SF2716 TORIC GEOMETRY
TAKE HOME ASSIGNMENT II
DUE MAY 10 2008

(1) Let V, W be two irreducible affine varieties, $p \in V$ and $q \in W$.
  (a) Let U_0, V_0 be two Zariski open subspaces of V and W respectively , with $p \in U_0$. Let $\phi : U_0 \to W_0$ be an isomorphism, such that $\phi(p) = q$. Show that there is a natural isomorphism $T_pU_0 \to T_qV_0$ and hence $\dim(T_pU_0) = \dim(T_qV_0)$.
  (b) Deduce that if two varieties are birationally equivalent, then they have the same dimension. (Two varieties are birationally equivalent if there is a rational map, whose inverse is a rational map.)

(2) Let $P = \cap_1^d H^+_{n_{F_i},a_i} \subset \mathbb{R}^n \cong M_{\mathbb{R}}$ be a maximal dimensional lattice polytope, with facets $F_1, ..., F_d$. For any $k \geq 1$ we denote by kP the polytope:

$$kP = \cap_1^d H^+_{n_{F_i},ka_i}.$$

If $m \in M$, then $P + m$ is the polytope translated by the translation $x \mapsto x + m$.

Show that for any $m \in M$ and any integer $k \geq 1$,

$$\Sigma_P = \Sigma_{P+m} = \Sigma_{kP},$$

and thus they define the same projective toric variety.

(3) Consider the Segre-embedding:

$$s_{1,1} : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$$

defined as $s_{1,1}[(x_0, x_1), (y_0, y_1)] = (x_0y_0, x_0y_1, x_1y_0, x_1y_1)$.
  (a) Show that it defines an isomorphism of $\mathbb{P}^1 \times \mathbb{P}^1$ with the quadric hypersurface $Q = V(x_0x_3 - x_1x_2) \subset \mathbb{P}^3$, where x_0, x_1, x_2, x_3 are homogeneous coordinates on $\mathbb{P}^3$.
  (b) Describe the image in Q of the two families of lines:

$$\{p \times \mathbb{P}^1 \text{ s.t. } p \in \mathbb{P}^1\}, \{\mathbb{P}^1 \times q \text{ s.t. } q \in \mathbb{P}^1\}.$$

  (c) Use (2) to show that $\mathbb{P}^1 \times \mathbb{P}^1 \ncong \mathbb{P}^2$.

(4) Explain as much as you can of exercise (3) using toric geometry.