MARCH 31

Contents

1. Projective toric varieties

The projective space $\mathbb{P}^n$ is a an irreducible projective variety. The quotient map induces the following exact sequence:

$$(1) \to \mathbb{C}^* = \mathbb{C}(1, ..., 1) \to (\mathbb{C}^*)^{n+1} \to T \to (1).$$

Where $T \cong (\mathbb{C}^*)^n \subset \mathbb{P}^n$.

Moreover we have an induced exact sequence:

$$0 \to Hom_{AG}(T, \mathbb{C}^*) \to Hom_{AG}((\mathbb{C}^*)^{n+1}, \mathbb{C}^*) \cong \mathbb{Z}^{n+1} \to Hom_{AG}(\mathbb{C}^*, \mathbb{C}^*) = \mathbb{Z}$$

which gives $Hom_{AG}(T, \mathbb{C}^*) = \{(a_0, ..., a_n) \in \mathbb{Z}^{n+1} \text{ s.t. } \sum a_i = 0\}$.

The torus T acts on $\mathbb{P}^n$. The action of the torus $(\mathbb{C}^*)^{n+1}$ on $\mathbb{C}^{n+1}$ is a well defined action on $\mathbb{P}^n$, because: $(t_0, ..., t_n)(\lambda(x_0, .., x_n)) = \lambda(t_0x_0, ..., t_nx_n)$. Moreover The diagonal sutorus acts trivially: $(t, ..., t)(x_0, .., x_n) = (x_0, .., x_n)$.

One can use a similar construction with lattice points.

Let $\mathcal{A} = \{m_0, ..., m_s\} \subset \mathbb{Z}^n$ be a finite subset with $|\mathcal{A}| = s + 1$. Then:

$$(\mathbb{C}^*)^n \to \mathbb{C}^{s+1} \to \mathbb{P}^s.$$

Observe that the image is in fact a torus $T_{\mathcal{A}} \subset \mathbb{P}^s$ and thus

$$X_{\mathcal{A}} = \overline{T_{\mathcal{A}}} \subset \mathbb{P}^s$$

is a projective toric variety, containing a torus, which acts on it. One sees exactly as in the previous example, considering the torus and the action induced by the first map on $\mathbb{C}^{s+1}$ and then observing that it defines an action of the image in $\mathbb{P}^s$.

Also:

$$Hom(T_{\mathcal{A}}, \mathbb{C}^*)_{AG} = \{\sum_0^s a_i m_i \text{ s.t. } \sum a_i = 0\}.$$

2. Affine patches

Consider $X_{\mathcal{A}} \cap U_i = X_i \subset \mathbb{C}^n$. Because $\mathbb{P}^s = \cup U_i$ we have

$$X_{\mathcal{A}} = \cup X_i.$$

The restriction map is given by:

$$(\mathbb{C}^*)^n \to \mathbb{C}^{s+1} \to \mathbb{P}^s \to X_i$$

$$t = (t_1, ..., t_n) \to (t^{m_0}, ..., t^{m_s}) \to (t^{m_0} : ... : t^{m_s}) = (t^{m_0 - m_i}, ..., t^{m_s - m_i}).$$

Then $X_i = Y_{\mathcal{A}_i}$, where $\mathcal{A}_i = \{m - m_i \text{ s.t. } m \in \mathcal{A}\}$.

In the construction of X_i we "invert" χ^{m_i}. For this reason

$$X_i = Spec(\mathbb{C}[S]_{\chi^{m_i}}) = Spec(\mathbb{C}[S_i]),$$

where $S_i = \mathbb{N}\mathcal{A}_i$.

Similarly:

$$X_i \cap X_j = Spec(\mathbb{C}[S_j]_{\chi^{m_i - m_j}}) = Spec(\mathbb{C}[S_i]_{\chi^{m_j - m_i}}).$$

3. Polytopes

A lattice polytope $P \subset \mathbb{R}^n$ is defined as the convex hull of a finite number of lattice points:

$$P = Conv(m_1, ..., m_k) = \{\sum r_i m_i, r_i \in \mathbb{Q}, \sum r_i = 1\},\ m_i \in \mathbb{Z}^n.$$

The dimension of a Polytope is given by the dimension of the smallest affine subspace containing it. Let $\mathbb{R}^n = M_{\mathbb{R}}$ and $N = \check{M}$.

