FEBRUARY 4

Contents

1. some algebraic torminology

For more details we refer to [H]. First some notation.

Recall that a *lattice* (for us) is a discrete subgroup of $\mathbb{R}^n$. So the element of a lattice $M \subset \mathbb{R}^n$ can be identified with all integral liear combinations of a basis of $\mathbb{R}^n$. Under this identification the lattice M is isomorphic (as lattice) to $\mathbb{Z}^n$.

2. algebraic tori

Definition 2.1. A *linear algebraic group* is an affine variety G having the structure of a group such that the multiplication map and the inverse map are morphisms of affine varieties.

Let G, G' be two linear algebraic groups, a morphism $G \to G'$ of algebraic groups is a map which is a morphism of affine varieties and a homomorphism of groups. We will indicate the SET of such morphisms with $Hom_{AG}(G, G')$.

Recall that when G, G' are abelian $Hom_{AG}(G, G')$ is an abelian group.

Example 2.2. The classical examples of algebraic groups are:$(\mathbb{C}^*)^n, GL_n, SL_n$.

Definition 2.3. An n-dimensional *algebraic torus* is an affine variety T, isomorphic to $(\mathbb{C}^*)^n$.

An algebraic torus is a group, with the group operation that makes the isomorphism (of affine varieties) a group-homomorphism. Hence an algebraic torus is a linear algebraic group.

From now on we will drop the adjective algebraic in algebraic torus.

Definition 2.4. Let T be a torus.

- An element of the abelian group $Hom_{AG}(\mathbb{C}^*, T)$ is called a *one parameter subgroup* of T.
- An element of the abelian group $Hom_{AG}(T, \mathbb{C}^*)$ is called a *character* of T.

Lemma 2.5.
$$Hom_{AG}(\mathbb{C}^*, \mathbb{C}^*) \cong \mathbb{Z}.$$

Proof. First note that an invertible element of the ring $\mathbb{C}[t, t^1]$ is of the form at^k for a constant $a \in \mathbb{C}$ and an integer $k \in \mathbb{Z}$. Let $\chi : \mathbb{C}^* \to \mathbb{C}^*$ be an element of $Hom_{AG}(\mathbb{C}^*, \mathbb{C}^*)$. It induces a ring homomorphism:
$$\chi^* : \mathbb{C}[t, t^{-1}] \to \mathbb{C}[t, t^{-1}]$$
where $\chi^*(t) = \chi(t) = at^k$ is an invertible polynomial. Observe that the moltiplication morphism induces the ring homorphism:
$$\mathbb{C}[t, t^{-1}] \to \mathbb{C}[t_1, t_1^{-1}, t_2, t_2^{-1}],\ t \mapsto t_1 t_2.$$
Moreover since χ is a group homorphism, we have the following diagram:
$$\begin{array}{ccc} \mathbb{C}[t, t^{-1}] & \longrightarrow & \mathbb{C}[t_1, t_1^{-1}, t_2, t_2^{-1}] \\ \downarrow{\scriptstyle \chi^*} & & \downarrow{\scriptstyle \chi^*} \\ \mathbb{C}[t, t^{-1}] & \longrightarrow & \mathbb{C}[t_1, t_1^{-1}, t_2, t_2^{-1}] \end{array}$$
Which implies that $\chi^*(t^2) = \chi^*(t)^2$. It follows that $a = 1$ and thus that $\chi(t) = t^k$. □

Note that:
$$Hom_{AG}(\mathbb{C}^*, T) \cong Hom_{AG}(\mathbb{C}^*, (\mathbb{C}^*)^n) = \sum_1^n Hom_{AG}(\mathbb{C}^*, \mathbb{C}^*) = \mathbb{Z}^n = N$$
as abelian groups. An important consequence of this simple fact is that the two abelian groups $Hom_{AG}(\mathbb{C}^*, T), Hom_{AG}(T, \mathbb{C}^*)$ can be identified as dual lattices.

