SF2722 Differential geometry, spring 2009

Homework problems 1 — To be handed in 10/3

1

(*Exercise 2.5.9 in the book.*) The map $\Phi : \mathrm{Gl}(n) \to \mathrm{Gl}(n)$, defined by $\Phi(Y) = Y^T Y$, is smooth since its coordinate functions are quadratic polynomials. Prove the following.

(1) Relative to the standard identification $T_I(\mathrm{Gl}(n)) = \mathfrak{M}(n)$, the differential $d\Phi_I : T_I(\mathrm{Gl}(n)) \to T_I(\mathrm{Gl}(n))$ has the formula

$$d\Phi_I(A) = A^T + A.$$

(2) The map Φ has constant rank $n(n+1)/2$.

(3) Using the above, conclude that the orthogonal group $\mathrm{O}(n) \subset \mathrm{Gl}(n)$ is a smooth, compact submanifold of dimension $n(n-1)/2$.

(4) Show that the vector subspace $T_I(\mathrm{O}(n)) \subset \mathfrak{M}(n)$ is the space of skew symmetric matrices.

2

Let $0 < b < a$. Show that the set of points $(x, y, z) \in \mathbb{R}^3$ satisfying

$$(a - \sqrt{x^2 + y^2})^2 + z^2 = b^2$$

is a smooth submanifold. Show that it is parallelizable (Definition 2.5.12, page 62.)

3

Let $V_1 = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ and $V_2 = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$ be vector fields on $\mathbb{R}^2$, show that they do not commute. Compute the flows Φ_t^1, Φ_t^2 of V_1, V_2 and verify that they also do not commute.

4

(*Similar to Exercise 2.8.21 in the book.*) Given $A \in \mathfrak{M}(n)$, explain how the right translation operation $R_A : \mathrm{Gl}(n) \to \mathfrak{M}(n)$ can be viewed as a vector field $R_A \in \mathfrak{X}(\mathrm{Gl}(n))$.

(1) Compute the Lie bracket $[R_A, R_B]$ for $A, B \in \mathfrak{M}(n)$.

(2) Compute $e^{tB} = \sum_{i=0}^{\infty} \frac{1}{i!}(tB)^i$ for the matrix $B = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, and explain how to find the flow on $\mathrm{Gl}(2)$ generated by R_B.

5

(*Similar to Exercise 3.1.2 in the book.*) Show that the sphere S^n has a C^∞ atlas consisting of two charts, the stereographic projections from the north resp. south poles (see Example 1.2.3).