HOMEWORK II

To be handed in on Monday, December 15[1]. Collaboration is encouraged, but you may NOT copy another students solutions. It is important that you show all your work and write complete proofs. There are 3 problems for a total of 20 points. If you are stuck on a problem you may ask me for hints.

(1) Let $f(x) = 1 + x + 2x^2 + 4x^3 + 3x^4 + x^5 \in \mathbb{Z}_5[x]$.
  a) Write $f(x)$ as a product of irreducible polynomials,
  b) Determine which of the following polynomials are invertible in $\mathbb{Z}_5[x]/(f(x))$:

$$1 + 3x + 2x^2, \quad 2 + 4x + x^2 + 3x^3, \quad x^3 + 2x^2 + 4x + 3$$

(2) a) Let p be a prime number and let a_n be the number of permutations π in $\mathcal{S}_n$ of order p. Prove that

$$\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n = e^{x + \frac{x^p}{p}} - e^x$$

Hint: Note that $\pi^p = e$ if and only if the cycles in π are of lengths 1 or p. You may use the formula on the bottom of page 134 in Biggs.
  b) Let p be a prime number and let b_n be the number of permutations π in $\mathcal{S}_n$ of order p that have no fixed points. Determine

$$\sum_{n=0}^{\infty} \frac{b_n}{n!} x^n$$

(3) Let X be a multi-set of size n whose elements are integers such that $\sum_{x \in X} x = 1$. Let

$$O(X) = \{(x_1, \dots, x_n) \in \mathbb{Z}^n : \{x_1, \dots, x_n\} = X\}$$

be all vectors obtained by re-ordering X.
  a) Define a relation $\sim$ on $O(X)$ by $\mathbf{x} = (x_1, \dots, x_n) \sim (y_1, \dots, y_n) = \mathbf{y}$ if $\mathbf{y}$ can be obtained from $\mathbf{x}$ by a cyclic shift, that is, if there is an integer $0 \le k \le n$ such that $y_i = x_{k+i}$ for all $1 \le i \le n - k$ and $y_i = x_{i-n+k}$ if $n - k < i \le n$ (i.e., if the sequences are the same when the numbers are arranged on a circle). Prove that $\sim$ is an equivalence relation on $O(X)$ for which every equivalence class has exactly n elements.
  b) Let S is an equivalence class of $\sim$. Prove that there is a unique element $(x_1, \dots, x_n) \in S$ for which

$$x_1 + \cdots + x_k \ge 1 \quad \text{for all } k \in \{1, \dots, n\}.$$

  c) Let X consist of $n+1$ 1s and n (-1)s. Prove that the number of equivalence classes of $\sim$ is the nth Catalan number by using (a) and (b) above. How is this related to Dyck paths?

Example. If $X = \{-1, 1, 1, 0\}$ then $O(X) = \{(-1, 1, 1, 0), (-1, 1, 0, 1), (-1, 0, 1, 1),$ $(0, -1, 1, 1), (0, 1, -1, 1), (0, 1, 1, -1), (1, -1, 0, 1), (1, -1, 1, 0), (1, 0, -1, 1), (1, 0, 1, -1),$ $(1, 1, -1, 0), (1, 1, 0, -1)\}$. The unique elements as in (b) in each equivalence class are $(1, 1, 0, -1), (1, 0, 1, -1), (1, 1, -1, 0)$.

[1]May be extended if special circumstances, but the very latest date is January 6