1) $$\int_{-2}^{-1}\frac{dx}{x^2+4x+5} = \int_{-2}^{-1}\frac{dx}{(x+2)^2+1} \underset{u=x+2}{\overset{\uparrow}{=}} \int_0^1 \frac{du}{u^2+1}$$

$$= \arctan u \Big|_0^1 = \pi/4$$

2) $$y_H'' + 4y_H = 0 \to \lambda^2 + 4 = 0, \quad \lambda = \pm 2i$$

$$\Rightarrow y_H(x) = A\cos 2x + B\sin 2x$$

$$y_p'' + 4y_p = \cos x \;; \quad \text{Ansatz } y_p(x) = C\cos x + D\sin x$$

$$\to y_p''(x) = -C\cos x - D\sin x$$

$$\to 3C\cos x + 3D\sin x = \cos x \Rightarrow C = \tfrac{1}{3}, \; D = 0$$

$$\Rightarrow y(x) = A\cos 2x + B\sin 2x + \tfrac{1}{3}\cos x$$

3) $$1 + 2 + 4 + \dots + 2^{63} = \frac{1-2^{64}}{1-2} = 2^{64} - 1$$

4) $$\sin x = x - x^3/6 + \dots, \quad \cos x = 1 - x^2/2 + \dots$$

$$(1+x)^4 = 1 + 4x + 6x^2 + \dots$$

$$\Rightarrow \lim_{x\to 0} \frac{(1+x)^4 - 4\sin x - \cos x}{x^2} = \lim_{x\to 0}\left(\frac{6x^2 + x^2/2 + \dots}{x^2}\right) = \frac{13}{2}$$

5) $$e^{i(\alpha-\beta)} = e^{i\alpha}e^{-i\beta} = (\cos\alpha + i\sin\alpha)(\cos\beta - i\sin\beta)$$

$$\overset{\shortparallel}{\cos(\alpha-\beta) + i\sin(\alpha-\beta)} \qquad = \cos\alpha\cos\beta + \sin\alpha\sin\beta + i(\dots)$$

$$1 + \cos 2\varphi = 1 + \cos(\varphi - (-\varphi)) = 1 + \cos^2\varphi - \sin^2\varphi = 2\cos^2\varphi$$

(oder direkt: $= 1 + \frac{e^{2i\varphi} + e^{-2i\varphi}}{2} = \frac{1}{2}\left(2 + e^{2i\varphi} + e^{-2i\varphi}\right)$

$= \frac{1}{2}\left(e^{i\varphi} + e^{-i\varphi}\right)^2 = \frac{1}{2}\cdot 4(\cos\varphi)^2$)

6) $f'(x) = \frac{1}{2} - \frac{2}{x^2} \stackrel{!}{=} 0 \Rightarrow x_0 = 2$ möjlig extrempunkt

$(\tilde{x}_0 = -2 \notin [1, 3])$

$f''(2) = +\frac{4}{x^3}\Big|_{x=2} > 0 \Rightarrow x_0 = 2$ är en minimipunkt $(f(2) = 2)$

Randpunkterna $x_- = 1$, $x_+ = 3$ är lokala maximipunkter

$\left(f(1) = 5/2, \quad f(3) = \frac{13}{6}\right)$

7) $J := \int_6^\infty \frac{dx}{x^2 - 9x + 20} = \lim_{\Lambda \to \infty} \int_6^\Lambda \frac{dx}{x^2 - 9x + 20} =: \lim_{\Lambda \to \infty} J_\Lambda$

$\frac{1}{x^2 - 9x + 20} = \frac{1}{(x-4)(x-5)} = \frac{1}{x-5} - \frac{1}{x-4}$ (via $\frac{A}{x-5} + \frac{B}{x-4} \stackrel{!}{=} \dots$)

partialbråkuppdelning

$\Rightarrow J_\Lambda = \left(\ln|x-5| - \ln|x-4|\right)\Big|_6^\Lambda$

$= \ln\left|\frac{x-5}{x-4}\right|\Big|_6^\Lambda = \ln\left|\frac{\Lambda-5}{\Lambda-4}\right| - \ln\frac{1}{2} \Rightarrow J = \ln 2 + \underbrace{\lim_{\Lambda \to \infty} \frac{1 - 5/\Lambda}{1 - 4/\Lambda}}_{=0}$

8) $\lambda^2 - 3\lambda + 2 \stackrel{!}{=} 0 \Rightarrow y_H(x) = Ae^x + Be^{2x}$

Ansats $y_1(x) = C \cdot x \cdot e^x$ (Resonans!), $y_1' = e^x C\{1 + x\}$

$Ce^x\{2 + x - 3(1 + x) + 2x\} \stackrel{!}{=} 2e^x$ $\quad y_1'' = C \cdot e^x\{2 + x\}$

$\Rightarrow C = -2 \Rightarrow y_1(x) = -2xe^x$

Ansats $y_2(x) = ax + b$, $y_2' = a$, $y_2'' = 0 \rightarrow 0 - 3a + 2(ax + b) \stackrel{!}{=} 3x$

$\Rightarrow a = 3/2$, $b = 9/4 \Rightarrow y(x) = Ae^x + Be^{2x} + \left(\frac{3}{2}x + \frac{9}{4}\right) - 2xe^x$

9) $f'(x) := \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = 3x^2 - \frac{1}{x^2}$ $(x \neq 0)$, eftersom

$\lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h \to 0} \frac{x^3 + 3hx^2 + \dots - x^3}{h} = \lim_{h \to 0} (3x^2 + \dots) = 3x^2$

$\lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h \to 0} \frac{x - (x+h)}{h \cdot x(x+h)} = -\lim_{h \to 0} \frac{1}{x(x+h)} = -\frac{1}{x^2}$

10) $L = \int_0^{2\pi} \sqrt{\dot{x}^2 + \dot{y}^2}\, dt$

$\dot{x} = -\frac{1}{2}\sin t - \frac{1}{2}\sin(2t)$, $\dot{y} = \frac{1}{2}\cos t + \frac{1}{2}\cos 2t$

$\Rightarrow \dot{x}^2 + \dot{y}^2 = \frac{1}{4} + \frac{1}{4} + \frac{1}{2}\left(\sin t \sin 2t + \cos t \cos 2t\right)$

$= \frac{1}{2}\left(1 + \cos t\right) = \left(\cos t/2\right)^2$

$\Rightarrow L = \int_0^{2\pi} |\cos t/2|\, dt \underset{\substack{u = \frac{t}{2} \\ dt = 2du}}{=} 2 \int_0^{\pi} |\cos u|\, du$

$= 4 \int_0^{\pi/2} \cos u\, du = 4 \sin u \Big|_0^{\pi/2} = 4$