Matematiska Institutionen, KTH Tuesday, June 3, 08:00 – 13:00 in Q36

Final Exam in Diskret Matematik och algebra (SF2714)

WITH SOLUTIONS

Examiner: Petter Brändén.
No calculator, textbooks or notes are allowed.
For full credit you should show all work.
There are 5 problems for a total of 58 points.

Problem 1. (10p).

(a) State the fundamental homomorphism theorem for ringss (you don't have to prove it).

(b) Find all integer solutions to the Diophantine equation

$$21x + 9y = 45$$

Solution. Note that $\gcd(21, 9) = 3 \mid 45$ so the equation is solvable. Divide by 3 and solve the equation $7x + 3y = 15$. By performing Euclid's algorithm backwards we get a particular solution $(x_0, y_0) = (15, -30)$. The general solution is then given by the theorem on Diophantine equations as

$$x = 15 + 3n, \quad y = -30 - 7n, \quad n \in \mathbb{Z}.$$

Problem 2. (12p).

(a) Determine if the polynomial $p(x) = x^4 + x + 1$ is irreducible in $\mathbb{Z}_2[x]$.

(b) Find the inverse (if possible) of $x^2 + 1$ in $\mathbb{Z}_2[x]/(p)$.

Solution. (a). If $p(x)$ is reducible then it has a linear factor or it is the product of two factors of degree 2. In the first case $p(x) = 0$ would have a root, but $p(0) = p(1) = 1$ so this is ruled out. In the second case we would get a factorization (since $p(x)$ is monic, has constant term 1)

$$x^4 + x + 1 = (x^2 + ax + b)(x^2 + cx + d) = x^4 + (a + c)x^3 + (ac + b + d)x^2 + (ad + bc)x + bd,$$

for some $a, b, c, d \in \mathbb{Z}_2$. Now, $bd = 1$ so $b = d = 1$ which gives $a + c = 0$ and $a + c = 1$. Hence $p(x)$ has no factors of degree two either and is therefore irreducible.

(b). Since $p(x)$ is irreducible $\gcd(p(x), x^2 + 1) = 1$ so the inverse of $x^2 + 1$ exists. Performing Euclid's algorithm backwards we find

$$(x^2 + 1)(x^3 + x + 1) + x(x^4 + x + 1) = 1$$

so the inverse of $x^2 + 1$ is $x^3 + x + 1$.

Problem 3. (12p). Prove the following theorem.

Theorem. Suppose that $a(x), b(x)$ are two polynomial in $K[x]$ (with $b(x) \neq 0$), where K is a field. Then there are unique polynomials $q(x), r(x) \in K[x]$ such that

$$a(x) = b(x)q(x) + r(x)$$

with either $r(x) = 0$ or $\deg(r(x)) < \deg(b(x))$.

Problem 4. (10p). Determine how many words of length n one can form using the letters a, b, c so that the number of a's and the number of b's are odd. (For $n = 4$ we have $abcc, aaba, \ldots$). Let A_n be this number and find the generating function

$$A(x) = \sum_{n=0}^{\infty} A_n x^n.$$

(Note that $A_0 = A_1 = 0$).
Solution. Let

$$F(x, y, z) = \sum_{k,\ell,m=0}^{\infty} a(k, \ell, m) x^k y^\ell z^m$$

where $a(k, \ell, m)$ is the number of words with letters a, b, c and with k a's, ℓ b's and m c's (and no other restrictions). Since a word is either empty, starts with an a, starts with a b, or starts with a c we have $F = 1 + xF + yF + zF$, that is,

$$F = \frac{1}{1 - x - y - z}.$$

The words with an odd number of a's are generated by

$$G(x, y, z) = (F(x, y, z) - F(-x, y, z))/2,$$

since all coefficients with an even k will cancel and the remaining have odd k. Similarly, the words with an odd number of a's and an odd number of b's are generated by

$$H(x, y, z) = (G(x, y, z) - G(x, -y, z))/2.$$

Hence

$$A(x) = H(x, x, x) = \frac{1}{4}\left(\frac{1}{1 - 3x} - \frac{2}{1 - x} + \frac{1}{1 + x}\right).$$

Problem 5. (14p). For $k \geq 3$ let C_k be the cycle with k vertices, i.e., $C_k = (V_k, E_k)$ where $V_k = \{1, 2, \ldots, k\}$ and $E_k = \{\{1, 2\}, \{2, 3\}, \ldots, \{k - 1, k\}, \{k, 1\}\}$.

(a) Find the chromatic number of C_k, for $k \geq 3$,

(b) Prove that the chromatic polynomial of C_k is

$$\chi_{C_k}(n) = (n - 1)^k + (-1)^k (n - 1), \quad k \geq 3,$$

(c) Find the the number of acyclic orientations of C_k, for $k \geq 3$.

Solution. (b). Applying the general recursion for chromatic polynomials gives

$$\chi_{C_k}(n) = n(n - 1)^{k-1} - \chi_{C_{k-1}}(n).$$

The formula then follows by induction over k.

(c). Set $n = -1$.