SF2708 KOMBINATORIK, 2008
PROBLEM SET 1/4

Each exercise is worth five points. You may get partial credit for non-useless, non-perfect solutions. Hand in your solutions no later than March 14.

1. How many k-tuples $(S_1, \ldots, S_k)$ satisfy $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_k \subseteq [n]$?

2. Recall that a polynomial is called *homogeneous* if all its monomials are of the same degree. For example, $2xyz + 3x^2z - z^3$ is homogeneous of degree 3 whereas $2xyz + 3x^2z - z$ is inhomogeneous. Together with 0, the homogeneous polynomials of degree k in n variables (with coefficients in $\mathbb{R}$, say) form a vector space (over $\mathbb{R}$) which we call V_k^n. Find $\dim(V_k^n)$ and a simple expression for the *Hilbert series* $\sum_{k\geq 0} \dim(V_k^n)x^k$.

3. Let p be a prime. The goal of this exercise is to show in a combinatorial way that p divides $(p-1)! + 1$.[1] You get full credit if you do so, regardless of whether or not you follow the approach which is now sketched:

For notational convenience we shall consider $\mathfrak{S}_p$ to be the set of permutations of $\{0, 1, \ldots, p-1\}$ throughout this exercise.

Define a bijection $s : \mathfrak{S}_p \to \mathfrak{S}_p$ by declaring $s(\pi)(i+1) = \pi(i) + 1$ for all $i \in \{0, \ldots, p-1\}$, where we sum modulo p.

(a) Let $X = \{\pi \in \mathfrak{S}_p \mid \pi^p = e\}$, where $e \in \mathfrak{S}_p$ is the identity permutation. Show that X is mapped to itself by s.

(b) Show that s^p is the identity map.

(c) Given $\pi \in \mathfrak{S}_p$, (b) shows that the set $\{\pi, s(\pi), s^2(\pi), \ldots\}$ either has cardinality 1 or p. Characterize the π that satisfy $s(\pi) = \pi$.

(d) Prove that p divides $(p-1)! + 1$.

4. Let $f(n)$ be the number of fixed point free involutions in $\mathfrak{S}_{2n}$, i.e. permutations $\pi \in \mathfrak{S}_{2n}$ such that $\pi^2(i) = i$ but $\pi(i) \neq i$ for all $i \in [2n]$. Define $f(0) = 1$. Find simple formulae for $f(n)$ and for the exponential generating function $\sum_{n\geq 0} f(n)\frac{x^n}{n!}$.

5. Given a prime p, let $f(n)$ be the number of permutations $\pi \in \mathfrak{S}_n$ such that $\pi^p = e$, where e denotes the identity permutation in $\mathfrak{S}_n$. Consider the exponential generating function $G(x) = \sum_{n\geq 0} f(n)\frac{x^n}{n!}$. Prove that $G(x) = \exp(x + \frac{x^p}{p})$

[1] In order to combinatorially prove that m divides n, one typically produces a partition of a set of cardinality n into blocks that each have size m.

6. Let $M_n \subseteq \mathfrak{S}_n$ be the set of permutations $\pi \in \mathfrak{S}_n$ such that for all $2 \leq i \leq n$ there exists $1 \leq j < i$ with $|\pi(i) - \pi(j)| = 1$. For example, $M_3 = \{123, 213, 231, 321\}$. Furthermore, for $S \subseteq [n-1]$, let M_n^S be the set of permutations in M_n with descent set S. Find $|M_n^S|$ and $|M_n|$.

7. Prove combinatorially that $\sum_{i=0}^{k} \binom{n+i}{i} = \binom{n+k+1}{k}$ for all $k, n \in \mathbb{N}$.

8. In how many ways (as a function of n) can you first choose a composition α of n and then a composition of each part in α? The proof should be combinatorial and elegant.

9. Let $a(n)$ be the number of compositions of n into parts of size 2 or 3. For example, $9 = 3+3+3 = 3+2+2+2 = 2+3+2+2 = 2+2+3+2 = 2+2+2+3$, so that $a(9) = 5$. Find a simple expression for the generating function $F(x) = \sum_{n \geq 0} a(n) x^n$.