MARCH 3

Contents

1. The tangent space of an irreducible variety

We will now see that "being singular" is a local property.

Let V be an irreducible affine variety and $\mathbb{C}(V)$ be the ring of rational functions. Let $P = (a_1, ..., a_n) \in V$. The following subring:

$$\mathcal{O}_{V,P} = \{f \in \mathbb{C}(V) \text{ s.t. } P \in Dom(f)\}$$

is called the local ring of V at P and the evaluation map

$$ev_p : \mathcal{O}_{V,P} \to \mathbb{C}$$

is a surjective ring homomorphism, whose kernel is the maximal ideal:

$$\mathfrak{m}_P = \{f \in \mathcal{O}_{V,P} \text{ s.t. } f(P) = 0\}.$$

Notice that $m_P = \{f \in \mathbb{C}[V] \text{ s.t. } f(P) = 0\} = (x_1 - a_1, ..., x_n - a_n)$, and that since for every $f \in \mathfrak{m}_P$ we have $f = g \cdot unit$, where $g \in m_P$ it is $\mathfrak{m}_P = (x_1 - a_1, ..., x_n - a_n)$ in $\mathcal{O}_{V,P}$ and (check)

$$\mathfrak{m}_P/\mathfrak{m}_P^2 \cong m_P/m_P^2$$

as $\mathbb{C}$ vector spaces.

Example 1.1. Let $P \in \mathbb{C}^n$, after a change of coordinates we can assume that $P = 0$ and thus $m_P = (x_1, ..., x_n)$. Consider the map (of abelian groups)

$$\phi_P : (x_1, ..., x_n) \to (\mathbb{C}^n)^*, f \mapsto \sum_i f^{(1)}(P)x_i.$$

It is surjective since $x_i = \phi(x_i)$. Moreover

$$Ker(\phi) = \{f(x) \text{ s.t. } \frac{\partial f}{\partial x_i}(P) = 0 \text{ for all } i\} = m_P^2.$$

It follows that for every $P \in \mathbb{C}^n$

$$m_P/m_P^2 \cong (\mathbb{C}^n)^*.$$

Proposition 1.2. *Let V be an irreducible affine variety and $P \in V$. Then*
$$(T_P V)^* \cong m_P/m_P^2$$
as $\mathbb{C}$ vector spaces.

Proof. We wil follow the lines of the previous example. By definition it is $(T_P V)^* \subset (\mathbb{C}^n)^*$ and thus the restriction map induces a surjective map:
$$\phi_{V,P} : M_P = (x_1, ..., x_n) \to (T_P V)^*$$
where we use the different notation M_P to recall that the ideal is in $\mathbb{C}[x_1, ..., x_n]$. Notice that if $f \in Ker(\phi_{V,P})$ then it is $f^{(1)} = \sum a_i g_i^{(1)}$ for $g_i \in I(V)$. Then it would be $f - \sum_i g_i \in M_P^2$ and thus $f \in M_P^2 + I(V)$. Since both $M_P^2 \subset Ker(\phi_{V,P})$ and $I(V) \subset Ker(\phi_{V,P})$ it is:
$$Ker(\phi_{V,P}) = M_P^2 + I(V)$$
Because $M_P/M_P^2 + I(V) \cong m_p/m_p^2$ the assertion is proved. □

The vector space $(m_P/m_P^2)^*$ is called the *Zariski tangent space*.

Recalling that a Zariski-open is isomorphic to an affine variety, we can speak of $T_P U_o$, where U_o is a Zariski-open.

Exercise 1.3. (Assignment 2) Let V be an irreducible affine variety and $P \in V$. If U_o, V_o are two open neighborhoods (Zariski open)and $\phi : U_o \to W_o$ is an isomorphism, such that $\phi(P) = Q$, then there is a natural isomorphism $T_P U_o \to T_P V_o$ and hence $\dim(T_P U_o) = \dim(T_Q V_o)$.

Two varieties are birationally equivalent if there is a rational map, whose inverse is a rational map. Deduce that if two varieties are birationally equivalent, they they have the same dimension.

Example 1.4. (Blow up) Consider the map $p : \mathbb{C}^2 \to \mathbb{C}^2$ defined by $p(x, y) = (x, xy)$. the rational inverse is $p^{-1} : \mathbb{C}^2 - - > \mathbb{C}^2$ defined by $(x, y) \mapsto (x, y/x)$. It defines an isomorphism $\mathbb{C}^2 \setminus \{0\} \cong \mathbb{C}^2 \setminus V(x)$.