First observe that the composition morphism of abelian varieties:
$$Hom_{AG}(\mathbb{C}^*, T) \times Hom_{AG}(T, \mathbb{C}^*) \to \mathbb{Z}$$
is a perfect pairing of $\mathbb{Z}$-modules, in fact it induces the isomorphism:
$$Hom_{AG}(T, \mathbb{C}^*) \to Hom(N, \mathbb{Z}), \lambda(\chi) = \chi \circ \lambda.$$
(In the assignment I you will verify that it is an isomorphism). It follows that
$$Hom_{AG}(T, \mathbb{C}^*) = M \cong N^*$$

Remark 2.6. Given a map $\phi : X \to Y \subset \mathbb{C}^m$ of affine toric varietie, we define the image variety as
$$V(I_\phi), \text{ where } I_\phi = Ker(f^*) = \{f \in \mathbb{C}[x_1, ..., x_m] \text{ s.t. } f \circ \phi = 0\}.$$
In the assignment you will show that I_ϕ is an ideal such that $Im(\phi) \subset V(I_\phi)$.

Moreover you will prove that if $T_1 \cong (\mathbb{C}^*)^k, T_2 \cong (\mathbb{C}^*)^t$ are algebraic tori and $\phi \in Hom_AG(T_1, T_2)$, then $Im(\phi) = V(I_\phi)$ and $V(I_\phi)$ is an algebraic torus.

Another important fact, whose proof can be found in [H] is that:

Lemma 2.7. *Any irreducible closed subgroup of a torus (i.e. an irreducible affine sub-variety which is a subgroup) is a sub-torus.*

3. Some examples of affine toric varieties

Recall that:

Definition 3.1. An affinie toric variety is an irreducible affine variety, containing a torus as a Zariski-open subset and such that the multiplicative action of the torus on itself extends to the whole variety.

In other words and affine toric variety is an affine variety endowed with an action of a torus, such that the action has a free open orbit (i.e. without fixed points).

Let Xbe an affine toric variety, containing the torus $T \cong (\mathbb{C}^*)^n$as Zariski open set. We define:

$$\dim(X) = n.$$

Example 3.2. Consider the affine variety $V(x^3 - y^2)$.

Because $x^3 - y^2$ is an irreducible polynomial, the ideal $(x^3 - y^2)$ is prime and hence the variety is irreducible. The map: $f : \mathbb{C}^* \to \mathbb{C}^2$ defined by $t \mapsto (t^2, t^3)$ is a morphism such that $Im(f) = V(x^3 - y^2)$. The rational map $(x, y) \mapsto \frac{y}{x}$ whose domain is $\mathbb{C}^2 \setminus V(x) \cap V(x^3 - y^2)$ is its inverse.

It follows that $(\mathbb{C}^2 \setminus V(x)) \cap V(x^3 - y^2)$ is a Zariski open subset of $V(x^3 - y^2)$ isomorphic to the 1-dimensional torus $\mathbb{C}^*$.

Moreover the self action of the torus extends to the algebraic action:

$$t(x, y) = (t^2 x, t^3 y).$$

Example 3.3. $V(xy - zw)$ as seen before.

4. Affine toric varieties from lattice points

In this section we will see that:

Any subset $\mathcal{A} = \{m_1, ..., m_s\}$ consisting of s elements of a lattice M defines an affine toric variety $Y_{\mathcal{A}} \subset \mathbb{C}^s$ of dimension $rank(\mathcal{A})$. The coordinate ring $\mathbb{C}[Y_{\mathcal{A}}] \cong \mathbb{C}[\chi^{m_i}, ..., \chi_{m_s}]$ (as $\mathbb{C}$-algebras) and $Y_{\mathcal{A}} = V(I_{\mathcal{A}})$ where $I_{\mathcal{A}} = Ker(\phi^*_{\mathcal{A}})$ and

$$\phi^*_{\mathcal{A}} : \mathbb{C}[x_1, ..., x_s] \to \mathbb{C}[Y_{\mathcal{A}}], x_i \mapsto \chi^{m_i}$$

Let $M \cong \mathbb{Z}^n$ be a lattice and let $N = M^*$. The lattice M can be identified with the character group of a torus T_N.

$$T_N \cong (\mathbb{C}^*)^n, Hom_{AG}(\mathbb{C}^*, T_N) = N, Hom_{AG}(T_N, \mathbb{C}^*) = N^* = M.$$

We will denote the character corresponding to the point $m \in M$ with χ^m.

