MARCH 10

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1. Strongly rational cones

The following is a key result for the teory of toric varieties:

Proposition 1.1. *(Gordon's Lemma) Let σ be a rational polyhedral convex cone. Then $S_\sigma = \check{\sigma} \cap M$ is an affine semigroup.*

Proof. Let $\check{\sigma} = Cone(S')$ where $S' \subset M$ is finite. Then every element $w \in S_\sigma$ is of the form:

$$w = \sum \lambda_m m, m \in S', \lambda_m \geq 0.$$

Let $[\lambda]$ be the biggest positive integer before λ. Then $0 \leq \gamma = \lambda - [\lambda] < 1$ and

$$w = \sum_{m \in S'} [\lambda_m] m + \sum_{m \in S'} \gamma_m m.$$

the set $K = \{\sum_{m \in S'} \gamma_m m, 0 \leq \gamma < 1\}$ is a bounded region of $M_\mathbb{R}$ and thus $K \cap M$ is finite. It follows that $S_\sigma = \mathbb{N}(S') \cup (K \cap M)$, which means that it is finitely generated. □

The associated affine toric variety $Spec(\mathbb{C}[S_\sigma])$ will be denoted by U_σ.

Let $\mathcal{H}$ be the subset of irreducible elements of S_σ :

$$\mathcal{H} = \{m \in S_\sigma \text{ s.t. } m \neq m_1 + m_2 \text{ with } m_1, m_2 \neq 0\}.$$

Example 1.2. Consider the cone in the previous example $\sigma = Cone(e_2, 2e_1 - e_2)$. It defines a two-dimensional affine toric variety $spec(\mathbb{C}[S_\sigma])$ where

$$S_\sigma = \mathbb{N}(e_1^*, e_1^* + e_2^*, e_1 + 2e_2^*).$$

It is then the closure in $\mathbb{C}^3$ of the image of the map: $(t_1, t_2) \mapsto (t_1, t_1 t_2, t_1 t_2^2)$, which is $V(xz - y^2)$, a quadratic cone.

Definition 1.3. A cone $\sigma \subset \mathbb{R}^n$ is called *strongly convex* if $\{0\}$ is a face of σ. Equivalently σ is strongly convex if and only if $\sigma \cap (-\sigma) = 0$, if and only if $\dim(\check{\sigma}) = n$.

Let σ be a strongly convex cone and let ρ be an edge. Then $\rho = \mathbb{R}^+ e$, where e is the generator of the semigroup $\rho \cap N$. The lattice point e is called the ray of ρ.

Any strongly convex cone is generated by the ray generating its edges, called a minimal set of generators. Moreover

$$|\{\text{ edges of } \check{\sigma}\}| \geq n.$$

Definition 1.4. A strongly convex polytope σ is said to be *smooth* if its minimal set of generators is part of a lattice base of N.

A strongly convex polytope σ is said to be *simplicial* if its minimal set of generators are linearly independent over $\mathbb{R}$.

Notice that σ is smooth if and only of $\check{\sigma}$ is smooth. (Check!)

Let σ be a smooth cone in $\mathbb{R}^n$. Then it is $\sigma = Cone(e_1, ..., e_r)$ for a lattice basisi $(e_1, ..., e_r)$. It follows that $\check{\sigma} = Cone(e_1, ..., e_r, \pm e_{r+1}, ..., \pm e_n)$, and thus

$$U_\sigma \cong \mathbb{C}^r \times (\mathbb{C}^*)^{n-r}.$$

We see that if σ is smooth then:

- $\dim(\check{\sigma}) = n$ and thus σ a strongly rational smooth cone.
- U_σ is a non singular (smooth) affine variety.

Proposition 1.5. *Let N be an n-dimensional lattice and let $\sigma \subset N_{\mathbb{R}}$ be a rational cone. Then*

(1) If σ is strongly rational cone then the semigroup S_σ is saturated, i.e., if $km \in S_\sigma$ for some positive p, then $m \in S_\sigma$.
(2) If σ is strongly rational cone then $\mathcal{H}$ is a minimal generating set of S_σ, w.r.t. inclusion.
(3) σ is strongly rational cone if and only if $Spec(\mathbb{C}[S_\sigma])$ is an n-dimensional affine toric variety.

