Final Exam in Diskret Matematik och Algebra (SF2714)

With Solutions

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No calculator, textbooks or notes are allowed
For full credit you should show all your work
There are 5 problems for a total of 60 points

Problem 1. (12p).

(a) State the Chinese remainder theorem (you don't have to prove it),
(b) Find all integer solutions to the Diophantine equation

$$36x + 15y = -9.$$

Solution. A particular solution to the equation is $(x_0, y_0) = (6, -15)$. All integers solutions are then given by

$$\begin{aligned} x &= x_0 + (15/3)n = 6 + 5n \\ y &= y_0 - (36/3)n = -15 - 12n \end{aligned}$$

where $n \in \mathbb{Z}$.

Problem 2. (12p).

(a) Write $p(x) = x^5 + 2x^2 + x + 2 \in \mathbb{Z}_3[x]$ as a product of irreducible polynomials,
(b) Determine if $x^4 + 1$ is invertible in $\mathbb{Z}_3[x]/(p)$. If so, find the inverse.

Solution. (a). We see that $p(1) = 0$ and dividing by $x - 1$ yields

$$p(x) = (x - 1)(x^4 + x^3 + x^2 + 1).$$

Let us prove that $q(x) = x^4 + x^3 + x^2 + 1$ is irreducible. Assume otherwise. One checks that $q(x) = 0$ has no root in $\mathbb{Z}_3$ so by the Factor Theorem q has no factors of degree 1. The only way $q(x)$ can factor into polynomials of lower degree is then (pull out leading coefficients) as:

$$q(x) = (x^2 + ax + b)(x^2 + cx + d) = x^4 + (a + c)x^3 + (b + d + ac)x^2 + (bc + ad)x + bd$$

By identifying coefficients we see that $bd = 1$, so either $b = d = 1$ or $b = d = -1$. By comparing the coefficients infront of x and x^3 in the two expansions we see that

$$1 = a + c, \quad 0 = b(a + c)$$

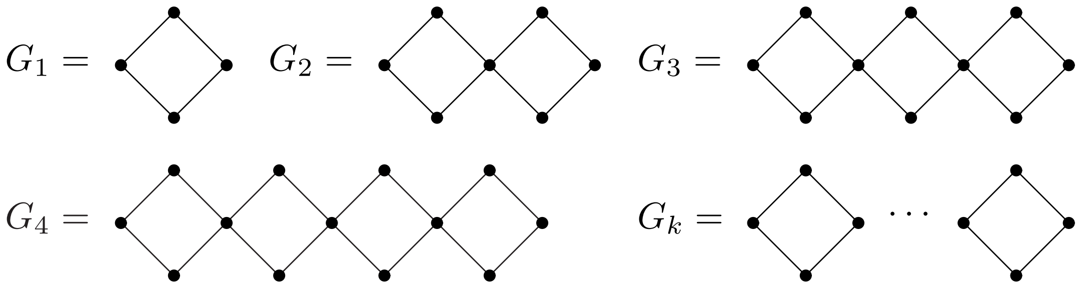
which is a contradiction since $b = \pm 1$. This proves that q is irreducible and that $p(x) = (x+2)(x^4+x^3+x^2+1)$ is the factorization of $p(x)$ into a product of irreducible polynomials.

(b). By performing Euclid's algorithm with $p(x)$ and x^4+1 we get a final remainder 1 and "solving backwards" in the algorithm we get

$$(2x^2+1)p(x) + (x^3+2x+2)(x^4+1) = 1.$$

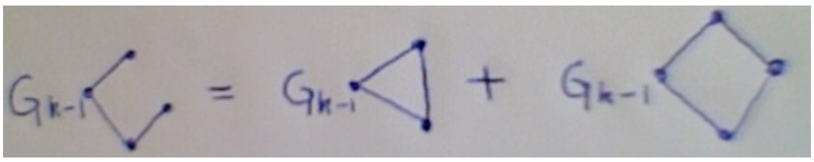
Hence x^4+1 is invertible in $\mathbb{Z}_3[x]/(p)$ and its inverse is x^3+2x+2.

Problem 3. (12p). Let G_k be the graph obtained by gluing together k copies of the 4-cycle in a row as indicated below.



(a) Let $f_k(n)$ be the chromatic polynomial of G_k. Find a formula for $f_k(n)$,
(b) Determine how many Eulerian walks there are in G_k.

Solution. (a). Let $G_0 = \bullet$ be the graph with just one vertex and no edges. Let $k \geq 1$. The "deletion/contraction"-recursion for chromatic polynomials applied to G_k yields in symbols:



so

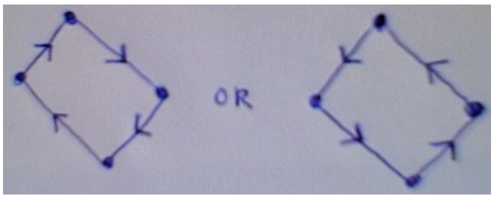
$$(n-1)^3 f_{k-1}(n) = (n-1)(n-2)f_{k-1}(n) + f_k(n)$$

that is $f_k(n) = (n-1)(n^2-3n+3)f_{k-1}(n)$. Since $f_0(n) = n$ we get

$$f_k(n) = n(n-1)^k(n^2-3n+3)^k.$$

(b). Associate to each Eulerian walk w in $G_k = (V_k, E_k)$ a function $F_w : E_k \to \{\to, \leftarrow\}$ by letting $F_w(e) = \to$ if in the walk the left endpoint of e is visited first and $F_w(e) = \leftarrow$ otherwise. The functions that arise in this way are precisely those for which in each square

(1) the top edges have the same direction,
(2) the bottom edges have the same direction, and
(3) the bottom edges have opposite direction from the top edges (see figure below).



