SF2708 KOMBINATORIK, 2008
PROBLEM SET 4/4

Each exercise is worth five points. You may get partial credit for non-useless, non-perfect solutions. Hand in your solutions no later than May 23.

1. Suppose P is a finite poset with $\hat{0}$ and $\hat{1}$. Show that P is a lattice if and only if the following property holds: $x \vee y$ exists whenever there is a z which is covered by both x and y.

2. Let P be a finite poset, and define $\hat{P} = P \cup \{\hat{0}, \hat{1}\}$ as usual. Suppose P has a fixed point free automorphism of order p, where p is a prime. Show that

$$\mu(\hat{0}, \hat{1}) \equiv -1 \pmod{p},$$

where μ is the Möbius function of $\hat{P}$.

3. Find a new proof of the third problem of the first problem set by applying the result from Problem 2 to the partition lattice.

4. Let A_n denote an antichain with n elements. Consider the ordinal sum of posets $P = A_2 \oplus A_3 \oplus A_2$ and the corresponding $\hat{P} = P \cup \{\hat{0}, \hat{1}\}$. Determine $\mu(\hat{0}, \hat{1})$ in two different ways:

(a) By using the recurrence (14) in Stanley's Section 3.7.

(b) By using Philip Hall's theorem in conjunction with the Euler-Poincaré formula (Proposition 1.2.6 and Theorem 1.2.8 in Wachs' text).

5. Consider the set of acyclic directed graphs[1] on vertex set $[n]$. Ordering these graphs by containment of the edge sets gives us a poset. Note that the edgeless graph is the bottom element $\hat{0}$. Add an artificial top element $\hat{1}$ and call the resulting poset G_n.

Now construct another poset by ordering all partial orders on $[n]$ by inclusion of relations. The antichain is the bottom element $\hat{0}$, and again we add an artificial top element $\hat{1}$. Denote this poset by P_n. Show that

$$\mu_{G_n}(\hat{0}, \hat{1}) = \mu_{P_n}(\hat{0}, \hat{1}).$$

[1] A directed graph is acyclic if it contains no directed cycles.

6. Consider a poset P. A *matching* on P is an involution $M : P \to P$ (i.e. $M^2 = \mathrm{id}$) such that either $M(x)$ covers x or $M(x)$ is covered by x for all $x \in P$.

Now suppose P is a finite poset with $\hat{0}$ in which every interval is Eulerian, and define $\hat{P} = P \cup \{\hat{1}\}$. Suppose $\mu_{\hat{P}}(\hat{0}, \hat{1}) \neq 0$. Prove that no matching on P exists.

7. Let $d(n)$ denote the dimension of the incidence algebra of the partition lattice Π_n. Show that the exponential generating function satisfies

$$\sum_{n\geq 0} d(n)\frac{x^n}{n!} = e^{e^{e^x-1}-1}.$$

8. A binary word is called *primitive* if it cannot be obtained by concatenating two or more copies of a shorter word. Thus, 110101 is primitive whereas 110110 and 101010 are not. Let $p(n)$ be the number of primitive binary words on n bits. Show that

$$p(n) = \sum_{d|n} \mu(d, n) 2^d,$$

where μ denotes the Möbius function of the divisibility lattice D_n.

9. Let P be a finite poset. Show that the following properties are equivalent:

(i) For all $x < y$, $[x, y]$ has an odd number of atoms.

(ii) For all $x < y$, $[x, y]$ has an odd number of coatoms.

(iii) The Möbius function satisfies

$$\mu(x, y) \equiv \begin{cases} 1 & \text{if } x = y \text{ or } y \text{ covers } x, \\ 0 & \text{otherwise} \end{cases} \pmod 2.$$