SF2716 TORIC GEOMETRY
TAKE HOME ASSIGNMENT I
DUE MARCH 10 2008

(1) Let T be an algebraic torus. Let $M = Hom_{AG}(T, \mathbb{C}^*), N = Hom_{AG}(\mathbb{C}^*, T)$. Show that there is an isomorphism of lattices

$$M \cong Hom_{AbGr}(N, \mathbb{Z}) = \check{N}(\text{dual of N})$$

(Note that the composition function $\alpha : N \times M \to \mathbb{Z}$ defines a bilinear map of abelian groups.)

(2) Let $V \subset \mathbb{C}^n, W \subset \mathbb{C}^m$ be two affine varieties and let $f : V \to W$ be a morphism of affine varieties. We will denote by Let

$$I_f = \{g \in \mathbb{C}[x_1, ..., x_m], g \circ f = 0\}$$

  (a) Show that I_f is an ideal such that $V(I_f) \subseteq W$.
  (b) Let $f \in Hom_{AG}(T_1, T_2)$ be a morphism between two tori. Show that $Im(f) = \{w \in W \text{ s.t. } w = f(v) \text{ for some } v \in V\} = V(I_f)$ and that it is a torus.

(3) Let T be an algebraic torus.
  (a) Consider the self action of T, by multiplication, and the induced action on $\mathbb{C}[T]$. For every t let $H_t : \mathbb{C}[T] \to \mathbb{C}[T]$ be the induced map of $\mathbb{C}$-algebras. Show that the only possible non constant simultaneous eigenvectors of H_t (for all t) are characters χ^m, with eigenvalue $\chi^m(t)$.
  (b) Assume that T acts (the action is denoted by $\star$) algebraically on a finite dimensional vector space W. This means that the map:

$$T \to GL(W), t \mapsto H_t(w) = t \star w,$$

  is a morphism of affine algebraic groups. Let M be the lattice of characters of T. Show that

$$W = \bigoplus_{m \in M} W_m, \text{ where}$$

$$W_m = \{w \in W \text{ s.t. } t \star w = \chi^m(t)w \text{ for all } t \in T\}$$

(4) Give an example of an affine toric variety $V \neq \mathbb{C}^2$, of dimesion 2.
  Find the lattice points $\mathcal{A}$ so that $V = Y_{\mathcal{A}}$. Give the toric ideal $I_{\mathcal{A}}$ defining V, and the semigroupg S such that $V = Spec(\mathbb{C}[S])$.