 1. Results for the operator of T_1

N	R. W	Iter.	Sec.	R. OCFSW	Iter.	Sec.	R. SCFSW	Iter.	Sec.	R. WW	Iter.	Sec.
16	0.9e-1	11	0.021	1.2e-3	6	0.024	1.9e-2	6	0.025	7.3e-1	17	0.019
32	0.7e-2	19	0.100	2.3e-4	11	0.131	4.6e-2	12	0.141	5.6e-1	25	0.085
64	7.4e-3	31	0.325	1.7e-4	15	0.490	8.3e-3	17	0.545	3.9e-2	37	0.225
128	3.0e-3	53	0.749	3.9e-5	21	0.970	7.8e-4	23	1.292	4.7e-2	79	0.493
256	9.2e-4	72	1.220	1.3e-5	29	1.921	4.3e-5	32	2.351	6.3e-3	110	0.935
512	3.9e-4	129	3.211	9.1e-6	45	4.573	5.6e-5	51	5.393	8.9e-4	172	1.579

Table 2. Reslts for operator of T_2

N	R. W	Iter.	Sec.	R. OCFSW	Iter.	Sec.	R. SCFSW	Iter.	Sec.	R. WW	Iter.	Sec.
16	2.7e-1	18	0.033	3.5e-2	10	0.036	5.3e-2	11	0.038	9.2e-1	28	0.028
32	1.9e-1	30	0.151	8.7e-3	18	0.197	1.3e-2	20	0.224	7.7e-1	42	0.128
64	2.1e-2	49	0.489	4.9e-3	25	0.735	2.1e-3	29	0.817	0.1e-1	63	0.337
128	4.3e-2	84	1.124	0.0e-3	34	1.455	4.5e-3	39	1.935	7.5e-2	134	0.741
256	2.7e-3	115	1.832	7.3e-4	49	2.881	0.1e-3	54	3.525	2.1e-2	181	1.402
512	1.2e-3	204	4.817	1.6e-4	77	6.856	9.6e-4	87	5.893	5.3e-3	182	2.369

Table 3. Results for the operator of T_3

N	R. W	Iter.	Sec.	R. OCFSW	Iter.	Sec.	R. SCFSW	Iter.	Sec.	R. WW	Iter.	Sec.
16	0.6e-1	21	0.042	0.5e-2	13	0.054	1.7e-2	15	0.060	5.1e-1	33	6.037
32	7.3e-2	36	0.211	0.6e-3	23	0.295	2.9e-3	28	0.358	0.1e-1	54	3.157
64	4.6e-2	55	0.686	0.1e-3	32	1.103	1.6e-3	40	1.307	7.9e-2	82	0.444
128	2.3e-2	100	1.573	0.8e-4	44	2.183	0.2e-3	55	3.095	9.1e-2	174	0.904
256	0.0e-2	138	2.564	0.0e-4	63	4.325	0.9e-4	74	4.532	2.1e-2	235	1.570
512	5.4e-3	245	6.741	0.1e-5	100	10.284	0.7e-5	120	7.823	9.2e-3	312	2.605

Table 4. Results for the operator of T_4

N	R. W	Iter.	Sec.	R. OCFSW	Iter.	Sec.	R. SCFSW	Iter.	Sec.	R. WW	Iter.	Sec.
16	4.1e-1	23	0.040	1.1e-2	14	0.062	1.9e-2	17	0.068	3.1e-1	31	0.034
32	9.2e-1	39	0.171	0.7e-3	26	0.342	7.3e-3	31	0.392	7.1e-2	56	0.172
64	8.7e-2	57	0.596	0.1e-3	35	1.378	0.4e-3	43	1.457	9.2e-2	90	0.456
128	9.9e-3	92	1.293	0.4e-4	53	2.399	5.9e-4	62	3.806	4.1e-2	168	1.138
256	4.1e-3	128	2.143	0.2e-4	70	4.844	1.6e-4	91	4.721	9.3e-3	210	2.038
512	0.1e-3	238	5.293	0.1e-5	107	11.806	0.0e-4	126	8.096	0.4e-3	378	3.186

Also, in the following Figs 1, 2, 3 and 4 will be observed variations of residual in terms of α for the above examples with $N = 512$. Therefore, the results show that SCFSW and OCFSW are stable algorithms so that OCFSW can be strongest algorithms.

