SYSTEMS CONTROLLED BY ROUGH PATHS

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It is an understatement to say that classical calculus has proved itself a very successful tool for expressing the relationships between evolving systems. Although many of the applications come from Applied mathematics and particularly from control theory, the development of a path into a manifold using a gauge, the development of a path from a manifold into a frame bundle using a connection, and the development of a path in a Lie algebra into a Lie group all provide examples from classical pure mathematics.

However, the classical approach does not easily deal with paths that occur when a random source modifies or perturbs the evolution of a dynamical system. Most of the natural models for noisy evolving systems are far from differentiable. This has stimulated the evolution of stochastic calculus and stochastic analysis with Ito's theory of stochastic integration, and Doob's martingale theory, both developed in the middle of the last century, being the decisive tools. Together they have transformed Finance and represent one the most successful areas of Applied mathematics.

However, Ito theory is essentially probabilistic and can only be applied to the sample paths of a special kind of stochastic process called a semi martingale (a condition which, in the one-dimensional case, is broadly equivalent to saying that the process can be re-parameterised to be a Brownian motion).

Motivated by the inroads these and other techniques in stochastic analysis have made in extending our concepts of calculus to non-standard settings, it seems worth reconsidering the possibility that the classical analytic theory could be extended (without probability) so as to provide a theory of differential equations driven by rough paths; a theory rich enough to capture the classical and the probabilistic examples.

In the end, a rather simple idea works. These differential equations obviously have meaning when the input or controlling path is smooth. If the Ito functional takes a smooth control to the resultant evolving state of the non-linear system, then one can reasonably ask: Are there metrics on smooth paths for which the Ito functional can be shown to be uniformly continuous or even Frechet differentiable. If there are and if the completions of the smooth paths in these metrics are not complicated then it would seem reasonable to consider these ”generalised paths” as the space in which the paths that drive differential equation should live.

It is a well-known idea, coming from K. T. Chen (geometry) Fleiss (control theory) Platen (stochastic differential equations) and many others, to consider the sequence of iterated integrals of a path in order to obtain a path wise Taylor series of arbitrary order for the solution to such equations. In effect, they lift the path up into the free nilpotent group of step K over the original vector space carrying a path (or in Chen's case, the path is lifted into the space of formal tensor series).

It has been shown that Ito functional is uniformly continuous if we consider the holder metric of order $1/p$ with respect to the Caratheodory metric on any of the free nilpotent group with step $K > p - 1$. For fixed p, these metrics are all equivalent. These are the rough or generalised paths.

The proof of the continuity theorem has led to a number of different applications. For example, within classical probabilistic theory, it has led to new proofs of the support theorem as well as techniques for proving the existence of solutions to infinite dimensional stochastic differential equations by careful studies of tensor norms on products of Gaussian random variables. It has extended the classical theory, and particularly work of Qian and Coutin allow us to consider stochastic differential equations driven by fractional Brownian motion.

The theorem is also provoked a number of new theoretical questions concerning iterated integrals – even of smooth paths.

The talk aims to look at simple examples which might motivate the main intuitions underpinning the development, and if time permits, look in detail at the mathematics of one of these applications.