Hence there are 2^k different such functions. Given such a function we may start our Eulerian walk in any vertex. However,

(1) if the starting vertex has degree two there is just one direction to go, and there are $2k+2$ such vertices,
(2) if the starting vertex has degree four we have a choice to go either left or right. There are $k-1$ such vertices.

Hence the number of Eulerian walks is

$$2^k(2k+2+2(k-1)) = k2^{k+2}.$$

Problem 4. (12p). Prove the following theorem.

Theorem. Suppose that $a(x)$ and $b(x)$ are polynomials in $K[x]$ (with $b(x) \neq 0$), where K is a field. Then there are unique polynomials $q(x), r(x) \in K[x]$ such that

$$a(x) = b(x)q(x) + r(x),$$

where $r(x) = 0$ or $\deg(r(x)) < \deg(b(x))$.

Problem 5. (12p).

(a) Let k be a positive integer and let a_n^k be the number of permutations in $\mathcal{S}_n$ whose disjoint cycle form only contains cycles of length k (we define $a_0^k := 1$). Prove that

$$\sum_{n=0}^{\infty} \frac{a_n^k}{n!} x^n = e^{\frac{x^k}{k}}.$$

(b) Prove that a permutation $\pi \in \mathcal{S}_n$ has order 6 if and only if (I) or (II) below are satisfied

(I) There are only cycles of length $1, 2, 3$ and 6 in the disjoint cycle form of π and there is at least one cycle of length 6;
(II) There are only cycles of length $1, 2$ and 3 in the disjoint cycle form of π and there is at least one cycle of length 2 and at least one of length 3.

(c) Let b_n be the number of permutations in $\mathcal{S}_n$ of order 6. Determine

$$\sum_{n=0}^{\infty} \frac{b_n}{n!} x^n.$$

(See the hint below.)

Hint on Problem 5 (c). Use (a) and (b) above and the combinatorial interpretation of products of exponential generating functions (several times): If

$$F(x) = \sum_{n=0}^{\infty} \frac{f_n}{n!} x^n, \quad G(x) = \sum_{n=0}^{\infty} \frac{g_n}{n!} x^n$$

then

$$F(x)G(x) = \sum_{n=0}^{\infty} \frac{h_n}{n!} x^n \quad \text{where} \quad h_n = \sum_{k=0}^{n} \binom{n}{k} f_k g_{n-k}.$$

Solution. (a). If the disjoint cycle form of $\pi \in \mathcal{S}_n$ only has cycles of length k then $n = mk$ for some $m \in \mathbb{N}$ and there are

$$\frac{1}{k!} \binom{mk}{k, \dots, k} (k-1)! \cdots (k-1)! = \frac{(mk)!}{k! k^m}$$

such permutations in $\mathcal{S}_n$ since there are $\frac{1}{k!}\binom{mk}{k,\dots,k}$ ways of choosing which integers should be in the same cycle, and there are $(k-1)!$ different cycles of k letters. Hence the exponential generating function is

$$\sum_{m=0}^{\infty} \frac{(mk)!}{k! k^m} \frac{x^{mk}}{(mk)!} = e^{\frac{x^k}{k}}.$$

(b). Since disjoint cycles commute and the order of a cycle of length k is k, the cycles in π are of lengths $1, 2, 3$ or 6 if the order of π is 6. Also, if all cycles in $\pi \in \mathcal{S}_n$ are of lengths $1, 2, 3$ or 6 then the order of π is either $1, 2, 3$ or 6. If one of the cycles in π has length 6 then the order of π is 6, and if no cycle in π has length 6 then the order of π is 6 if and only if there are cycles of lengths 2 and 3 in π.

(c). Let $S = \{s_1, \dots, s_\ell\}$ be a set of positive integers and let $a_n(S)$ be the number of permutations $\pi \in \mathcal{S}_n$ for which all the cycles in π have lengths in S. Let

$$E_S(x) = \sum_{n=0}^{\infty} \frac{a_n(S)}{n!} x^n.$$

Now, if $\ell \geq 2$, then

$$a_n(S) = \sum_{k=0}^{n} \binom{n}{k} a_n(\{s_1, \dots, s_{\ell-1}\}) a_{n-k}(\{s_\ell\}),$$

since we get a permutation with cycle lengths in S by specifying which integers should be in cycles of length s_ℓ, and then the remaining integers are in cycles of cycle lengths in $\{s_1, \dots, s_{\ell-1}\}$. Using the hint we see that

$$E_S(x) = E_{\{s_1,\dots,s_{\ell-1}\}} E_{\{s_\ell\}}. \quad (1)$$

Iterating (1) using $E_{\{k\}}(x) = e^{\frac{x^k}{k}}$ we have

$$E_S(x) = e^{\frac{x^{s_1}}{s_1}} \cdots e^{\frac{x^{s_\ell}}{s_\ell}}.$$

The number of permutations satisfying (I) is precisely $a_n(\{1,2,3,6\}) - a_n(\{1,2,3\})$ so the corresponding generating function is

$$e^{x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^6}{6}} - e^{x+\frac{x^2}{2}+\frac{x^3}{3}}.$$

Similarly the number of permutations satisfying (II) is

$$a_n(\{1,2,3\}) - a_n(\{1,2\}) - a_n(\{1,3\}) + a_n(\{1\}),$$

which has generating function

$$e^{x+\frac{x^2}{2}+\frac{x^3}{3}} - e^{x+\frac{x^2}{2}} - e^{x+\frac{x^3}{3}} + e^x.$$

Hence

$$\sum_{n=0}^{\infty} \frac{b_n}{n!} x^n = e^{x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^6}{6}} - e^{x+\frac{x^2}{2}} - e^{x+\frac{x^3}{3}} + e^x.$$