KTH Teknikvetenskap

SF2723 Topics in Mathematics - Matrix groups
Homework Assignment 11
2008-11-19

The solutions should be handed in no later than November 26, 2008. The final grade will be based upon the total score on the homework and on the oral exam. The total maximal score on the homework assignments is 200 and in order to pass, at least 100 is required.

In order to get full score on each problem, the written presentation of the solution should be clear and the arguments easy to follow.

---

1. For positive integers $m \leq n$, let $M_{m,n}(\mathbb{C})$ be the space of $m \times n$ matrices of rank m with complex entries. The general linear group, $\mathrm{Gl}_m(\mathbb{C})$ acts on $M_{m,n}$ by multiplication on the left.

   (a) Show that the set of orbits under this action, has a natural structure of an analytic manifold - the *Grassmannian*, $\mathrm{Grass}(m, n)$. (Hint: Given a matrix X in $M_{m,n}(\mathbb{C})$. After multiplication by an element A of $\mathrm{Gl}_n(\mathbb{C})$ on the right, we can assume that the first $m \times m$-minor of XA is non-singular. Hence the orbit under $\mathrm{Gl}_m(\mathbb{C})$ containing XA has a unique representative whose first m columns is the identity matrix. Use the remaining entries to define coordinates on the chart around the matrix X we start with.) **(4)**

   (b) Show that the Grassmannian, $\mathrm{Grass}(m, n)$, is a union of affine spaces. (Hint: Use the Gauss-Jordan elimination to row-echelon form.) **(3)**

2. Let $B_n \subseteq \mathrm{Gl}_n = \mathrm{Gl}_n(\mathbb{C})$ be the subset of upper triangular matrices.

   (a) Show that B_n is a Lie subgroup of Gl_n and compute the center of B_n. **(2)**

   (b) Compute the Lie algebra $\mathfrak{b}_n$ of B_n as a sub Lie algebra of $\mathfrak{gl}_n$. **(1)**

   (c) Show that the exponential map $\exp\colon \mathfrak{b}_n \longrightarrow B_n$ is surjective. **(2)**

   (d) Show that B_n acts on Gl_n by left multiplication and that all orbits are homeomorphic to B_n. **(1)**

   (e) The set of left cosets, $F\ell_n = \mathrm{Gl}_n / B_n$, has a natural structure of an analytic manifold - the *Flag manifold*. Determine the dimension of its tangent space. **(2)**