Proof. (1) It is a consequence of convexity.

(2) We have to prove that every element of S_σ is a sum of elements oh $\mathcal{H}$. The cone $\check{\sigma}$ is strongly rational and thus $\{0\}$ is a face, which means that there is $u \in \sigma \cap N \setminus \{0\}$ such that $< m, u >= 0$ only if $m = 0$, and $< m, u >> 0$ if $m \neq 0$.

If $m = m_1 + m_2, m_1, m_2 \neq 0$ in S_σ, then

$$< m, u >=< m_1, u > + < m_2, u >, i.e., < m, u >\geq< m_i, u > .$$

Induction on $< m, u >$, concludes that every element is the sum of irreducible elements. It implies that $\mathcal{H}$ generates S_σ. Check that it is indeed a minimal set of generators!

(3) If σ is strongly rational cone then $\dim(\check{\sigma}) = n$ which implies that $rank(\mathbb{Z}S_\sigma) = n$. The other direction is left as exercise.

□

In particular we have that if σ is a strongly rational cone then:

$$|\{H\}| = |\{\text{ edges of } \check{\sigma}\}| \geq n.$$

Example 1.6. $\sigma = Cone(de_1 - e_2, e_2)$.

2. Normal toric varieties

The property of the semigroup S_σ being saturated translates in Algebraic Geometry to the property of the affine variety being *normal*.

Definition 2.1. Let V be an irreducible affine variety. The ring $\mathbb{C}[V]$ is integrally closed if every element of $\mathbb{C}(V)$ which is a root of a monic polynomial in $\mathbb{C}[V][x]$, is in $\mathbb{C}[V]$. V is said to be *normal* if $\mathbb{C}[V]$ is integrally closed.

In particular smooth affine varieties are normal

The ring $\mathbb{C}[x_1, .., x_k, x_{k+1}, x_{k+1}^{-1}, ..., x_n, x_n^{-1}]$ is integrally closed, which implies that smooth toric varieties are normal,

Let $Y_\mathcal{A}$ be an affine toric variety of dimension n, with $\mathbb{N}\mathcal{A} = S$. The cone $Cone(\mathcal{A})^*$ is a rational convex cone. Moreover the cone has dimension equal to $rank(\mathbb{Z}\mathcal{A}) = n$ and thus $Cone(\mathcal{A})^*$ is a strongly rational convex cone in $N = \mathbb{Z}^n$, and

$$S \subset S_\sigma = \check{\sigma} \cap M$$

S is saturated, implies $S = S_\sigma$ (check!).

We can conclude:

Corollary 2.2. */Definition.*

An affine toric variety X is normal if and only is $X = Spec(\mathbb{C}[S_\sigma])$ where σ is a strongly convex polyhedral cone.

Recall that an affine toric variety $Spec(\mathbb{C}[S]) \subset \mathbb{C}^s$ has a fixed point if and only if $S \cap (-S) = \{0\}$. In this case the fixed point is the 0. In the case $S = S_\sigma$, this translates to the cone being strongly convex.

Consider the point $0 \in U_\sigma$. If the cone σ is strongly rational then $\mathbb{Z}S_\sigma = M$. In particular:

$$m_0 =< \chi^m \text{ s.t. } m \in S_\sigma \setminus \{0\} >= \bigoplus_{m \neq 0} \mathbb{C}\chi^m.$$

Bceause $\mathcal{H}$ is a minimal set of generators it is:

$$m_0^2 = \bigoplus_{m \notin \mathcal{H}} \mathbb{C}\chi^m.$$

We then get that:

$$m_0/m_0^2 = \bigoplus_{m \in \mathcal{H}} \mathbb{C}\chi^m$$

and thus that $\dim(T_0(U_\sigma)) = |\mathcal{H}|$.

This is a very important property of normal affine toric varieties because it says that the embedding in $\mathbb{C}^{|\mathcal{H}|}$ defined by $\mathcal{H}$ is the "best" affine embedding. In particular $|\mathcal{H}| \leq s$.

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