Institutionen för matematik
**KTH**
Chaotic Dynamical Systems, Fall 2008
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**Homework assignment 2**

*This exercise set is due October 21, 2008*

1. (Devaney p. 93, 2) Let $E_\lambda(x) = \lambda e^x$. Determine the phase portrait for $E_\lambda$ for $\lambda > 0$.
2. (Devaney p. 93, 3) Consider $E_\lambda$ as in the previous excercise but for $\lambda < 0$. Prove that $E_\lambda$ has a unique fixed point that is attracting for $0 > \lambda > -e$ and is repelling for $\lambda < -e$.
3. (Devaney p. 93, 4)

   a) Prove that $E_\lambda^2$ is convex for $x$ such that $E'_\lambda(x) > -1$ and concave if $E'_\lambda(x) < -1$.

   b) Consider $E_\lambda(x)$ for $x < 0$. Use the information from a) to prove that $E_\lambda$ has a unique attracting fixed point if $-e < \lambda < 0$, and that for $\lambda < -e$ it has a repelling fixed point coexisting with a stable two-orbit.
4. Define a metric on $\Sigma_N = \{\mathbf{s} = (s_k)_{k=0}^\infty \mid s_k \in 1, \ldots, N\}$ by

   $$d_N(\mathbf{s}, \mathbf{t}) = \sum_{k=0}^{\infty} \frac{|s_k - t_k|}{N^k}.$$

   a) Prove that $d_N$ is a metric.

   b) Prove that if $s_i = t_i$ for $i = 0, \ldots, k$ then $d_N(\mathbf{s}, \mathbf{t}) \leq 1/N^k$. Similarly if $d_N(\mathbf{s}, \mathbf{t}) < 1/N^k$, then $s_i = t_i$ for $i \leq k$.

   For symbols $s$ and $t$ in the symbol space $\{1, 2, \ldots, N\}$, define the discrete metric by

   $$\delta(s,t) = \begin{cases} 0 \text{ if } s = t \\ 1 \text{ if } s \neq t. \end{cases}$$

   Define a metric $d'$ on $\Sigma_N$ by

   $$d'(\mathbf{s}, \mathbf{t}) = \sum_{k=0}^{\infty} \frac{\delta(s_i, t_i)}{3^i}$$

c) Prove that the metrics $d_N$ and $d'$ are *not equivalent*, i.e. that there *do not exist constants* $c$ and $C$ so that

$$c\, d_N(\mathbf{s}, \mathbf{t}) \leq d'(\mathbf{s}, \mathbf{t}) \leq C d_N(\mathbf{s}, \mathbf{t}),$$

where $c$ and $C$ do not depend on $\mathbf{s}$ and $\mathbf{t}$.

Does one of the relations between the metrics stated above hold?

d) State and prove a version of the statement in b) for the metric $d'$.