JANUARY 21

Contents

The aim of the next few lectures is to make precise the following definition:

Definition 0.1. An **affine toric variety** is an IRREDUCIBLE AFFINE VARIETY, containing a TORUS as Zariski-open space and such that the multiplicative action of the torus on itself extends to the whole variety.

1. Some notation

Let $\mathbb{C}$ be the field of complex numbers.

Definition 1.1. $\mathbb{C}^n = \{\underline{a} = (a_1, ..., a_n) \| a_i \in \mathbb{C}\}$ is called the n-dimensional affine space.

Recall that $\mathbb{C}[x_1, ..., x_n]$ is a U.F.D., i.e. every polynomial can be uniquely written as product of irreducible polynomials. Given a polynomial $f \in \mathbb{C}[x_1, ..., x_n]$, $f = 0$ could mean:

(1) f is the zero polynomial, i.e. $f = \sum h_\alpha \underline{x}^\alpha$, where $h_\alpha = 0$.
(2) f is the zero-function, i.e. $f(\underline{a}) = 0, \forall \underline{a} \in \mathbb{C}^n$.

Note also that:

Lemma 1.2.
- *(1) and (2), because $\mathbb{C}$ is infinite.*
- *We can then say that for $f, g \in \mathbb{C}[x_1, ..., x_n]$, $f = g$ as polynomials iff $f = g$ as functions.*

2. Some algebra

For more details see [AM].

Recall that a field $\mathbb{C}$ is *algebraically closed*, i.e. every non-constant $g \in \mathbb{C}[x_1, ..., x_n]$ has at least one root in $\mathbb{C}$.

Definition 2.1. Let R be a ring (like $\mathbb{C}[x_1, ..., x_n]$)

(1) A set $I \subseteq R$ is an *ideal* if:
  (a) $0 \in I$,
  (b) $f, g \in \Rightarrow f + g \in I$,

   (c) $f \in I, h \in R \Rightarrow hf \in I$.

(2) The ideal generated by $f_1, ..., f_k \in R$ is the smallest ideal containing $f_1, ..., f_k$ and it is defined by:

$$(f_1, ..., f_k) = \{\sum_1^s h_i f_i, h_i \in R\}.$$

(3) An ideal I is finitely generated if there are $f_1, ..., f_k \in R$ so that $I = (f_1, ..., f_k)$.

(4) An ideal I is *prime* if $I \neq 1$ and $xy \in I \Rightarrow x \in I$ or $y \in I$. An ideal I is prime iff the quotient ring R/I is an *integral domain*(this means that no element $r \neq 0$ is a zero divisor. ex. $\mathbb{Z}, \mathbb{C}[x_1, ..., x_n]$).

(5) An ideal I is *maximal* if $I \neq 1$ and there is no ideal $M \in R$ such that $m \subset M \subset (1) = R$. An ideal I is maximal iff the quotient ring R/I is a *field*.

Exercise 2.2.
- The only ideals of a field are (1) and (0).
- Let $R = \mathbb{C}[x_1, ..., x_n]$ and let f be an irreducible polynomial, then (f) is prime.

Definition 2.3. A ring R is said to be *Noetherian* if equivalently:
- Every non empty set of ideals has a maximal element.
- Every ideal is finitely generated.

The ring $\mathbb{C}[x_1, ..., x_n]$ is Noetherian (Hilbert basis theorem).

Definition 2.4. Let I be an ideal in R, the radical of I is defined as:

$$\sqrt{I} = \{f \in R | f^n \in R, \text{ for some } n\}.$$

For example $\sqrt{(x^2)} = (x)$.

3. Affine varieties

Let $I \in \mathbb{C}[x_1, ..., x_n]$ be an ideal, the solution set

$$V(I) = \{\underline{a} \in \mathbb{C}^n \| f(\underline{a}) = 0 \text{ for all} f \in I\}$$

is called an *affine variety*. From the Hilbert basis theorem $I = (f_1, ..., f_k)$ for $f_i \in \mathbb{C}[x_1, ..., x_n]$ and thus every affine variety

$$V(I) = V((f_1, ..., f_k))$$

is the solution set of a system $f_1 = 0, ..., f_k = 0$.

Example 3.1.
- $V((0)) = \mathbb{C}^n$.
- $V((1)) = \emptyset$.
- $V((y - x^2, z - x^3)) \subset \mathbb{C}^3$ is called the *twested cubit*.
- $V((x^2 + y^2 + z^2)) \subset \mathbb{C}^3$ is the smooth quadric surface.
- $V(xy - 1) \subset \mathbb{C}^2$ can be ideantified with $\mathbb{C}^* = \{0 \neq x \in \mathbb{C}\}$.

Lemma 3.2. *Let $V = V(f_1, ..., f_k), W = V(g_1, ..., g_s)$ be two non-empty affine varieties, then:*

(1) $V \cap W = V(f_1, ..., f_k, g_1, ..., g_s) \subset \mathbb{C}^n$, *where* $V \subset \mathbb{C}^n, W \subset \mathbb{C}^n$.
(2) $V \cup W = V(f_i g_j, 1 \leq i \leq k, 1 \leq j \leq s)$, *where* $V \subset \mathbb{C}^n, W \subset \mathbb{C}^n$.
(3) If $V \subset \mathbb{C}^n, W \subset \mathbb{C}^m$, $V \times W = V(f_1, ..., f_k, g_1, ..., g_s) \subset \mathbb{C}^{n+m}$, *where the* f_i *and* g_j *are considered as polynomials in* $\mathbb{C}[x_1, ..., x_n, y_1, ..., y_n]$.

An important fact, which is a consequence of the fact that $\mathbb{C}$ is algebraically closed is that:

$$V(I) = \emptyset \Leftrightarrow I = \mathbb{C}[x_1, ..., x_n].$$

Definition 3.3. An affine variety V is *irredubible* if it cannot be written as the union of two non-empty proper affine subvarieties.

Given an affine variety $V \subseteq \mathbb{C}^n$, the set:

$$I(V) = \{f \in \mathbb{C}[x_1, ..., x_n] \| f(x) = 0 \forall x \in V\}$$

is an ideal of $\mathbb{C}[x_1, ..., x_n]$, with following proprties:

Lemma 3.4. *Let* V, W *be non empty affine varieties*

(1) $V \subseteq W \Leftrightarrow I(W) \subseteq I(V)$.
(2) $V = W \Leftrightarrow I(W) = I(V)$.
(3) $I(V)$ *is prime* $\Leftrightarrow V$ *is irreducible;*
(4) (Weak Nullstellensatz) For $(a_1, ..., a_n) = x \in V$, *then* $I(\{x\}) := m_x$ *is a maximal ideal. Moreover all the maximal ideal of* $\mathbb{C}[x_1, ..., x_n]$ *are of the form* $m_x = ((x_1 - a_1, ..., x_n - a_n)$. *This means that there is a one to one correspondence:*

$$\{ \textit{ maximal ideals of } \mathbb{C}[x_1, ..., x_n]\} \leftrightarrow \{ \textit{ points of } \mathbb{C}^n\}.$$

References

[AM] Atiyah, M. F.; Macdonald, I. G. Introduction to commutative algebra. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont. 1969 ix+128 pp.