FEBRUARY 18

Contents

1. Fixed points

Using all these different characterizations, points on an affine toric variety can be seen in different ways:

Lemma 1.1. *Let V be an affine toric variety- Let M be the character lattice and $\mathbb{C}[V] = \mathbb{C}[S]$ the coordinate ring, where $S \subset M$ is the associated semigorup. There are one to one correspondences:*

$$\{p \in V\} \leftrightarrow \{m_p, \text{ maximal ideals in } \mathbb{C}[V]\} \leftrightarrow \{f_P : S \to \mathbb{C} \text{ s.g. morphism}\}$$

Where $\mathbb{C}$ has the multiplicative semigroup structure.

Proof. We have already seen the first correspondence. Given a maximal ideal $m_p \in \mathbb{C}[V]$ we can define a semigroup morphism by

$$f_P(m) = \chi^m(p).$$

Vice versa, given a semigroup morphism $f : S \to \mathbb{C}$, consider the induced morphism of $\mathbb{C}$-algebras: $f^* : \mathbb{C}[S] \to \mathbb{C}$ by $f^*(\chi^m) = f(m)$. It is a surjective morphism ans thus

$$\mathbb{C}[S]/Ker(f^*) \cong \mathbb{C}$$

which implies that $Ker(f^*)$ is a maximal ideal and hence $Ker(f^*) = m_p$. □

Remark 1.2. Notice that the torus action induces a map of $\mathbb{C}$-algebras:

$$\mathbb{C}[V] \to \mathbb{C}[T \times V]$$

The image of χ^m is denoted by $\chi^m \otimes \chi^m$ where $(\chi^m \otimes \chi^m)(t, p) = \chi^m(t)\chi^m(p)$. It implies that the semigroup morphism associated to tp is:

$$f_{tp}(m) = \chi^m(t) f_p(m)$$

Using this characterization one can understand the fixed points of the torus action:

Proposition 1.3. *Let* $Y_{\mathcal{A}} = Spec(\mathbb{C}[S])$ *be an affine toric variety. Then the torus action has a fixed point if and only if* $0 \in Y_{\mathcal{A}}$ *and* $S \cap (-S) = \{0\}$, *in which case* 0 *is the unique fixed point.*

Proof. Let $V = Y_{\mathcal{A}} = Spec(\mathbb{C}[S])$, where $\mathcal{A} \subset S \setminus \{0\}$. A point $p \in V$ is a fixed point if $tp = p$ for every $t \in T_N$. Equivalently p is a fixed point if $f_p(m) = f_{tp}(m)$ for all $m \in M$ and for all $t \in T_N$. This means that

$$f_p(m) = \chi^m(t) f_p(m) \text{ for all } m \in S \text{ and for all } t \in T_N.$$

This is clearly satisfied if $m = 0$. If $m \neq 0$ then it has to be $f_p(m) = 0$. It follows that the function f_p is defined as

$$f_p(m) = \begin{cases} 1 & m = 0 \\ 0 & m \neq 0 \end{cases}$$

Moreover this is a morphism of semigroup only if 0 is the only invertible element of S, i.e. $S \cap (-S) = \{0\}$. Notice that the corresponding point is then p such that $\chi^m(p) = 0$ for all $m \neq 0$, and thus $p = 0$.

□

2. The tangent space of a hypersurface

Let $f \in \mathbb{C}[x_1, ..., x_n]$ be an irreducible polynomial. The zero set of f, $V(f)$, is an irreducible affine variety. Let $p \in V(f)$.

Definition 2.1. $V(f)$ is singular at P if and only it $\frac{\partial f}{\partial x_i}(P) = 0$ for $i = 1, ..., n$.

If $P = (a_1, ..., a_n)$, the Jacobian matrix

$$Jac(f)_P = (\frac{\partial f}{\partial x_i}(P))$$

defines a linear map whose kernel is by definition the tangent space of V at p, $T_P V$.

It is $T_P V \cong \mathbb{C}^n$ if V is singular at p and $T_P V \cong \mathbb{C}^{n-1}$ if the V is non singular at p. Than $T_P V$ is an affine variety isomorphic to $\mathbb{C}^n$ if p is singular and it is an hyperplane otherwise.