Consider now the irreducible variety $C = V(y^2 - x^3)$ which is an irreducible curve, singular at 0. $p^{-1}(C) = V(x^2(y^2 - x))$ which we can write as the union of two irreducible components:
$$p^{-1}(C) = V(x^2) \cup V(y^2 - x) = E \cup C'$$
where C' is not singular at 0 and it is isomorphic to C outside 0. This process is called *resolution of singularities* and the map p^{-1} is the blow-up of $\mathbb{C}^2$ at the origin.

2. Rational convex polyhedral cones

The proofs of what follows can be found in [F] and [CLS].

Let M, N be dual lattices of dimension n. We will denote by $M_\mathbb{R}, N_\mathbb{R}$ the associated dual real vector spaces of dimension n. Let $S = \{u_1, ..., u_s\} \subset N_\mathbb{R}$ be a finite subset. The set
$$Cone(S) = \mathbb{R}^+ u_1 + ... + \mathbb{R}^+ u_s = \{\sum_{s \in S} \lambda_s s \text{ s.t. } \lambda \geq 0\}$$

is called a *rational polyhedral convex cone*. We will use the term cone for simplicity.

Notice that such a cone is indeed a cone ($x \in \sigma \Rightarrow \lambda x \in \sigma,$ for all $\lambda \geq 0$) and it is indeed convex ($x, y \in \sigma \Rightarrow \lambda x + (1-\lambda)y \in \sigma,$ for all $0 \leq \lambda \leq 1$).

The dual cone of a cone σ is:

$$\check{\sigma} = \{m \in M_{\mathbb{R}} \text{ s.t } < m, u > \geq 0 \text{ for all } u \in \sigma\}$$

The dual cone is also a rational convex polyhedral cone: $\check{\sigma} = Cone(S')$ for some finite $S' \subset M_{\mathbb{R}}$.

Example 2.1. Let $\{e_1, e_2\}$ be the lattice base of a two-dimenasional lattice N and let $\{e_1^*, e_2^*\}$ be the lattice base of the dual two-dimenasional lattice M . Consider

$$\sigma = Cone(e_2, 2e_1 - e_2) \subset N_{\mathbb{R}}.$$

The dual cone is:

$$\check{\sigma} = Cone(e_1^*, e_1^* + 2e_2^*) \subset M_{\mathbb{R}}.$$

Definition 2.2.
- The dimension of a cone σ is equal to the dimension of its linear span: $\dim(\sigma) = \dim(\mathbb{R}\sigma) = \dim(\sigma + (-\sigma))$.
- A face of σ is given by the intersection with a supporting hyperplane:
$$\tau = \sigma \cap H_v = \{u \in \sigma \text{ s.t. } < u, v >= 0\}.$$
- A face of codimesion 1 is called a facet and a face of dimension 1 is called an edge.

The following properties hold:

(1) A face is again a rational convex cone. An edge is than generated by one element:
$$\rho = \mathbb{R}^+ v_i.$$
(2) The intersection of faces is again a face.
(3) If $\tau \subset \sigma$ is a face, then any face of τ is a face of σ.
(4) Any proper face is contained in a facet.
(5) Any proper face is the intersection of the facets containing it.

If $\tau \subset \sigma$ is a face, we define

$$\tau^\perp = \{m \in M_{\mathbb{R}} \text{ s.t. } < m, u >= 0 \text{ for all } u \in \tau\}$$

Then $\tau^* = \tau^\perp \cap \check{\sigma}$ is called the dual face, because it is a face of the dual cone $\check{\sigma}$. The correspondence:

$$\tau \mapsto \tau^*$$

defines an inclusion reversing one-to-one corespondence between faces on σ and faces of $\check{\sigma}$, such that

$$\dim(\tau) = codim(\tau^*) \text{ i.e. } \dim(\tau) + \dim(\tau^*) = \dim(\sigma).$$

References

[AM] Atiyah, M. F.; Macdonald, I. G. Introduction to commutative algebra. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont. 1969 ix+128 pp.

[CLS] D. Cox, J. Little, H. Scenck. Toric Varieties.

[F] W. Fulton.Introduction to toric varieties.Annals of Math. Princeton Univ. Press, 131.