References

1. A. Aldroubi and M. Unser, Families of multiresolutin and wavelet spaces with optimal properties, *Numerical Functional Analysis and Optimization*, **14**, No. 5-6,(1993) pp. 417-446.
2. G. Beylkin, R. Coifman and V. Rokhlin, Fast wavelet transforms and numerical analysis, *J. Comm. Pure Appl. Math.*, **44** (1992) pp. 141-193.
3. R. Chan, Circulant preconditioners for Hermitian Toeplitz systems, *SIAM J. Matrix Anal. Appl.* **10** (1989) pp. 542-550.
4. D. Cohen, *Partial Differential Equations: An Introduction*, Rendom House, New York 1998.
5. G. David, *Wavelets and Singular Integral on Curves and Surface*, Springer-Verlag 1992.
6. P. Davis, *Circulant Matrices*, John Wiley and Sons, New York 1979.
7. W. Dahmen, S. Pröβdorf and R. Schneider, Wavelet approximation methods for pseudodifferential equations II: Matrix compression and fast solution, *Advances in Computational Mathematics* **1** (1993) pp. 259-335.
8. L. Elden, Algorithms for the regularization of ill- conditioned least squares problems, *BIT* **17** (1977) pp. 134-145.
9. A. Friedman, *Variational Principles and Free- Boundary Problems*, John Wiley & Sons 1982.
10. W. Hackbusch and Z. P. Novak, On the fact matrix multiplication in the boundary element method by panel clustering, *Numerische Mathematik* **54** (1989) pp. 229-245.
11. G. Kaiser, *A Friendly Guide to Wavelets*, Birkhäuser, 1995.
12. M. A. Malcolm and J. Palmer, A fast method for solving a class of tridiagonal linear systems, *Comm. ACM* **17** (1974) pp. 14-17.
13. J. T. Marti, *Introduction to Sobolev spaces and finite element solution of elliptic boundary value problems*, Academic Press Inc. (London) LTD, 1986.

14. Y. Meyer and R. Coifman, *Wavelets Calderon-Zygmund and multilinear operators*, Cambridge University Press, 1997.

15. Y. Meyer, *Wavelets and operators*, Analysis at Urbana, Vol 1, edited by E. Borkson, N.T. Peck and J. Uhl, London Math. Society, Lecture Notes Series 137, 1989.

16. M. Mitrea,*Clifford Wavelets Singular Integrals and Hard Spaces*, Springer-Verlag 1994.

17. B. S. Pariiski, An economical method for the numerical solution of convolution equations,*USSR Comput. Math. and Math. Phy.*, **18** (1978) pp. 208-211.

18. V. Rokhlin, Rapid solution of integral equations of classical potential theory, *J. Comp. Phys.* **60** (1985) pp. 187-207.

19. D. Rostami V. F. and K. Maleknejad, Fast iterative methods for solving of boundary nonlinear integral equations with singularity, *Journal of Computational Analysis and Applications*, **1** (1999) No. 2. pp.219-234.

20. D. Rostami V. F. and K. Maleknejad, Preconditioners for solving stochastic boundary integral equations with weakly singular kernels, *Computing*, **63** (1999) pp.47-67.

21. D. Rostami V. F.,*Preconditioned singular and hypersingular integral equations*, Ph.D Thesis, Iran University of science and Technology, 1999.

22. E. Tyrtyshnikov, Optimal and Super-optimal circulant preconditioners, *SIAM J. Matrix Anal. Appl.* **13** (1992) pp. 459-473.

23. E. Tyrtyshnikov and V.V. Strela, Which circulant preconditioner is better?, *Math of Comp.* **65** (1996) pp.135-150.

24. M. Unser and T. Blu, Fractional splines and wavelets, *SIAM Review,* **42** NO. 1 (2000) pp. 43-67.

25. M. Unser and T. Blu, Construction of fractional spline wavelet bases, *Proceedings of Wavelet Applications in Signal and Image,* **3813**, Processing VII, Denver Co, USA July 19-23, 1999, pp. 422-431.

26. W. L. Wendland, *Strongly elliptic boundary integral equations in the state of the art in numerical analysis*, Clarendon Press, Oxford (1987) pp. 511-561.

27. Y. Yan, Sparse preconditioned iterative methods for dense linear systems, *SIAM J. Sci. Comput.*, **15** (1994) pp. 1190-1200.

28. D. M. Young, A historical overview of iterative methods, *Proceedings of the Minnesota Supercomputer Institute Workshop on Practical Methods for Large Scale Computations*, 1988, pp. 1-18.