The faces of the polytope are lower dimensional lattice polytopes cut out by affine hyperplanes. For $u \in N_{\mathbb{R}}, a \in \mathbb{R}$,

$$H_{u,a} = \{m \in M_{\mathbb{R}} \text{ s.t. } < m, u >= -a\}.$$

defines an affine hyperplane. We will denote by

$$H^+_{u,a} = \{m \in M_{\mathbb{R}} \text{ s.t. } < m, u > \geq -a\}$$

the closed half-space.

Definition 3.1. Let $P = Conv(S)$ be a lattice polytope. A face $F \subset P$ is $P \cap H_{u,a}$ and $P \subset H^+_{u,a}$ for some $u \in N_{\mathbb{R}}, a \in \mathbb{R}$. $H^+_{u,a}$ is called a supporting hyperplane.

Notice that $F = P \cap H_{u,a} = Conv(S \cap H_{u,a})$. Faces of dimension $\dim(P) - 1, 1, 0$ are called *facets, edges and verteces* resp.

An important fact to remember is that, when $\dim(P) = n$ (maximal), then the facets, F, have a unique supporting hyperplane H_F, and

$$P = \bigcap_{F\, facet} H^+_F = \{m \in M_{\mathbb{R}} \text{ s.t. } < m, u_F > \geq -a_F, F\, facet\}$$

This means that there are unique (up to multiplication with a positive real number) vector $u_F \in N_{\mathbb{R}}, a_F \in \mathbb{R}$ giving $H_F = H_{u_F, a_F}$. The vector u_F is called the inward-pointing facet normal.

Example 3.2. Let $e_0 = 0$ and $e_1, ..., e_n$ be the standard basis of $\mathbb{R}^n$. $\Delta_n = Conv(e_0, ..., e_n)$ is called the n-dimensional standard simplex.

For $n = 2$, $\Delta_2 = H_{(1,0),0} \cap H_{(0,1),0} \cap H_{(-1,-1),1}$.

4. The affine patches given by the Polytope

Lat $X_{\mathcal{A}}$ be the projective toric variety defined above, and let $P_{\mathcal{A}} = Conv(\mathcal{A})$. Notice that

$$\dim(P_{\mathcal{A}}) = \dim(X_{\mathcal{A}}).$$

Lemma 4.1. *Let J be the subset of $\mathcal{A}$ given by the verteces of $P_{\mathcal{A}}$, then*

$$X_{\mathcal{A}} = \cup_{i \in J} X_i.$$

Proof. We have that

$$m_i = \sum_{j \in J} r_{i,j} m_j, r_{i,j} \in \mathbb{Q}, \sum_j r_{i,j} = 1.$$

After clearing denominators we can write

$$km_i = \sum_{j \in J} k_{i,j} m_j, k_{i,j} \in \mathbb{Z}, \sum_j r_{i,j} = k.$$

It follows that $\sum k_{i,j}(m_i - m_j) = 0$. Then, for every $k_{i,j} \neq 0$ we have $m_j - m_i \in S_i$ and therefore

$$X_i = Spec(\mathbb{C}[S_i]) = Spec(\mathbb{C}[S_i]_{\chi^{m_j - m_i}} = X_i \cap X_j$$

after choosing one $k_{i,j} \neq 0$. It follows that $X_i \subset X_j$ for some $i \in J$. □

Example 4.2. $\mathcal{A} = \{(0,0), (1,0), (2,0), (0,1), (1,1), (1,2)\}$.

Recall that if S_i is saturated, then X_i is normal and $S_i = \check{\sigma_i} \cap M$ for an n-dimensional strongly convex cone σ_i.

Let $C_i = Cone(P \cap M - m_i)$. If $\dim(P) = n$ the cone C_i is stongly convex.

It follows that if the following is satisfied:

- the polytope is of maximal dimension, $\dim(P) = n$,
- For every vertex m, the semigroup $\mathbb{N}(\mathcal{A} - m)$ is saturated,

Then

$$X_{\mathcal{A}} = \bigcup_{m\, vertex} Spec(\mathbb{C}[\check{\sigma_m} \cup M])$$

where $\check{\sigma_m} = Cone(P \cap M - m)$.

References

[AM] Atiyah, M. F.; Macdonald, I. G. Introduction to commutative algebra. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont. 1969 ix+128 pp.

[CLS] D. Cox, J. Little, H. Scenck. Toric Varieties.

[F] W. Fulton.Introduction to toric varieties.Annals of Math. Princeton Univ. Press, 131.