A choice of lattice points $\mathcal{A} = \{m_1, ..., m_s\} \subset M$ determines a morphism of affine varieties:

$$\psi_{\mathcal{A}} : T_N \to \mathbb{C}^s, t \mapsto (\chi^{m_1}(t), ..., \chi^{m_s}(t))$$

Let $\mathcal{A}$ also denote the $n \times s$ matrix where the lattice points m_i are columns.

This map can be seen as a map between two tori and hence the image will be a torus: $\phi_{\mathcal{A}}(T_N) \cong (\mathbb{C}^*)^k$. Observe that $\phi_{\mathcal{A}}(T_N)$ is closed in $(\mathbb{C}^*)^k$, but open in $\mathbb{C}^s$.

Definition 4.1. Let X be a subset of $\mathbb{C}^s$. The Zariski-clousure $\overline{X}$ is defined to be the smallest affine variety of $\mathbb{C}^s$ containing X.

So we have:

$$\phi_{\mathcal{A}}(T_N) \subset \overline{\phi_{\mathcal{A}}(T_N)} \subset \mathbb{C}^s$$

Notice that, because the torus $\phi_{\mathcal{A}}(T_N)$ is irreducible, so is $\overline{\phi_{\mathcal{A}}(T_N)}$.

The torus $\phi_{\mathcal{A}}(T_N)$ acts on $\mathbb{C}^s$ via multiplication (it is an algebraic action extending the self action) and $\overline{\phi_{\mathcal{A}}(T_N)}$ is invariant under this action, i.e. the action restricts to an algebraic action:

$$\phi_{\mathcal{A}}(T_N) \times \overline{\phi_{\mathcal{A}}(T_N)} \to \overline{\phi_{\mathcal{A}}(T_N)}.$$

This is because the image of the action map is a variety containing the torus $\phi_{\mathcal{A}}(T_N)$.

After identifying $Hom_{AG}(\phi_{\mathcal{A}}(T_N), \mathbb{C}^*) = M(A)$ with $\mathbb{Z}^r$, we see that the map $\phi_{\mathcal{A}}$induces a map:

$$\phi'_{\mathcal{A}} : \mathbb{Z}^s \to M$$

where the stantard basis-element e_i i mapped to m_i. The image can be identified with $M(A)$ and it has dimension equal to $rank(\mathcal{A}) = r$.

All the above proves that $\overline{\phi_{\mathcal{A}}(T_N)} = Y_{\mathcal{A}}$ is an affine toric variety of dimension $rank(\mathcal{A}) = r$.

It follows that the associated map of $\mathbb{C}$-algebras is

$$\phi^*_{\mathcal{A}} : \mathbb{C}[x_1, ..., x_s] \to \mathbb{C}[Y_{\mathcal{A}}], x_i \mapsto \chi^{m_i}$$

and that $Y_{\mathcal{A}} = V(Ker(\phi^*_{\mathcal{A}}))$.

Because $\phi^*_{\mathcal{A}}$ is onto we can write:

$$\mathbb{C}[Y_{\mathcal{A}}] = \mathbb{C}[\chi^{m_1}, ..., \chi^{m_s}].$$

The $\mathbb{C}$-algebra generated by the χ^{m_i}, where the multiplication is given by $\chi^{m_i}\chi^{m_j} = \chi^{m_1+m_j}$. One sees easily that if S is the finitely generated semigroup $\mathbb{N}\mathcal{A}$, then

$$\mathbb{C}[\chi^{m_1}, ..., \chi^{m_s}] = \{\sum_{m \in S} c_m \chi^m \text{ s.t. } c_m \in \mathbb{C} \text{ and } c_m = 0 \text{ for all but finitely many } m\}.$$

Example 4.2. Consider $\{(1,0,0), (0,1,0), (0,0,1), (1,1-1)\} = \mathcal{A} \subset \mathbb{Z}^3$. $Y_{\mathcal{A}} = V(xy - zw)$ is the quadric threefold $(rk(A) = 3)$ in $\mathbb{C}^4$.

References

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