SF2708 KOMBINATORIK, 2008
PROBLEM SET 3/4

Each exercise is worth five points. You may get partial credit for non-useless, non-perfect solutions. Hand in your solutions no later than April 28.

1. Let n be an even integer. Given $X \subseteq [n]$, define $\sigma(X) = \sum_{x \in X} x$. Show that

$$\sum_{X \in \binom{[n]}{k}} (-1)^{\sigma(X)} = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ (-1)^{k/2}\binom{n/2}{k/2} & \text{if } k \text{ is even.} \end{cases}$$

2. Let k and n be positive integers. Define a matrix M of size $n \times n$ with $\binom{k+i}{k+j-i}$ being the entry on row i, column j. Prove that all minors[1] of M are nonnegative.

3. This exercise gives a proof that a largest possible antichain in the Boolean lattice B_n is given by the subsets of cardinality $\lfloor n/2 \rfloor$.

Let P be a finite poset in which the maximum size of an antichain is w. Let $\varphi : P \to P$ be a poset automorphism. Say that a subset $S \subseteq P$ is *invariant* under φ if $\varphi(S) = \{\varphi(s) \mid s \in S\} = S$.

(a) Prove that P contains an antichain of size w which is invariant under every automorphism $P \to P$.

(b) Given two subsets $S, T \subseteq [n]$ with $|S| > |T|$, show that there exists a permutation $\pi \in \mathfrak{S}_n$ such that $\pi(S) \supset T$.

(c) Deduce from (b) that no antichain in B_n which contains subsets of different cardinalities is invariant under every automorphism of B_n.

Hint. In (a), use the following fact which is implied by Exercise 3.25 in Stanley: In a finite distributive lattice L, the subposet of L induced by those elements that cover the maximum possible number of elements is a distributive lattice, too.

4. Let L be a finite geometric lattice. A set $A \subset L$ of atoms is called *independent* if $|A|$ is the rank of $\vee A$. Now suppose $A, B \subset L$ are two independent atom sets such that $|A| > |B|$. Show that there exists some $a \in A \setminus B$ such that $B \cup \{a\}$ is independent.

5. Let L be a lattice. Recall that L is called *complete* if every subset $S \subseteq L$ has a join and a meet.[2] Suppose there exists an integer $\alpha \in \mathbb{N}$ such that $|C| < \alpha$ whenever $C \subseteq L$ is a chain. Prove that L is complete.

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[1] Recall that a *minor* of M is the determinant of some square matrix obtained by removing rows and columns from M.

[2] Clearly, finite lattices are always complete; cf. page 103 in Stanley.

6. Suppose $H_1, \dots, H_k$ are linear hyperplanes in $\mathbb{R}^n$ (i.e. linear subspaces of dimension $n-1$). Given $S \subseteq [k]$, define $H_S = \cap_{i \in S} H_i$.[3] We observe/define that $H_\emptyset = \mathbb{R}^n$.

Define a poset L on $\{H_S \mid S \subseteq [k]\}$ by declaring $H_S \leq H_{S'} \Leftrightarrow H_S \supseteq H_{S'}$. Prove that L is a geometric lattice.

7. We are given a finite poset P and want to partition it into antichains. Show that the minimum number of antichains required is the same as the maximum cardinality of any chain in P.

8. A poset P is a *forest* if $I_x \cap I_y = \emptyset$ whenever $x, y \in P$ are incomparable elements. (Here, we use the notation $I_p = \{q \in P \mid q \leq p\}$ for $p \in P$.)

Suppose P is a forest with n elements. Prove that

$$e(P) = \frac{n!}{\prod_{x \in P} |I_x|},$$

where $e(P)$ denotes the number of linear extensions of P.

9. Define a partial order on $2^{[n]}$ by $S \leq T$ if the elements of S are $s_1 > s_2 > \cdots > s_j$ and the elements of T are $t_1 > t_2 > \cdots > t_k$, where $j \leq k$ and $s_i \leq t_i$ for all $i \in [j]$. (Observe/define that the empty set is smaller than all other subsets in this order.) Prove that this poset is graded and describe the rank function.

[3]Notice that $H_S = H_{S'}$ does not necessarily imply $S = S'$.