MARCH 17

Contents

1. Smooth toric varieties

Example 1.1. $\sigma = Cone(de_1 - e_2, e_2)$.

Proposition 1.2. *Let $V \subset \mathbb{C}^n$ be an affine toric variety. It is non singular if and only if $V = Spec(\mathbb{C}[S_\sigma])$ for a regular strongly convex polytope $\sigma \in \mathbb{R}^n$.*

Proof. If $V = Spec(\mathbb{C}[S_\sigma])$ for a regular strongly convex polytope $\sigma \in \mathbb{R}^n$, then the associated toric variety U_σ is non singular as observed.

Let now $V \subset \mathbb{C}^n$ be a smooth affine toric variety. It is normal and thus $V = Spec(\mathbb{C}[S_\sigma])$ for a strongly convex polytope $\sigma \in \mathbb{R}^n$. If $\dim(\sigma) = n$ let 0 be the fixed point, which is smooth, and thus $|\mathcal{H}| = \dim(U_\sigma) = n$.

We have:

$$n \leq |\{ \text{ edges of } \check{\sigma}\}| = |\mathcal{H}| = n.$$

In particular $\mathbb{Z}S_\sigma = M$ is generated by n rays, which implies that they are a lattice basis for M, i.e. $\check{\sigma}$ is smooth and thus so is σ.

If $\dim(\sigma) = r < n$ we can embedd the cone in the appropriate lower dimensional lattice. Let N_1 be the smallest saturated sublattice containing all the generators of σ. In particular σ is a rationally convex polytope in N_1 and $\dim(U_{\sigma,N_1}) = r = rk(N_1)$. Moreover the abelian group N/N_1 is torsion free and finitely generated and thus $N/N_1 \cong \mathbb{Z}^{n-r}$ and $N = N_1 \oplus N_2$. It follows that $M = M_1 \oplus M_2$ and that

$$S_\sigma = S_{\sigma,N_1} \oplus M_2,$$

where $S_{\sigma,N_1} = \check{\sigma} \cap M_1$. In particular $U_\sigma \cong (U_{\sigma,N_1}) \times T_2$, where T_2 is a torus of dimension $n - r$. We conclude that U_σ is smooth at every point $(p, q) \in U_{\sigma,N_1} \times T_2$, which implies that U_{σ,N_1} is smooth hence that σ is smooth in N_1, because it is maximal dimensional.

□

The Example 1.1 gives a normal toric variety, not smooth. It is simplicial.

2. Projective varieties

Recall the definition of projective space:

$$\mathbb{P}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\sim, (a_0, .., a_n) \sim \lambda(a_0, .., a_n), 0 \neq \lambda \in \mathbb{C}.$$

A point, i.e. an equivalence class wil be denoted by $(a : 0 : ... : a_n)$.

It is covered by affine patches. In fact, consider:

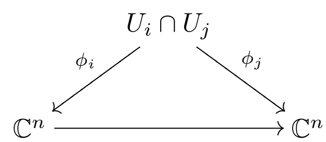
$$U_i = \{(a_0 : .. : a_n) \in \mathbb{P}^n \text{ s.t. } a_i \neq 0\}.$$

is obviously a cover and:

$$phi_i : U_i \to \mathbb{C}^n, (a_0 : ... : a_i : ... : a_n) \mapsto (a_0/a_i, .., a_n/a_i) \text{ and}$$

$$\phi_i : U_i \to \mathbb{C}^n, (a_1, ..., : a_n) \mapsto (a_1 : .. : 1 : ... : a_n)$$

define an isomorphism $U_i \cong \mathbb{C}^n$, for $i = 0, ..., n$. Moreover



where $\phi_i(U_i \cap U_j) \cong \phi_j(U_i \cap U_j)$ are identified.

Example 2.1. $\mathbb{P}^1$

Definition 2.2. A polynomial $f \in \mathbb{C}[x_o, ..., x_n]$ is homogeneous of degree d if it is a sum of monomials all of degree d.

Every polynomial is a sum of homogeneous polynomials.

A polynomial can be thought as a function on $\mathbb{P}^n$ only if it is homogeneous.

Definition 2.3. An ideal $I = (f_1, ..., f_k)$ is a homogeneous ideal is f_i are homogeneous for every i.

We can then define the correspondence V, I between homogeous ideals and what we will call projective varieties.

