SF2708 KOMBINATORIK, 2008
PROBLEM SET 2/4

Each exercise is worth five points. You may get partial credit for non-useless, non-perfect solutions. Hand in your solutions no later than April 11.

1. Let k and n be positive integers. Prove that the following are all equal:

- The number of partitions of n where each part occurs at most $2k-1$ times.
- The number of partitions of n with no part divisible by $2k$.
- The number of partitions of n where parts that are divisible by k occur at most once each.

2. Give a combinatorial proof of the recursion $D(n) = (n-1)(D(n-1)+D(n-2))$ for the derangement numbers (Stanley's §2, equation 14).

3. In this exercise you shall reprove the principle of inclusion-exclusion using generating functions.

Let $f, g : 2^{[n]} \to \mathbb{R}$ be functions that satisfy $g(S) = \sum_{T \supseteq S} f(T)$ for all $S \subseteq [n]$. Define

$$G(x_1, x_2, \ldots, x_n) = \sum_{S \subseteq [n]} f(S)(1+x_1)^{\chi_S(1)}(1+x_2)^{\chi_S(2)} \ldots (1+x_n)^{\chi_S(n)},$$

where $\chi_S : [n] \to \{0,1\}$ is the characteristic function which is one on elements of S and zero elsewhere.

a) Show that $G(x_1, \ldots, x_n) = \sum_{S \subseteq [n]} g(S) x_1^{\chi_S(1)} \ldots x_n^{\chi_S(n)}$.

b) Consider $G(x_1 - 1, \ldots, x_n - 1)$ and deduce a formula for $f(S)$ in terms of g.

4. Let $M_{m,n}$ be the set of $0,1$-matrices of size $m \times n$ such that every row and every column contains at least one 0. Show that

$$|M_{m,n}| = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (2^{n-k} - 1)^m.$$

5. Find the number of ways to choose k points from a collection of m distinguishable points arranged in a circle if each pair of chosen points must be separated by at least d points.

6. Let $f(n)$ be the number of permutations $\pi \in \mathfrak{S}_{2n}$ such that $\pi(i) > n - i$ for all $i \in [2n]$. Show that

$$f(n) = \sum_{k=0}^{n} (-1)^{n+k} S(n,k)(n+k)!.$$

7. Given any function $f : [m] \to [n]$ and $i \in [n]$, let $f^{-1}(i)$ denote the set $\{j \in [m] \mid f(j) = i\}$. Let $k \in \mathbb{N}$. Show that the number of functions $f : [m] \to [n]$ such that $|f^{-1}(i)| \neq k$ for all $i \in [n]$ is

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} \binom{m}{k, \dots, k, m-kj} (n-j)^{m-kj},$$

where "$k, \dots, k$" in the multinomial coefficient means that k appears j times.

8. Let $p = p(x_1, \dots, x_n) \in \mathbb{R}[x_1, \dots, x_n]$ be a polynomial of degree n. Given $k \in \{0, 1, \dots, n\}$, define $\sigma^k(p) \in \mathbb{R}[x_1, \dots, x_n]$ to be the sum of the $\binom{n}{k}$ polynomials obtained by letting k variables in p vanish in all possible ways. That is,

$$\sigma^k(p) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} p|_{x_{i_1} = \cdots = x_{i_k} = 0}.$$

Prove that

$$\sum_{k=0}^{n} (-1)^k \sigma^k(p) = c x_1 \dots x_n,$$

where c is the coefficient in front of $x_1 \dots x_n$ in p.

9. Consider the complete bipartite graph $K_{n,n}$, $n \geq 2$. Remove from it the edges that belong to some fixed Hamiltonian cycle. How many complete matchings does the resulting graph have?[1]

[1] Let us recall some graph-theoretic terminology. The graph $K_{n,n}$ has vertex set $\{x_1, \dots, x_n, y_1, \dots, y_n\}$ and edges $\{x_i, y_j\}$ for all $(i,j) \in [n]^2$. A *Hamiltonian cycle* is a connected subgraph containing all the vertices in which each vertex is contained in precisely two edges. A *complete matching* is a subgraph containing all the vertices in which each vertex is contained in precisely one edge.