FEBRUARY 11

Contents

1. some algebraic torminology

For more details we refer to [H]. First some notation.

Recall that a *lattice* (for us) is a discrete subgroup of $\mathbb{R}^n$. So the element of a lattice $M \subset \mathbb{R}^n$ can be identified with all integral linear combinations of a basis of $\mathbb{R}^n$. Under this identification the lattice M is isomorphic (as lattice) to $\mathbb{Z}^n$.

A *semigroup* is a set S with an associative binary operation and an identity 0.

A semigroup is finitely generated if there is a finite subset $\mathcal{A} \subset S$ such that

$$S = \mathbb{N}\mathcal{A} = \{\sum_{m \in \mathcal{A}} a_m m \text{ s.t. } a_m \in \mathbb{N}\}.$$

Definition 1.1. A finitely generated semigroup $S = \mathbb{N}\mathcal{A}$ is called an *affine semigroup* if

- the binary operation is commutative
- It can be embedded in a lattice.

To an affine semigroup, we associate a $\mathbb{C}$-algebra, called the semigroup algebra:

$$\mathbb{C}[S] = \{\sum_{m \in S} c_m \chi^m \text{ s.t. } c_m \in \mathbb{C} \text{ and } c_m = 0 \text{ for all but finitely many } m\}$$

Example 1.2. If $\mathcal{A} = \{\pm e_1, ..., \pm e_n\}$, where $(e_1, ..., e_n)$ is a basis for $\mathbb{Z}^n = M$, then $S = M$ and

$$\mathbb{C}[M] = \mathbb{C}[t_1, t_1^{-1}, ..., t_n, t_n^{-1}] = \mathbb{C}[(\mathbb{C}^*)^n].$$

2. Affine toric varieties via affine semigroups

Let $S = \mathbb{N}\mathcal{A}$ be an affine semigroup, fix an embedding in a lattice $M \cong \mathbb{Z}^n$, and let $\mathcal{A} = \{m_1, ..., m_s\} \subset M$. After identifying M with the chacter lattice of a torus $T_N \cong (\mathbb{C}^*)^n$, the elements of $\mathcal{A}$ can be identified with the characters χ^{m_i}. The semigroup algebra is (as seen above):

$$\mathbb{C}[S] = \mathbb{C}[\chi^{m_1}, ..., \chi^{m_s}].$$

It is almost immediate to see that

> *Every finitely generated affine semigroup S defines an affine toric variety Y_S, whose coordinate ring is $\mathbb{C}[S]$.*

We will use the notation:

$$Y_S = Spec(\mathbb{C}[S]).$$

Observe that because $\mathbb{C}[\chi^{m_1}, ..., \chi^{m_s}] \subset \mathbb{C}[T_N]$, it is an integral domain and hence the associated affine variety is irreducible.

After the identifications above we see that the associated variety is a toric variety: $Y_S = Y_{\mathcal{A}} = V(I_{\mathcal{A}})$, where $I_{\mathcal{A}} = Ker(F)$ and

$$F : \mathbb{C}[x_1, ..., x_s] \to \mathbb{C}[\chi^{m_i}, ..., \chi^{m_s}] \subset \mathbb{C}[T], x_i \mapsto \chi^{m_i}.$$

We have then sen that:

> *The two constructions of defining an affine toric variety from a sublattice $\mathcal{A}$ or a finitely generated affine semigroup S are equivalent.*

3. toric ideals

We will now see yet another construction which will turn out to be equivalent to the two previous ones. We will prove that

> *An affine variety is a toric variety $Y_{\mathcal{A}}$, associated to the a lattice subset (or equivalently to a f.g. affine semigroup) if anf only if it is defined by a toric ideal.*

In the notation of the previous section, consider the two sequences: Consider the two sequences:

$$0 \longrightarrow I_{\mathcal{A}} \longrightarrow \mathbb{C}[x_1, ..., x_s] \xrightarrow{\phi^*_{\mathcal{A}}} \mathbb{C}[\chi^{m_i}, ..., \chi^{m_s}]$$

$$0 \longrightarrow L \longrightarrow \mathbb{Z}^s \xrightarrow{\phi'_{\mathcal{A}}} M$$

where $\phi^*_{\mathcal{A}}(x_i) = \chi^{m_i}$ and $\phi'_{\mathcal{A}}(e_i) = m_i$.

Proposition 3.1.

$$I_{\mathcal{A}} =< x^{\alpha} - x^{\beta}, \alpha, \beta \in \mathbb{N}^s \text{ and } \alpha - \beta \in L >$$

Proof. Let $I = \{x^{\alpha} - x^{\beta}, \alpha, \beta \in \mathbb{N}^s$ and $\alpha - \beta \in L\}$. It is not difficult to see that $I \subseteq I_{\mathcal{A}}$. Assume that $I_{\mathcal{A}} \setminus I \neq \emptyset$ and let $f \in I_{\mathcal{A}} \setminus I \neq \emptyset$ be the element with minimal (after having set a term ordering) leading coefficient x^{α}. Then after possibly rescaling:

$$f = x^{\alpha} + f_2 \text{ and } f(t^{m_1}, ..., t^{m_s}) = 0$$

This implies that f_2 has a term x^{β} cancelling x^{α}, i.e.

$$f = x^{\alpha} - x^{\beta} + f_3 \text{ and } \Pi(t_i^{m_i})^{\alpha_i} = \Pi(t_i^{m_i})^{\beta_i}$$

where $\alpha = (\alpha_1, ..., \alpha_s), \beta = (\beta_1, ..., \beta_s)$. which says that $x^{\alpha} - x^{\beta} \in I$ and thus $f_3 \in I_{\mathcal{A}} \setminus I$.

