APRIL 14

Contents

1. Toric variety associated to a fan

Let $\Sigma \subseteq N_{\mathbb{R}}$ be a complete fan. Let $rank(N) = n$, and let M be the dual lattice.

We have seen that each cone of maximal dimension, σ, defines an affine toric variety:

$$U_\sigma = Spec(\mathbb{C}[\check{\sigma} \cap M])$$

Recall that if σ is regular, then U_σ is non singular and it is $U_\sigma \cong \mathbb{C}^n$.

Let $\tau \subset \sigma$ be a face of dimension k. Then there is an element $u \in \check{\sigma} \cap M$ such that $\tau = \sigma \cap u^\perp$ and $\check{\tau} = \check{\sigma} + \mathbb{R}_{\geq 0}(-u)$. It follows that $S_\tau = S_\sigma + \mathbb{Z}(-u)$. which gives

$$U_\tau = Spec(\mathbb{C}(S_\sigma)_{\chi^u})$$

as an open subset of U_σ.

Similarly we see that $U_\sigma \cap U_{\sigma'} = U_{\sigma \cap \sigma'}$.

We then define the corresponding toric variety:

$$X(\Sigma) = \cup_{\sigma \in \Sigma} U_\sigma$$

as the disjoint union of the affine toric varieties corresponding to cones in Σ, glued together the open subsets $U_{\sigma\cap\sigma'}$ in U_σ and U'_σ. (This identifications are compatible, by construction). Moreover the action of T_N on the maximal affine patches is compatible with the intersection and thus defines an action of T_N over $X(\Sigma)$.

Recall that if σ, with $\dim(\sigma) = k$ is regular if and only if it is generated by part of a basis of the lattice N. This implies that then $U_\sigma = \mathbb{C}^k \oplus (\mathbb{C}^*)^{n-k}$.

Moreover if σ is regular then U_σ contains only one fixed point

$$x_\sigma \text{ corresponding to } x_\sigma(m) = 1 \text{ if } m \in \sigma^\perp, \text{ and 0 otherwise.}$$

2. Orbits and subvarieties

Recall that a variety with a group action is always the disjoint union of its orbits.

Example 2.1. Consider the affine toric variety $\mathbb{C}^n$. The torus is the unique orbit of any point inside, let $1 = (1, ..., 1)$, then

$$O_1 = (\mathbb{C}^*)^n.$$

For any $I \subset \{1, ..., n\}$ let $x_I = \sum_{i \in I} e_i$, where $e_1, .., e_n$ is an affine basis. Then

$$O_{x_I} = \{(z_1, ..., z_n) \text{ such that } z_i \neq 0 \text{ if and only if } i \in I\} \cong (\mathbb{C}^*)^{|I|}.$$

and

$$\overline{O_{x_I}} = V(x_i, i \notin I) \subset \mathbb{C}^n \setminus O_1.$$

Moreover they form a disjoint cover of $\mathbb{C}^n$.

One can describe the orbits of $\mathbb{P}^n$ similarly by intersecting with the affine patches.

Toric varieties coming from a complete fan, $X(\Sigma)$ behave similarly. Assume that every cone is regular.

Let $\sigma \in \Sigma$ be a maximal dimensional cone. Then $U_\sigma \cong \mathbb{C}^n$ ha a unique fixed point x_σ and a big dense open orbit $O_1 = T_N$, such that $\bar{(}O_1) = U_\sigma$.

Let $\tau \subset \sigma$ be a face of dimension k. Then $U_\tau \cong \mathbb{C}^k \times (\mathbb{C}^*)^{n-k}$. let

$$O_\tau = T_{N(\tau)} = N(\tau) \otimes \mathbb{C}$$

be the following $(n-k)$ dimensional torus. Recall that $N_\tau = (\tau \cap M) + (-\tau \cap M)$ is the sublattice giving the $(n-k)$-dimensional sublattice $N(\tau) = N/N_\tau$ whose dual is $M(\tau) = \tau^\perp \cap M$.

Let $\pi : N \to N(\sigma)$ be the projection map, then

$$\pi(\sigma) = \overline{\sigma} = (\sigma + N(\sigma)_\mathbb{R})/N(\sigma)_\mathbb{R} \in N(\sigma).$$

We define the following fan (it is a complete fan):

$$Star(\tau) = \{\bar{(}\sigma) \text{ such that } \tau \subset \sigma\} \subset N(\tau).$$

And the corresponding $n-k$ dimensional toric variety is denoted by

$$V(\tau) = X(Star(\tau)).$$

Note that it is the closure of O_τ.

The affine patches are given by cones in the fan, i.e. cones σ containing τ. Because $\check{\overline{\sigma}} = \check{\sigma} \cap \tau^\perp$ we have that:

$$U_\sigma(\tau) = Spec(\mathbb{C}[\check{\overline{\sigma}} \cap M(\tau)]) = Spec(\mathbb{C}[\check{\sigma} \cap \tau^\perp \cap M]).$$

and thus

$$V(\tau) = \bigcup_{\tau \subseteq \sigma} Spec(\mathbb{C}[\check{\sigma} \cap \tau^\perp \cap M])$$

and O_τ is the big dense open orbit in each affine patch.

Observe that $O_{\{0\}} = T_N$ and thus $V(\{0\}) = X(\Sigma)$. Also because they are closure of orbits they are invariant under the action of T. (T acts on $V(\tau)$ via the projection $T \to T(\tau)$.) This defines a correspondence (order reversing, see below) between

$$k-\text{dimensional cones in } \Sigma \text{ and } \text{ codimension } (n-k) \text{ invariant subvarieties }.$$

The closed embedding of $V(\tau)$ in $X(\Sigma)$ is defined as

$$i : U_\sigma(\tau) = Hom_{sg}(\check{\sigma} \cap \tau^\perp \cap M, \mathbb{C}) \to U_\sigma = Hom_{sg}(\check{\sigma} \cap M, \mathbb{C})$$

where $i(f)$ is the extension by zero:

$$i(f) = \begin{cases} f(m) & m \in \check{\sigma} \cap \tau^\perp \\ 0 & \text{otherwise} \end{cases}$$

Notice that this map is well defined because $\check{\sigma} \cap \tau^\perp$ is a face of $\check{\sigma}$.

Let $\tau \subset \sigma \subset \gamma$, and thus $\check{\gamma} \subset \check{\sigma}$. The following diagram commutes:

$$\begin{array}{ccc} Hom_{sg}(\check{\sigma} \cap \tau^\perp \cap M, \mathbb{C}) & \longrightarrow & Hom_{sg}(\check{\gamma} \cap \tau^\perp \cap M, \mathbb{C}) \\ \downarrow & & \downarrow \\ Hom_{sg}(\check{\sigma} \cap M, \mathbb{C}) & \longrightarrow & Hom_{sg}(\check{\gamma} \cap M, \mathbb{C}) \end{array}$$

where the horizontal maps are restrictions and the vertical are extensions. Then the induced diagram also commutes:

$$\begin{array}{ccc} U_\sigma(\tau) & \hookrightarrow & U_\gamma(\tau) \\ \downarrow & & \downarrow \\ U_\sigma & \hookrightarrow & U_\gamma \end{array}$$

The maps glue to define the closed inclusion

$$V(\tau) \hookrightarrow X(\Sigma) = V(\{0\})$$

Similarly if τ is a face of τ' then $V(\tau') \hookrightarrow V(\tau)$.

References

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