The only main difference is that: $\emptyset = V((1)) = V(x_0, ..., x_n)$. One proves (similarly as in the affine case) that there is a bijection :

$$\{ \text{ homog. radical ideals } I \neq (1)\} \Leftrightarrow \{ \text{ projective varieties } \}$$

and

$$\{ \text{ homog. prime radical ideals } I \neq (1)\} \Leftrightarrow \{ \text{ irreducible projective varieties } \}$$

With the homogeneus coordinate ring:

$$\mathbb{C}[V] = \mathbb{C}[V]/I = \sum_d (\mathbb{C}[V]/I)_d = \sum_d (\mathbb{C}[V])_d/I_d.$$

The corresponding affine variety $V(I) \subset \mathbb{C}^{n+1}$ is called the *affine cone* .

Example 2.4.

$$V(xy - zy) \subset \mathbb{P}^3.$$

Definition 2.5. Let $V \subset \mathbb{P}^n$ be a projective variety. A rational function $h : V - - > \mathbb{C}$ is defined as $h(P) = f(P)/g(P)$, where f, g are homogeneous polynomial of the same degree.

If $h(P) \neq 0$, then $f(kP)/g(kP) = f(P)/g(P)$,and $f/g = f'/g'$ if and only if $fg' = f'g$. We can then define the function field of V as:

$$\mathbb{C}(V) = \{f/g \text{ s.t. } f, h \in \mathbb{C}[x_0, ..., x : n]_d, h \notin I(V)\} \sim$$

where $f/g \sim f'/g'$ if $fg' = f'g \in I(V)$.

A rational function h is regular at P if there is an expression $h = f/g$ such that $h(P) \neq 0$. The domain, $Dom(f)$, is the set of all regular points.

Example 2.6. Let $V \subset \mathbb{P}^n$ be an irreducible projective variety such that

$$V \not\subset (x_i = 0), \text{ for all } i.$$

Let $V_i = V \cap U_i$. then $V_i \subset U_i \cong \mathbb{C}^n$ are affine algebraic varieties, because

$$(x_0/x_i, ..., 1, ...x_n/x_i) \in V_i \Leftrightarrow f((x_0/x_i, ..., 1, ...x_n/x_i) = 0 \text{ for all } f \in I(V).$$

The V_i are called the standard affine patches of V.

Definition 2.7. A rational map $f : V - - > \mathbb{C}^n$ is defined as an n-tuple of rational functions, $f = (f_1 : ... : f_n)$.

Similarly a rational map $f : V - - > \mathbb{P}^n$ is defined by:

$$f(P) \mapsto (f_0(P) : ... : f_n(P)), f_i \in \mathbb{C}(V).$$

The map f is regular at P if each f_i is regular at P and $(f_0(P), ..., f_n(P)) \neq (0, ..., 0)$.

If f is regular at P, then take $U = \cap Dom(f_j/f_i)$ where $f_i(P) \neq 0$. We have a morphism:

$$U \to \mathbb{C}^n, P \mapsto (f_0/f_j, ..., f_n/f_j).$$

Example 2.8.

$$f : \mathbb{P}^1 \to \mathbb{P}^m, f(x : y) = (x^m, x^{m-1}y : x^{m-2}y^2 : ... : y^m.$$

It is a rational map and define the morphisms:

$$f_1(1, y/x) = (1, (y/x), (y/x)^2, ..., (y/x)^m, f_2(x/y, 1) = ((x/y)^m, ..., 1).$$

The Image V is given by the points $(a_0 : ... : a_m)$ such that $x_0/x_1 = x_1/x_2, ..., x_{m-1}/x_0$. It has an inverse: $(x_0 : ... : x_m) \mapsto (x_0 : x_1)$.

Example 2.9.

$$\pi : \mathbb{P}^3 \to \mathbb{P}^2, (x_0 : ... : x_3) \mapsto (x_1 : ... : x_3).$$

is a rational map, and a morphism outside $(1 : 0 : 0 : 0)$.

Example 2.10. The Segre embedding:

$$s_{n,m}\mathbb{P}^n \times \mathbb{P}^m\mathbb{P}^{(n+1)(m+1)}, [(x_0 : ... : x_n), (y_o : .. : y_m) = (x_iy_j).$$

defines a morphism, used to "define" the product of two projevtive varieties, V, W as $s_{n,m}(V \times W)$..

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