But this is impossible since f_3 has lower leading term than f. □

This type of ideals: generated by binomials and prime are called *toric ideals*:

Definition 3.2. Let $L \subset \mathbb{Z}^s$ be a sublattice. The ideal $I_L = \{x^{\alpha} - x^{\beta}$ s.t. $\alpha - \beta \in L\}$ is a *lattice ideal* . A prime lattice ideal is called a *toric ideal.*

Corollary 3.3. *An affine variety $V(I)$ is defined by a toric ideal if and only if $V(I) = Y_{\mathcal{A}}$ is a toric variety, defined by an appropriate $\mathcal{A}$.*

Proof. From the above proposition it follows that $Y_{\mathcal{A}}$ is an affine variety defined by a toric ideal.

Assume that $V(I) \subset \mathbb{C}^s$ is defined by a toric ideal. Because $(1, ..., 1) \in V(I)$, $V(I) \cap (\mathbb{C}^*)^s$ is a subvariety (closed) of the torus $(\mathbb{C}^*)^s$ and it is not difficult to see that it is a subgroup. It follows that $V(I) \cap (\mathbb{C}^*)^s = T \cong \mathbb{Z}^k$ is a torus. The inclusion $T \subset (\mathbb{C}^*)^s$ induces a map

$$\mathbb{Z}^s \to Hom_{AG}(T, \mathbb{C}^*), e_i \mapsto \chi^{m_i}$$

By construction it is $V(I) = Y_{\mathcal{A}}$, for $\mathcal{A} = \{m_1, ..., m_s\} \subset \mathbb{Z}^k$. □

4. EQUIVALENT CONSTRUCTIONS

So far we have seen that for an affine variety X, the follwing are euivalent:

$$X = Y_{\mathcal{A}} \Leftrightarrow X = V(I) \text{ where } I \text{ is toric } \Leftrightarrow X = Spec(\mathbb{C}[\mathbb{N}\mathcal{A}]) \Rightarrow X \text{ is toric}$$

The following proposition will reverse the last arrow and concluse the list of equivalent constructions of an affine toric variety. We will need the following result, which will be part of Assignment 1.

Lemma 4.1. *Let T be a torus.*

(1) Consider the self action of T, by multiplication, and the induced action on $\mathbb{C}[T]$. For every t let $H_t : \mathbb{C}[T] \to \mathbb{C}[T]$ be the induced map of $\mathbb{C}$-algebras defined by:

$$H_t(\chi^m) = \chi^m(t)\chi^m.$$

Show that the only possible eigenvectors of H_t are characters χ^m, with eigenvalue $\chi^m(t)$.

(2) *Assume that T acts (the action is denoted by $\star$) algebraically on a finite dimensional vector space W. This means that the map:*

$$T \to GL(W), t \mapsto H_t(w) = t \star w,$$

is a morphism of affine algebraic groups. Let M be the lattice of characters of T. Show that

$$W = \bigoplus_{m \in M} W_m, \text{ where}$$

$$W_m = \{w \in W \text{ s.t. } t \star w = \chi^m(t) w \text{ for all } t \in T\}$$

Proposition 4.2. *Let V be an affine toric variety. Then $V = Spec(\mathbb{C}[S])$ fpr an affine semigroup S.*

Proof. (As in [CLS]. Let $T_N \subset V$ be the Zariski open torus contained in V and let M be the character lattice. The induced map $\mathbb{C}[V] \to \mathbb{C}[T] = \mathbb{C}[M]$ is injective (because V is the Zariski-closure of T). Define:

$$S = \{m \in M \text{ s.t. } \chi^m \in \mathbb{C}[V]\}$$

It is a semigroup embedded in a lattice, $S \subset M$. To conclude the prove it remains to show that $\mathbb{C}[S] = \mathbb{C}[V]$ and that S is finitely generated.

It is clearly $\mathbb{C}[S] \subset \mathbb{C}[V]$. Let now $f \in \mathbb{C}[V] \subset \mathbb{C}[M]$. Then

$$f = \sum_{m \in B} c_m \chi^m$$

where $B \subset M$ and $c_m \neq 0$. Consider the finitely generated vector space $W = Span(\chi^m, m \in B) \subset \mathbb{C}[M]$ and notice that $f \in W \cap \mathbb{C}[V]$. The induced action of T_N on $\mathbb{C}[M]$ leaves invariant the subspace $W \cap \mathbb{C}[V]$ (check!) and thus induces an action on the finitely dimensional vector space $W \cap \mathbb{C}[V]$. It follows that

$$W \cap \mathbb{C}[V] = \oplus_{m \in M} W_m.$$

But since $W \in \mathbb{C}[M]$, the possible eigenvectors (hence factors of f) are the characters and because we are in $\mathbb{C}[V]$. all factors belong to $\mathbb{C}[V]$. This shows that $f \in \mathbb{C}[S]$.

Moreover, because $\mathbb{C}[V]$ is the quotient of a finitely generated $\mathbb{C}$- algebra, it is finitely generated: $\mathbb{C}[V] = \mathbb{C}[f_1, ..., f_t]$, where $f_i = \sum_{m \in B_i} c_m \chi^m$. It follows that S is generated by $B_1 \cup ... \cup B_t$. □

References

[AM] Atiyah, M. F.; Macdonald, I. G. Introduction to commutative algebra. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont. 1969 ix+128 pp.

[CLS] D. Cox, J. Little, H. Scenck. Toric Varieties.

[H] Humphreys, James E. Linear algebraic groups. Graduate Texts in Mathematics, No. 21. Springer-Verlag, New York-Heidelberg, 1975. xiv+247 pp.