SF2722 Differential geometry, spring 2009

Homework problems 3 — To be handed in 2/6

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1 Let ω be the $(n-1)$-form on $\mathbb{R}^n \setminus \{0\}$ defined by

$$\omega = \sum_{i=1}^{n} (-1)^{i-1} x_i \, dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n$$

where $\widehat{dx^i}$ is omitted. Let M be an n-dimensional submanifold with boundary in $\mathbb{R}^n$ (an open set with smooth boundary). Let $i : \partial M \to \mathbb{R}^n$ be the inclusion and compute

$$\int_{\partial M} i^*(\omega), \quad \text{and} \quad \int_{\partial M} i^*(\omega/r^n),$$

where r is the distance to the origin, and we assume $0 \notin \partial M$ for the second integral.

2 A manifold M is simply connected if every piecewise smooth map $S^1 \to M$ can be extended to a piecewise smooth map $D^2 \to M$. Show that $H^1(M) = 0$ if M is simply connected.

3 Let $a : S^n \to S^n$ be the antipodal map, $a(x) = -x$. Compute the effect of pullback a^* on forms and on cohomology.

a) Show that there are no nowhere zero vector fields on S^n when n is even. (*Hint:* There is a nowhere zero vector field on $S^n \Rightarrow$ There is a map $v : S^n \to S^n$ with $v(x) \cdot x = 0$ for all $x \in S^n \Rightarrow a$ is homotopic to the identity.)

b) Show that real projective space $\mathbb{R}P^n$ has a nowhere zero n-form (= is orientable) if and only if n is odd.

4 A symplectic form on a manifold M of dimension $2n$ is a closed two-form ω such that $\omega^n = \underbrace{\omega \wedge \cdots \wedge \omega}_{n}$ is never zero. Show that ω^n is not exact. Show that $H^{2k}(M^{2n}) \neq 0$ for $k = 0, \ldots, n$ if M^{2n} has a symplectic form. Show that S^2 is the only sphere which has a symplectic form.

5 Solve exercises 10.1.21, 10.1.22 in the book.

6 On the upper half-plane $H = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ we define a Riemannian metric by

$$g = \frac{1}{y^2}(dx \otimes dx + dy \otimes dy).$$

Compute the Levi-Civita connection and the curvature operator associated with this metric.