If a point P is non singular, then around P, V has the structure of a manifold, i.e. it has a local coordinate system $(y_1, ..., y_n)$ via an isomorphism of a neighborhood with $\mathbb{C}^{n-1}$. The function $P : (x_1, ..., x_n) \mapsto (f(x), x_2, ..., x_3)$ has a non vanishich Jacobian at P. Then by the inverse function theorem there is a neighborhood $P \in U \subset V$ and a diffeomorphism $U \cong P(U)$. This means that $(f(x), x_2, ..., x_n)$ forms a coordinate system around P and thus $(x_2, ..., x_n)$ is a coordinate system in $V \cap \mathbb{C}^n$ around P.

Example 2.2. $V(xy - zw)$ is non singular at $(1, 1, 1, 1)$ and singular at 0

An important fact in algebraic geometry is that any non empty Zariski-open subset of an irreducible affine variety V is DENSE. In fact, in any topology, a subset is dense if it meets all the open subsets. Because V is irreducible, any two proper Zariski open subsets must intersect.

Notice that the set of singular points of the irreducible affine variety $V = V(f)$ is

$$V_{sing} = V(f, \frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n}) \subset V.$$

It follows that the set of non singular points is a Zariski open subset. One sees that it is always non empty. If it were empty, then $V_{sing} = V$and thus all $\frac{\partial f}{\partial x_i}$ would vanish on V, and thus they would all be divisible by f. But if we view $\frac{\partial f}{\partial x_i}$ as a polynomial in x_i then $\deg(\frac{\partial f}{\partial x_i}) < \deg(f)$ and hence $f / \frac{\partial f}{\partial x_i}$ implies $\frac{\partial f}{\partial x_i} = 0$. But this means that x_i does not appear in f, i.e. $f = costant$.

We conclude that:

Lemma 2.3. *Let f be an irreducible polynomial, then the set of non singular points of $V(f)$ is a dense open Zariski open set.*

3. The tangent space of an irreducible variety

Given a $\mathbb{C}$ vector spaces we will denote by V^* the dual vector space, i.e. the vector space of linear functions to $\mathbb{C}$.

Now, let $V \in \mathbb{C}^n$ be an irreducible algebraic variety and let $I(V) = (f_1, ..., f_m)$. For an $f \in \mathbb{C}[x_1, ..., x_n]$ and $P = (a_1, ..., a_n) \in \mathbb{C}^n$ we define the linear polynomial:

$$f^{(1)} = \sum_i \frac{\partial f}{\partial x_i}(P)(x - a_i).$$

Definition 3.1. The tangent space of V at $P \in V$ is

$$T_P V = \bigcap_{i=1}^{m} V(f_i^{(1)}).$$

It can be identified with the kernel of the linear map defined by the $n \times m$ Jacobian matrix:

$$Jac_P = (\frac{\partial f_j}{\partial x_i}(P)).$$

and therefore $\dim(T_P V) = n - rank(Jac_P)$.

Equivalently $S_r = \{P \in V | \dim(T_P V) \geq r\}$ is a closed (subvariety) of V.

It follows that the function $V \to \mathbb{N}$ is upper-semicontinuous (in the Zariski-topology.)

Corollary 3.2. *Let V be an irreducible subvariety. Then there exists a dense Zariski open subset U_0 and an integer r such that:*

$$\dim_P V = r \textit{ for all } P \in U_0 \textit{ and } \dim_P V \geq r \textit{ for all } P \notin U_0$$

Definition 3.3. In the notation above we set

$$r = \dim(V)$$

and therefore we say that a point $P \in V$ is singular if and only if $\dim(T_P V) \geq \dim(V)$.

References

[AM] Atiyah, M. F.; Macdonald, I. G. Introduction to commutative algebra. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont. 1969 ix+128 pp.

[CLS] D. Cox, J. Little, H. Scenck. Toric Varieties.

[H] Humphreys, James E. Linear algebraic groups. Graduate Texts in Mathematics, No. 21. Springer-Verlag, New York-Heidelberg, 1975. xiv+247 pp.